Quadratic Equations

DISCLAIMER

Quadratic Equations

Quadratic Polynomial:

$y = ax^2 + bx + c;\ a \neq 0$

a = leading coefficient

b = coefficient of linear term

c = absolute term

$y = f(x) = ax^2 + bx + c$

In case

$a = 0,\ b \neq 0$	$\Rightarrow$	$y = bx + c$ is linear polynomial
$a = c = 0,\ b \neq 0$	$\Rightarrow$	$y = bx$ is odd linear polynomial

Cubic Polynomial:

$y = ax^3 + bx^2 + cx + d;\ a \neq 0$

a = leading coefficient

d = absolute term

Roots of quadratic equation:

$y = ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Where $D = b^2 - 4ac$ is called discriminant.

$ax^2 + bx + c = 0$

Sum of roots $= -\dfrac{b}{a}$

Product of roots $= \dfrac{c}{a}$

$D = b^2 - 4ac$

Different graphs of Quadratic Expression:

(i) Graph of $y = ax^2 + bx + c$; $(a \neq 0,\ a, b, c \in R)$ when $a > 0,\ D < 0$

$a > 0 \ \Rightarrow$ Mouth facing upward

$D < 0 \ \Rightarrow$ Parabola neither touch nor cut x– axis (no real root)

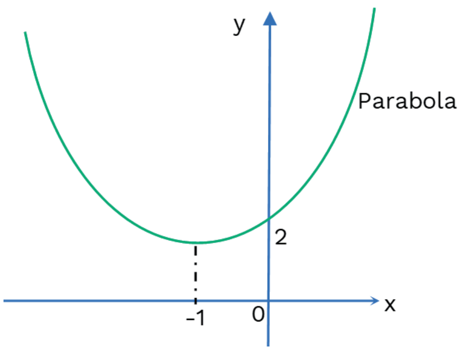
$y > 0,\ \forall\ x \in R$

Q. Plot $y = x^2 + 2x + 2$

A. $y = x^2 + 2x + 2 = (x + 1)^2 + 1$

$D = 2^2 - 8 = -4 < 0$

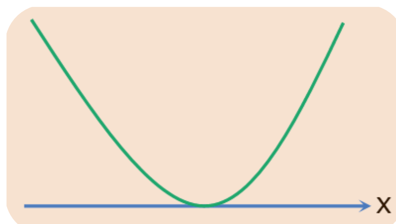
For $x = -1$, y is minimum



x	0	1	2	3	4	– 1	– 2	– 3	– 4	– 5	∞	-∞
y	2	5	10	17	26	1	2	5	10	17	∞	∞

(ii) Graph of $y = ax^2 + bx + c$; (a 0, $a, b, c, \in R$) when $a > 0$, $D = 0$

$a > 0 \Rightarrow$ Mouth facing upward,
$D = 0 \Rightarrow$ Parabola touches the x–axis (real and equal root)
$y = 0$ for only one value of x (root)
$y > 0, \forall x \in R - \{root\}$



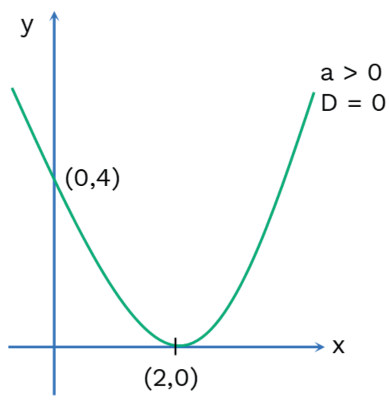
Q. Plot $y = x^2 - 4x + 4$

A. $y = x^2 - 4x + 4 = (x-2)^2$

$D = 0 \Rightarrow y \geq 0, x \in R$

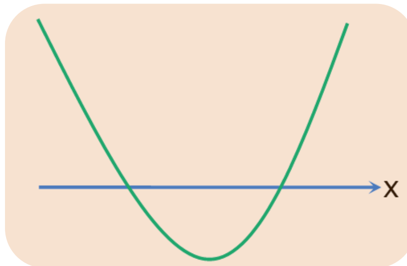
Leading coefficient > 0

x	0	1	2	3	4	5	6	– 1	– 2	∞	–∞
y	4	1	0	1	4	9	16	9	16	∞	∞



(iii) Graph of $y = ax^2 + bx + c$; $(a \neq 0, a, b, c \in R)$ when $a > 0$ and $D > 0$

$a > 0 \Rightarrow$ Mouth facing upward parabola.
$D > 0 \Rightarrow$ Parabola cuts the x–axis at 2 distinct points (two distinct real roots)



Q. Plot $y = x^2 - 3x + 2$

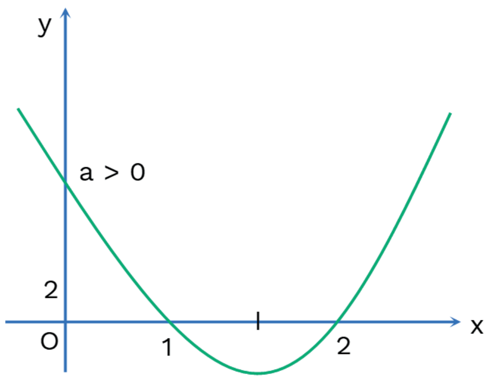
A. $D = 3^2 - 4(2) = 1 > 0$

x	0	1	2	3	4	3/2	∞	$-\infty$
y	2	0	0	2	6	–1/4	∞	∞

$y > 0 \Rightarrow x \in (-\infty, 1) \cup (2, \infty)$

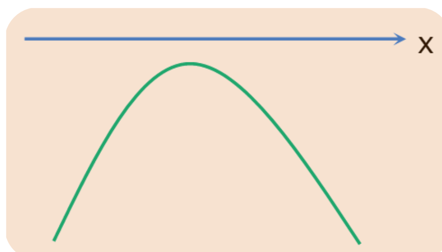
$y < 0 \Rightarrow x \in (1, 2)$

$y = 0 \Rightarrow x \in \{1, 2\}$



(iv) Graph of $y = ax^2 + bx + c$; $(a \neq 0, a, b, c \in R)$ when $a < 0$ and $D < 0$

$a < 0 \Rightarrow$ Mouth facing downward
$D < 0 \Rightarrow$ No real root
$y < 0, \forall x \in R$



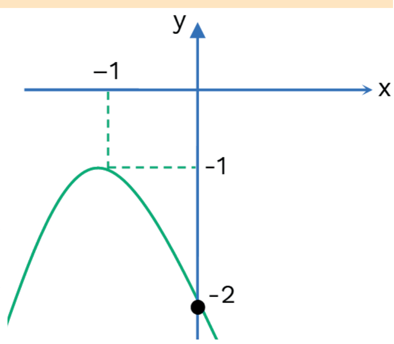
Q. Plot $y = -x^2 - 2x - 2$

A. $y = -(x + 1)^2 - 1$

$D < 0$

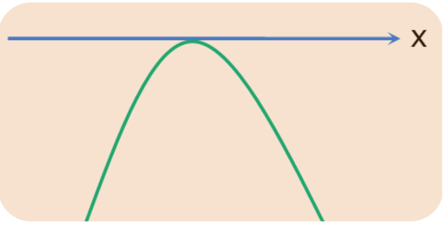
Leading coefficient < 0

x	0	1	2	3	–1	–2	–3	∞	$-\infty$
y	– 2	–5	–10	–17	–1	–2	–5	$-\infty$	$-\infty$



(v) Graph of $y = ax^2 + bx + c$; ($a \neq 0$, a, b, c $\in$ R) when $a < 0$ and $D = 0$

$a < 0 \Rightarrow$ Mouth facing downward
$D = 0 \Rightarrow$ Equal roots, i.e., parabola touches x–axis
$y \leq 0, \forall x \in R$



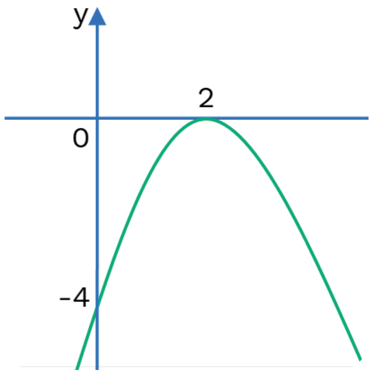
Q. Plot $y = -x^2 + 4x - 4$

A. $= -(x-2)^2$

$D = 0$

Leading coefficient < 0

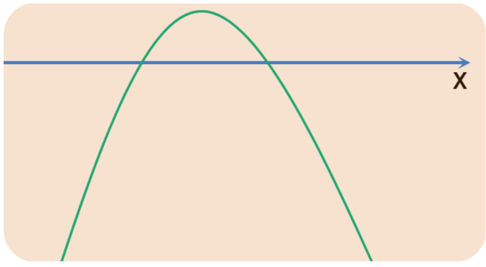
x	0	1	2	3	4	–1	∞	–∞
y	–4	–1	0	– 1	–4	–9	–∞	–∞



(vi) Graph of $y = ax^2 + bx + c$; ($a \neq 0$, a, b, c $\in$ R) when $a < 0$ and $D > 0$

$a < 0 \Rightarrow$ Mouth facing downward

$D > 0 \Rightarrow$ Two distinct real roots parabola cuts x– axis at two distinct points



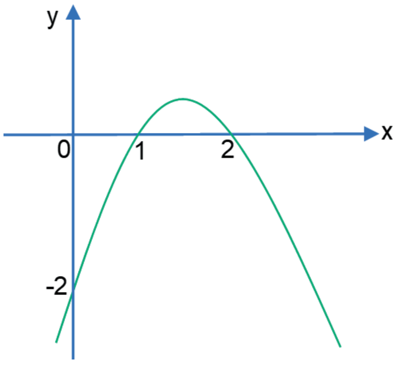
Q. Plot $y = -x^2 + 3x - 2$

A. $y = -(x-1)(x-2)$

$D > 0$

Leading coefficient < 0

x	0	1	2	3	4	–1	–2	∞	–∞
y	–2	0	0	–2	–6	–6	–12	–∞	–∞

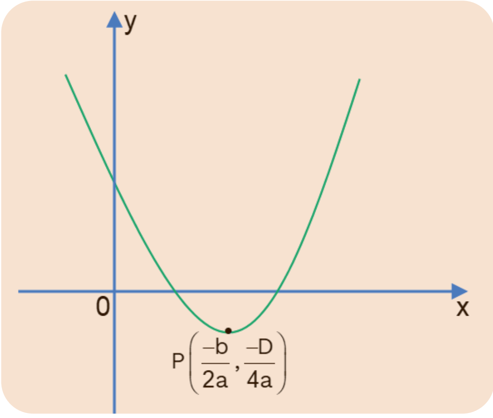


Co–ordinate of vertex:

$y = ax^2 + bx + c$ ($a \neq 0$, a, b, $c \in R$)

$$x = -\frac{b}{2a}$$

$$y = -\frac{D}{4a}$$



Nature of roots

$y = ax^2 + bx + c$, ($a \neq 0$, a, b, $c \in R$)

$D > 0 \Leftrightarrow$ roots are real and distinct (unequal)
$D = 0 \Leftrightarrow$ roots are real and coincident (equal)
$D < 0 \Leftrightarrow$ roots are imaginary

Know the facts

If p + iq is one root of a quadratic equation then the other root must be the conjugate p – iq and vice versa ($p, q \in R$ and $i = \sqrt{-1}$) provided coefficients are real.

Point to Remember!!!

Nature of roots

Consider the quadratic equation $ax^2 + bx + c = 0$, where a, b, $c \in Q$ and $a \neq 0$ then

(i) If D is perfect square, then roots are rational.

(ii) If $\alpha = p + \sqrt{q}$ is one root in this case (where p is rational and $\sqrt{q}$ is a surd) then other root will be $p - \sqrt{q}$.

Examples:

Q. Both the roots of the equation $(x - b)(x - c) + (x - c)(x - a) + (x - a)(x - b) = 0$ are always

(A) Positive (B) Negative (C) Real (D) None of these

A. **(C)**

Given equation is

$3x^2 - 2(a + b + c)x + (ab + bc + ca) = 0$

Now $D = 4(a + b + c)^2 - 4 \times 3(ab + bc + ca)$

$= 2\{(a - b)^2 + (b - c)^2 + (c - a)^2\}$

Clearly, $D \geq 0 \Rightarrow$ both roots are always real.

Q. The number of real solutions of the equation $|x|^2 - 3|x| + 2 = 0$ is

(A) 4 (B) 1 (C) 3 (D) 2

A. **(A)**

Let $|x| = t$

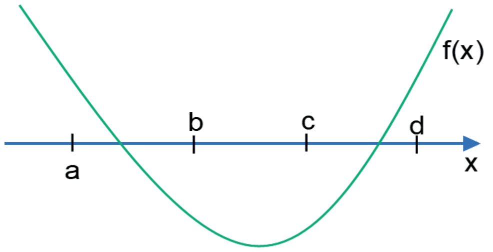
$\therefore$ given equation is $t^2 - 3t + 2 = 0$

$(t - 1)(t - 2) = 0$

$t = 1$, $t = 2$

$|x| = 1$, $|x| = 2$

$x = \pm 1$, $x = \pm 2$



Q. Let $f(x)$ be a quadratic expression which is positive for all real values of x. If $g(x) = f(x) + f'(x) + f''(x)$ then for any real x :

(A) $g(x) < 0$ (B) $g(x) > 0$ (C) $g(x) = 0$ (D) $g(x) \geq 0$

A. **(B)**

Let $f(x) = ax^2 + bx + c$ $(a \neq 0, a, b, c \in R)$

Also, $f(x) > 0 \ \forall x \in R$ $\Rightarrow a > 0$ and $D < 0$

Hence $b^2 - 4ac < 0$ ...(i)

Now $g(x) = (ax^2 + bx + c) + (2ax + b) + 2a = ax^2 + (b + 2a)x + (b + c + 2a)$

$D = (b + 2a)^2 - 4a(b + c + 2a)$

$= b^2 + 4a^2 + 4ab - 4ab - 4ac - 8a^2$

$= b^2 - 4a^2 - 4ac$

$= (b^2 - 4ac) - 4a^2 < 0$ {from (i)}

Hence for $g(x)$; $D < 0$, $a > 0$

$\Rightarrow g(x) > 0, \ \forall x \in R$

Q. Let α, β be the roots of the equation $(x - a)(x - b) = c$, $c \neq 0$. Then the roots of the equation $(x - \alpha)(x - \beta) + c = 0$ are

(A) a, c (B) b, c (C) a, b (D) a + c, b + c

A. **(C)**

As α, β are roots of equation $(x - a)(x - b) - c = 0$

hence $(x - a)(x - b) - c = (x - \alpha)(x - \beta)$

$\Rightarrow (x - a)(x - b) = (x - \alpha)(x - \beta) + c$

Clearly, roots of equation $(x - \alpha)(x - \beta) + c = 0$ are a, b

Q. **True/False**

If $a < b < c < d$ then the roots of the equation $(x - a)(x - c) + 2(x - b)(x - d) = 0$ are real and distinct

A. **True**

Let $f(x) = (x - a)(x - c) + 2(x - b)(x - d)$

Now $f(a) = 2(a - b)(a - d) > 0$

$f(b) = (b - a)(b - c) < 0$

$f(c) = 2(c - b)(c - d) < 0$

$f(d) = (d - a)(d - c) > 0$

Also, graph of f(x) is upward parabola.

Clearly, both roots of $f(x) = 0$ are real and distinct.

Q. The number of points of intersection of two curves $y = 2\sin x$ and $y = 5x^2 + 2x + 3$ is

(A) 0 (B) 1 (C) 2 (D) ∞

A. **A**

$y = 5x^2 + 2x + 3$

$D = 2^2 - 4(5)(3) = -56 < 0$

$a = 5 > 0 \Rightarrow y > 0 \ \forall \ x \in R$

Vertex $\left(\frac{-b}{2a}, \frac{-D}{4a}\right) = \left(\frac{-1}{5}, \frac{14}{5}\right)$

Clearly, $y = 5x^2 + 2x + 3 \geq \frac{14}{5}$ and $y = 2\sin x \leq 2$

Hence, both curves do not intersect at any point.

Q. For all x; $x^2 + 2ax + 10 - 3a > 0$ then interval in which a lies is:

(A) $a < -5$ (B) $-5 < a < 2$ (C) $a > 5$ (D) $2 < a < 5$

A. **B**

$D < 0$

$(2a)^2 - 4(10 - 3a) < 0$

$4(a^2 - 10 + 3a) < 0$

$a^2 + 5a - 2a - 10 < 0$

$a(a + 5) - 2(a + 5) < 0$

$(a - 2)(a + 5) < 0$

$\Rightarrow a \in (-5, 2)$

Q. If $b > a$ then the equation $(x - a)(x - b) - 1 = 0$ has

(A) Both roots in (a, b)

(B) Both roots in $(-\infty, a)$

(C) both roots in (b, ∞)

(D) one root in $(-\infty, a)$ and the other in (b, ∞)

A. **D**

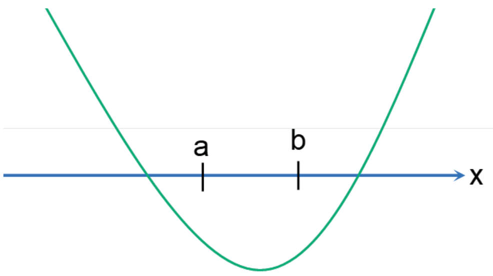
Let $f(x) = (x - a)(x - b) - 1$

Now $f(a) = -1$

$f(b) = -1$

As $f(x)$ is a upward parabola

Clearly, $f(x) = 0$ have one root in $(-\infty, a)$ and other in (b, ∞)



Q. If $a, b, c \in$ odd integers then prove that $ax^2 + bx + c = 0$ can't have rational roots.

A. Let $D = b^2 - 4ac = m^2$ ($m \in$ odd integer)

$\Rightarrow b^2 - m^2 = 4ac$

$\Rightarrow (2k_2 + 1)^2 - (2k+1)^2 = 4ac$

$\Rightarrow (2k_2 + 2k + 2)(2k_2 - 2k) = 4ac$

$\Rightarrow 4(k_2 + k + 1)(k_2 - k) = 4ac$

Clearly, LHS is multiple of 8 while RHS is not a multiple of 8

hence, D cannot be m^2, so roots cannot be rational.

Q. Prove that $x^8 - 24x^7 - 18x^5 + 39x^2 + 1155 = 0$ doesn't have integral roots.

A. $(x^8 + 39x^2) - 6(4x^7 + 3x^5) + 1155 = 0$ ...(i)

For $x \in I$

$x^8 + 39x^2 \Rightarrow$ even

$6(4x^7 + 3x^5) \Rightarrow$ even

$\therefore$ equation (i) cannot be true.

Q. If the equation $\sin^4 x - (K+2)\sin^2 x - (K+3) = 0$ has a solution then K must lie in the interval

(A) $(-4, -2)$ (B) $[-3, 2)$ (C) $(-4, -3)$ (D) $[-3, -2]$

A. **D**

$$\sin^2 x = \frac{(K+2) \pm \sqrt{(K+2)^2 + 4(K+3)}}{2}$$

$$= \frac{(K+2) \pm \sqrt{(K^2 + 8K + 16}}{2}$$

$$= \frac{(K+2) \pm (K+4)}{2}$$

$= K + 3, -1$

Clearly $\sin^2 x \neq -1$ hence the equation to have a solution $\sin^2 x = K + 3$

$0 \le K + 3 \le 1$

$-3 \le K \le -2$

Quadratic Equations

Q. If a, b $\in$ R, a $\neq$ 0 and the quadratic equation $ax^2 - bx + 1 = 0$ has imaginary roots then a + b+ 1 is

(A) Positive (B) Negative

(C) zero (D) Depends on sign of b

A. **A**

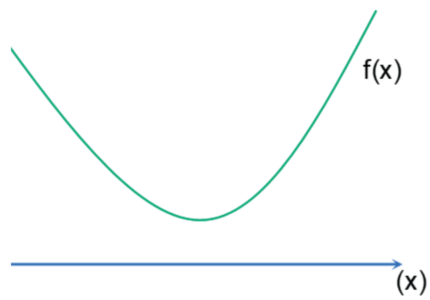
$\because$ roots are imaginary of given equation.

$\therefore$ D < 0 $\Rightarrow$ $b^2 - 4a < 0$

Now let $f(x) = ax^2 - bx + 1$

for which D < 0, f(0) = 1 > 0

Hence graph of f(x) is



$\Rightarrow$ $f(x) > 0 \quad \forall x \in R$

$\therefore$ $f(-1) > 0$

$a + b + 1 > 0$

Q. **Multiple Objective Type**

The graph of the quadratic polynomial; $y = ax^2 + bx + c$ is as shown in the figure then:

(A) $b^2 - 4ac > 0$ (B) b < 0 (C) a > 0 (D) c < 0

A. **ABCD**

$\because$ Graph cuts x –axis at two distinct point

$\therefore$ D > 0 $\Rightarrow$ $b^2 - 4ac > 0$

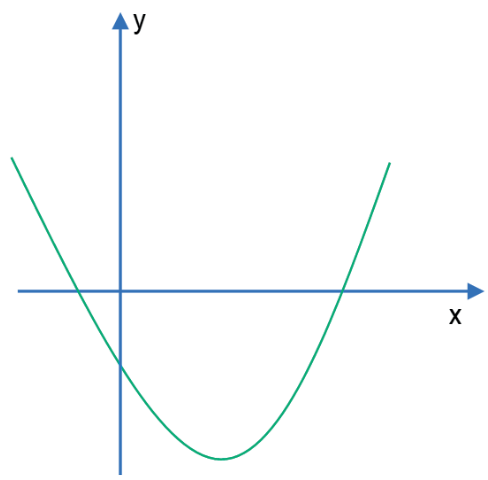
Graph is upward parabola

$\Rightarrow$ a > 0

Graph cuts negative y– axis

$\Rightarrow$ f(0) = c < 0

Also, vertex lies in IV quadrant



$$\Rightarrow \frac{-b}{2a} > 0$$

$$\Rightarrow \frac{b}{2a} < 0$$

$$\Rightarrow b < 0$$

Relation between roots and coefficient of quadratic equation

$ax^2 + bx + c = 0$; a $\neq$ 0, a, b, c $\in$ R

$\Rightarrow ax^2 + bx + c = a(x-\alpha)(x-\beta) = 0$

$\Rightarrow \alpha + \beta = \frac{-b}{a}$ and $\alpha\beta = \frac{c}{a}$

Know the facts

$$x^2 + \left(\frac{b}{a}\right)x + \left(\frac{c}{a}\right) = (x - \alpha)(x - \quad)$$

Formation of quadratic equation

$x^2 -$ (sum of roots)x + (product of roots) = 0

Q. Form a quadratic equation with rational coefficients whose one root is tan75°

A. $\because$ one root $\alpha = \tan 75° = 2 + \sqrt{3}$

$\therefore$ other root $\beta = 2 - \sqrt{3}$ as coefficients are rational.

Now $\alpha + \beta = 4$, $\alpha\beta = 2^2 - \left(\sqrt{3}\right)^2 = 1$ required equation is

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

$x^2 - 4x + 1 = 0$

Q. Form a quadratic equation with rational coefficients whose one root is cos 36°

A. $\because$ one root $\alpha = \cos 36° = \dfrac{\left(1+\sqrt{5}\right)}{4}$

$\therefore$ other root $\beta = \dfrac{\left(1-\sqrt{5}\right)}{4}$ as coefficients are rational

Now $\alpha + \beta = \dfrac{1}{2}$, $\alpha\beta = \dfrac{1^2 - \left(\sqrt{5}\right)^2}{16} = -\dfrac{1}{4}$

Required equation is:

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

$x^2 - \dfrac{1}{2}x - \dfrac{1}{4} = 0$

$\Rightarrow 4x^2 - 2x - 1 = 0$

Q. Form a quadratic equation with rational coefficients whose one root is $\tan \dfrac{\pi}{8}$

A. $\because$ one root $\alpha = \tan\dfrac{\pi}{8} = (-1) + \sqrt{2}$

$\therefore$ other root $\beta = (-1) - \sqrt{2}$ as coefficient are rational

Now $\alpha + \beta = -2$, $\alpha\beta = (-1)^2 - \left(\sqrt{2}\right)^2 = -1$

Required equation is:

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

$x^2 + 2x - 1 = 0$

Q. If α, β are the roots of the quadratic equation $x^2 - 2x + 5 = 0$ then form a quadratic equation whose roots are $\alpha^3 + \alpha^2 - \alpha + 22$ and $\beta^3 + 4\beta^2 - 7\beta + 35$

A. $\alpha^2 - 2\alpha + 5 = 0$, $\beta^2 - 2\beta + 5 = 0$

$\alpha^3 + \alpha^2 - \alpha + 22 = (\alpha^2 - 2\alpha + 5)(\alpha + 3) + 7 = 7$

$\beta^3 + 4\beta^2 - 7\beta + 35 = (\beta^2 - 2\beta + 5)(\beta + 6) + 5 = 5$

$\because$ equation with roots 7, 5 is $x^2 - 12x + 35 = 0$

Q. $x^2 - 17x - 6 = 0$, has roots α and β. Let $a_n = \alpha^{n+2} + \beta^{n-2}$ find the value of $\frac{a_{10} - 6a_8}{a_9}$

A. $a_{10} - 6a_8 = (\alpha^{12} + \beta^8) - 6(\alpha^{10} + \beta^6)$

$= (\alpha^{12} - 6\alpha^{10}) + (\beta^8 - 6\beta^6)$

$a_{10} - 6a_8 = \alpha^{10}(\alpha^2 - 6) + \beta^6(\beta^2 - 6)$ ...(i)

$\because$ $\alpha^2 - 17\alpha - 6 = 0 \Rightarrow \alpha^2 - 6 = 17\alpha$

Similarly, $\beta^2 - 6 = 17\beta$

From (i) $a_{10} - 6a_8 = \alpha^{10}(17\alpha) + \beta^6(17\beta)$

$= 17(\alpha^{11} + \beta^7) = 17a_9$

Hence $\frac{a_{10} - 6a_8}{a_9} = 17$

Q. $x^2 - ax + b = 0$, α, β are its roots. $V_n = \alpha^n + \beta^n$ then show that

$V_{n+1} = aV_n - b\,v_{n-1}$

A. $\because$ $\alpha^2 - a\alpha + b = 0$

Multiply by α^{n-1} then $\alpha^{n+1} = a\alpha^n - b\alpha^{n-1}$ ... (i)

Similarly $\beta^{n+1} = a\beta^n - b\beta^{n-1}$ ... (ii)

(i) + (ii)

$(\alpha^{n+1} + \beta^{n+1}) = a(\alpha^n + \beta^n) - b(\alpha^{n-1} + \beta^{n-1})$

$V_{n+1} = a\,V_n - b\,V_{n-1}$

Q. Find monic cubic polynomial with $f(1) = 1$, $f(2) = 4$, $f(3) = 9$

A. $f(1) = 1^2$, $f(2) = 2^2$, $f(3) = 3^2$

Then cubic is $f(x) = (x - 1)(x - 2)(x - 3) + x^2$

Q. Solve $(x - 7)(x - 3)(x + 5)(x + 1) = 1680$

A. $(x - 7)(x + 5)(x - 3)(x + 1) = 1680$

$(x^2 - 2x - 35)(x^2 - 2x - 3) = 1680$

Let $x^2 - 2x - 3 = \alpha$

Then $(\alpha - 32)\alpha = 1680$

$\alpha^2 - 32\alpha - 1680 = 0$

$(\alpha - 60)(\alpha + 28) = 0$

$\alpha - 60 = 0$, $\alpha + 28 = 0$

$x^2 - 2x - 63 = 0$ $\quad$ $x^2 - 2x + 25 = 0$

$(x - 9)(x + 7) = 0$ $\quad$ $D < 0$

$x = -7, 9$

Q. Solve $\left(5+2\sqrt{6}\right)^{x^2-3} + \left(5-2\sqrt{6}\right)^{x^2-3} = 10$

A. $\left(5+2\sqrt{6}\right)\left(5-2\sqrt{6}\right) = 1$

$$\therefore \left(5+2\sqrt{6}\right)^{x^2-3} = \frac{1}{\left(5-2\sqrt{6}\right)^{x^2-3}} = t \text{ (let)}$$

Now equation $t + \frac{1}{t} = 10 \Rightarrow t^2 - 10t + 1 = 0$

$$t = \frac{10 \pm \sqrt{96}}{2} = 5 \pm 2\sqrt{6}$$

$t = 5 \pm 2\sqrt{6}$,

$$\left(5+2\sqrt{6}\right)^{x^2-3} = \left(5+2\sqrt{6}\right), \left(5+2\sqrt{6}\right)^{x^2-3} = \left(5+2\sqrt{6}\right)^{-1}$$

$x^2 - 3 = 1$, $\quad x^2 - 3 = -1$

$x = \pm 2$, $\quad x = \pm\sqrt{2}$

$x = \{\pm\sqrt{2}, \pm 2\}$

Inequalities

Rule:

1. Adding positive number both sides inequality remains same.
Ex. $2 > 1 \Rightarrow 3 > 2$

2. Subtracting both sides by positive number inequality remains same.
Ex. $2 > 1 \Rightarrow 1 > 0$

3. Multiply and divide by positive number doesn't affect inequality but multiplying or dividing inequality with negative number changes sign of inequality.
Ex. $4 > 2 \Rightarrow -2 < -1$

Type – 1: Expression which can not be factorized

Q.1 $x^2 + x + 1 > 0$

A. $\because D = 1^2 - 4(1)(1) < 0$ and $a = 1 > 0$

$\therefore x^2 + x + 1 > 0, \ \forall x \in R$

$\therefore x \in R$

Q.2 $x^2 - 3x + 4 < 0$

A. $D = (-3)^2 - 4(1)(4) < 0$ and $a = 1 > 0$

$\therefore x^2 - 3x + 4 > 0, \forall x \in R$

$\therefore x \in \phi$

Q.3 **3.** $3x^2 - 7x + 6 > 0$

A. **Sol.** $D = (-7)^2 - 4(3)(6) < 0$ and $a = 3 > 0$

$\therefore 3x^2 - 7x + 6 > 0, \forall x \in R$

$\therefore x \in R$

Q.4 **4.** $-x^2 - 2x - 4 > 0$

A. **Sol.** $D = (-2)^2 - 4(-1)(-4) < 0$ and $a = -1 < 0$

$\therefore -x^2 - 2x - 4 < 0, \ \forall x \in R$

$\therefore x \in \phi$

Type–2: Expression which can be factorized:

Steps:

(i) Factorize in linear as much as possible.

(ii) Make coefficient of x as 1 in all linear by multiplying and dividing by appropriate number.

(iii) Mark zeroes of linear on number line.

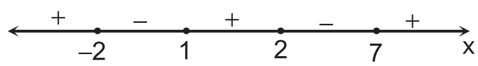
(iv) Give sign to respective area on number line.

Q.1 $(1-x)(4+2x)(x-2)(x-7) > 0$

A. $(x-1)\,2(x+2)(x-2)(x-7) < 0$

$(x-1)(x+2)(x-2)(x-7) < 0$

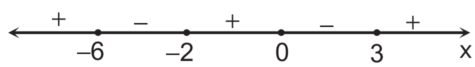
$x \in (-2, 1) \cup (2, 7)$



Q.2 $(x^2 - x - 6)(x^2 + 6x) > 0$

A. $(x-3)(x+2)\,x(x+6) > 0$

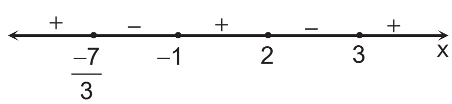
$x \in (-\infty, -6) \cup (-2, 0) \cup (3, \infty)$



Q.3 $(x+1)(x-3)(x-2)3(x+7/3) < 0$

A. $(x+1)(x-3)(x-2)(x+7/3) < 0$

$x \in \left(-\frac{7}{3}, -1\right) \cup (2, 3)$

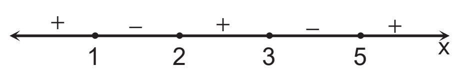


Type – 3:

Q.1 $(x^2 - 5x + 6)(x^2 - 6x + 5) \leq 0$

A. $(x-2)(x-3)(x-1)(x-5) \leq 0$

$x \in [1, 2] \cup [3, 5]$

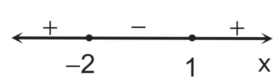


Q.2 $2 - x - x^2 \geq 0$

A. $x^2 + x - 2 \leq 0$

$(x+2)(x-1) \leq 0$

$x \in [-2, 1]$



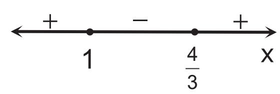
Q.3 $3x^2 - 7x + 4 \geq 0$

A. $3x^2 - 3x - 4x + 4 \geq 0$

$3x(x - 1) - 4(x - 1) \geq 0$

$(3x - 4)(x - 1) \geq 0$

$x \in (-\infty, 1] \cup \left[\frac{4}{3}, \infty\right)$



Type–4: Repeated Linear factor

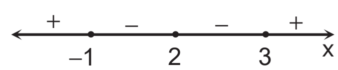
Rules:

(i) Factors with even power doesn't affect sign.

(ii) Factors with odd power affect sign as linear.

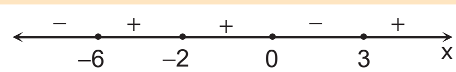
Q.1 $(x + 1)(x - 3)(x - 2)^2 > 0$

A. $x \in (-\infty, -1) \cup (3, \infty)$



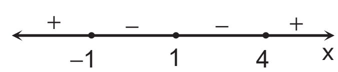
Q.2 $x(x + 6)(x + 2)^2 (x - 3) > 0$

A. $x \in (-6, -2) \cup (-2, 0) \cup (3, \infty)$



Q.3 $(x - 1)^2 (x + 1)^3 (x - 4) < 0$

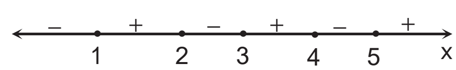
A. $x \in (-1, 1) \cup (1, 4)$



Type–5: Rational Inequality

Q. $\dfrac{(x-1)(x-2)(x-3)}{(x-4)(x-5)} < 0$

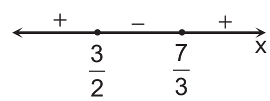
A. $x \in (-\infty, 1) \cup (2, 3) \cup (4, 5)$



Q. $\dfrac{2x-3}{3x-7} < 0$

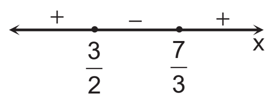
A. $\dfrac{2(x-3/2)}{3(x-7/3)} < 0$

$\dfrac{(x-3/2)}{(x-7/3)} < 0; \quad x \in \left(\frac{3}{2}, \frac{7}{3}\right)$



Q. $\dfrac{2x-3}{3x-7} \geq 0$

A. $\dfrac{2\left(x-\dfrac{3}{2}\right)}{3\left(x-\dfrac{7}{3}\right)} \geq 0$

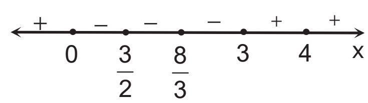


$\dfrac{(x-3/2)}{(x-7/3)} \geq 0$

$x \in \left(-\infty, \dfrac{3}{2}\right] \cup \left(\dfrac{7}{3}, \infty\right)$

Q. $\dfrac{x^3(2x-3)^2(x-4)^6}{(x-3)^3(3x-8)^4} \leq 0$

A. $x \in \left[0, \dfrac{8}{3}\right) \cup \left(\dfrac{8}{3}, 3\right) \cup \{4\}$



Q. $\dfrac{x^2-5x+12}{x^2-4x+5} > 3$

A. For $x^2 - 4x + 5$

$D = 16 - 4(5) < 0,\ a = 1 > 0 \Rightarrow$ always positive

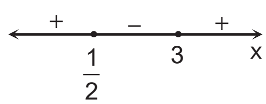
Hence By cross multiplication

$x^2 - 5x + 12 > 3x^2 - 12x + 15$

$2x^2 - 7x + 3 < 0 \Rightarrow 2x^2 - 6x - x + 3 < 0$

$(2x - 1)(x - 3) < 0$

$x \in \left(\dfrac{1}{2}, 3\right)$



Q. $\dfrac{x^2-5x+6}{x^2+x+1} < 0$

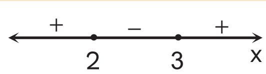
A. For $x^2 + x + 1$

$D = 1^2 - 4.1.1 < 0,$ $a > 0 \Rightarrow$ always positive.

Hence given inequality reduces to $x^2 - 5x + 6 < 0$

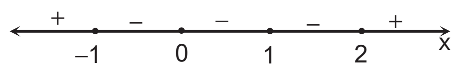
$(x - 2)(x - 3) < 0$

$X \in (2, 3)$



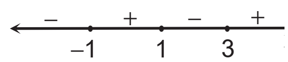
Q. $\dfrac{(x-1)^2(x+1)^3}{x^4(x-2)} < 0$

A. $x \in (-1, 0) \cup (0, 1) \cup (1, 2)$



Q. $\dfrac{x+1}{x-1} \geq \dfrac{x+5}{x+1}$

A. $\dfrac{x+1}{x-1} - \dfrac{x+5}{x+1} \geq 0$



$\Rightarrow \dfrac{(x+1)^2 - (x-1)(x+5)}{(x-1)(x+1)} \geq 0$

$\Rightarrow \dfrac{(x^2+2x+1) - (x^2+4x-5)}{(x-1)(x+1)} \geq 0$

$\Rightarrow \dfrac{-2x+6}{(x-1)(x+1)} \geq 0$

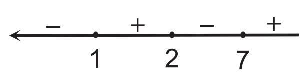
$\Rightarrow \dfrac{-2(x-3)}{(x-1)(x+1)} \geq 0$

$\Rightarrow \dfrac{(x-3)}{(x-1)(x+1)} \leq 0$

$x \in (-\infty, -1) \cup (1,3]$

Q. $\dfrac{2(x-4)}{(x-1)(x-7)} \geq \dfrac{1}{(x-2)}$

A. $\dfrac{2(x-4)}{(x-1)(x-7)} - \dfrac{1}{(x-2)} \geq 0$



$\Rightarrow \dfrac{2(x-4)(x-2) - (x-1)(x-7)}{(x-1)(x-7)(x-2)} \geq 0$

$\Rightarrow \dfrac{x^2 - 4x + 9}{(x-1)(x-7)(x-2)} \geq 0$

Consider $(x^2 - 4x + 9)$:

$D = 16 - 4(9) < 0,\ a = 1 > 0$ $\Rightarrow$ Always positive.

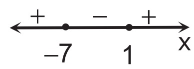
Hence inequality becomes

$$\frac{1}{(x-1)(x-7)(x-2)} \geq 0$$

$x \in (1, 2) \cup (7, \infty)$

Q. $\dfrac{x^2+6x-7}{|x+4|}<0$

A. Clearly $|x+4|>0, \forall x \in R-\{-4\}$



Hence the inequality becomes

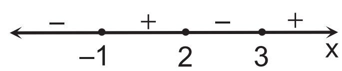
$x^2+6x-7<0, x \neq -4$

$(x+7)(x-1)<0, x \neq -4$

$x \in (-7, 1) - \{-4\}$

Q. Let $y=\sqrt{\dfrac{(x+1)(x-3)}{(x-2)}}$. Find the real values of x for which y takes real values.

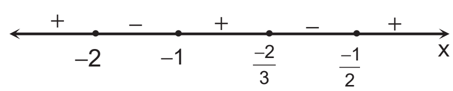
A. y to be real $\dfrac{(x+1)(x-3)}{(x-2)} \geq 0$



$x \in [-1, 2) \cup [3, \infty)$

Q. Find the set of all x for which $\dfrac{2x}{2x^2+5x+2} \geq \dfrac{1}{x+1}$

A. $\dfrac{2x}{2x^2+5x+2}-\dfrac{1}{x+1} \geq 0$



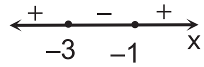
$\Rightarrow \dfrac{2x(x+1)-\left(2x^2+5x+2\right)}{\left(2x^2+5x+2\right)(x+1)} \geq 0$

$\Rightarrow \dfrac{(3x+2)}{(2x+1)(x+2)(x+1)} \leq 0$

$x \in (-2,-1) \cup \left[\dfrac{-2}{3}, \dfrac{-1}{2}\right)$

Q. Solve $\left|x^2+4x+3\right|+2x+5=0$

A. Consider $x^2+4x+3=(x+1)(x+3)$



Case– I:

Let $x^2+4x+3 \geq 0 \Rightarrow x \in (-\infty,-3] \cup [-1, \infty)$

So, given equation becomes $x^2 + 4x + 3 + 2x + 5 = 0$

$x^2 + 6x + 8 = 0$

$(x + 2)(x + 4) = 0$

$\Rightarrow x = -2, \quad x = -4 \quad$ but $x \in (-\infty, -3] \cup [-1, \infty)$

$\Rightarrow x = -4 \quad$ ...(i)

Case– II:

Let $x^2 + 4x + 3 < 0 \Rightarrow x \in (-3, -1)$

then equation becomes $-\left(x^2 + 4x + 3\right) + 2x + 5 = 0$

$$x = \frac{-2 \pm \sqrt{4 + 4(2)}}{2} = -1 \pm \sqrt{3}$$

$x = -1 + \sqrt{3}$, $x = -1 - \sqrt{3}$ but $x \in (-3, -1)$

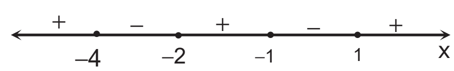
$\Rightarrow x = \left(-1 - \sqrt{3}\right)$ ...(ii)

Now, (i) U (ii)

So, x = $\left\{-4, -1 - \sqrt{3}\right\}$

Q. $\left(x^2 + 3x + 1\right)\left(x^2 + 3x - 3\right) \geq 5$

A. Let $x^2 + 3x = \alpha$



$(\alpha + 1)(\alpha - 3) - 5 \geq 0 \Rightarrow \alpha^2 - 2\alpha - 8 \geq 0$

$\Rightarrow (\alpha - 4)(\alpha + 2) \geq 0$

$\Rightarrow \left(x^2 + 3x - 4\right)\left(x^2 + 3x + 2\right) \geq 0$

$\Rightarrow (x + 4)(x - 1)(x + 1)(x + 2) \geq 0$

$x \in (-\infty, -4] \cup [-2, -1] \cup [1, \infty)$

Q. $1 < \dfrac{3x^2 - 7x + 8}{x^2 + 1} \leq 2$

A. $\because x^2 + 1 > 0 \ \forall x \in R$

$\therefore$ given inequality is

$x^2 + 1 < 3x^2 - 7x + 8 \leq 2x^2 + 2$

$\Rightarrow x^2 + 1 < 3x^2 - 7x + 8$ and $3x^2 - 7x + 8 \leq 2x^2 + 2$

$2x^2 - 7x + 7 > 0$ and $x^2 - 7x + 6 \leq 0$

$2x^2 - 7x + 7 > 0$ and $(x - 1)(x - 6) \leq 0$

(For $2x^2 - 7x + 7 > 0$; $D < 0$, $a > 0$)

$\therefore x \in R \cap x \in [1, 6]$

$x \in [1, 6]$

Q. Find the set of values of 'a' for which the quadratic polynomials

(i) $(a+4)x^2 - 2ax + 2a - 6 < 0, \ \forall x \in R$

(ii) $(a-1)x^2 - (a+1)x + (a+1) > 0, \ \forall x \in R$

A. **(i) Case–I:** $a + 4 \neq 0$

$D < 0$ and $(a + 4) < 0$

$4a^2 - 4(a+4)2(a-3) < 0$ and $a < -4$

$a^2 - 2(a^2 + a - 12) < 0$

$a^2 + 2a - 24 > 0$

$(a + 6)(a - 4) > 0$

$a \in (-\infty, -6) \cup (4, \infty)$

Also, $a < -4$

Hence $a \in (-\infty, -6)$ ...(i)

Case–II: $a + 4 = 0 \Rightarrow a = -4$

then given inequality becomes

$(0)x^2 + 8x - 14 < 0, \ \forall\!\!\!/ x \in R$

Which is not possible $\Rightarrow a \in \phi$ ...(ii)

(i) ∪ (ii)

$\therefore a \in (-\infty, -6)$

(ii) Case–I: $a - 1 \neq 0$

then $D < 0$ and $a - 1 > 0$

$(a + 1)^2 - 4(a - 1)(a + 1) < 0$ and $a > 1$

$(a + 1)\{(a + 1) - 4(a - 1)\} < 0$

$(a + 1)(5 - 3a) < 0$

$(a + 1)(3a - 5) > 0$

$a \in (-\infty, -1) \cup \left(\frac{5}{3}, \infty\right)$

Also, $a > 1 \ \therefore a \in \left(\frac{5}{3}, \infty\right)$ ...(i)

Case–II: $a - 1 = 0 \Rightarrow a = 1$

given inequality becomes

$(0)x^2 - 2x + 2 > 0, \ \forall x \in R$

which is not possible $\Rightarrow a \in \phi$ ...(ii)

(i) ∪ (iii)

$\therefore a \in \left(\frac{5}{3}, \infty\right)$

Q. Find the least integer value of 'm' for which the angle between the two vectors $\vec{v}_1 = x^2\hat{i} - 4\hat{j} + (3m+1)\hat{k}$ and $\vec{v}_2 = m\hat{i} - x\hat{j} + \hat{k}$ is acute for every $x \in R$

A. If there is the acute angle between $\vec{v}_1$ and $\vec{v}_2$ then $\vec{v}_1.\vec{v}_2 > 0$

$mx^2 + 4x + (3m + 1) > 0 \quad \forall\ x \in R$

Case–I: If $m \neq 0$

then $D < 0$	and	$m > 0$
$16 - 4m(3m + 1) < 0$	and	$m > 0$
$4 - 3m^2 - m < 0$	and	$m > 0$
$3m^2 + m - 4 > 0$	and	$m > 0$
$(3m + 4)(m - 1) > 0$	and	$m > 0$
$m \in \left(-\infty, \dfrac{-4}{3}\right) \cup (1, \infty)$	and	$m > 0$

$\therefore\ m \in (1, \infty)$ ... (i)

Case–II: If $m = 0$

then given inequality become

$(0)x^2 + 4x + 1 > 0,\ \forall\ x \in R$

which is not possible

hence m cannot be zero $\Rightarrow m \in \phi$ ... (ii)

(i) $\cup$ (ii)

$\therefore\ m \in (1, \infty)$

Q. The set of values of 'a' for which the inequality $(x-3a)(x-a-3) < 0$ is satisfied for all $x \in [1, 3]$ is

(A) $\left(\dfrac{1}{3}, 3\right)$ (B) $\left(0, \dfrac{1}{3}\right)$ (C) $(-2, 0)$ (D) $(-2, 3)$

A. **B**

Let $f(x) = (x - 3a)(x - a - 3)$

$f(x) < 0,\ \forall\ x \in [1, 3]$

$f(1) < 0$	and	$f(3) < 0$
$(1 - 3a)(1 - a - 3) < 0$	and	$(3 - 3a)(3 - a - 3) < 0$
$(3a - 1)(a + 2) < 0$	and	$3(a - 1)a < 0$

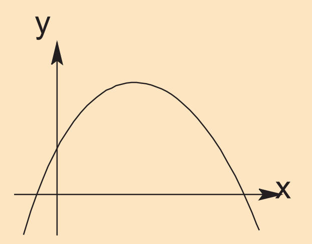
$a \in \left(-2, \dfrac{1}{3}\right)$ and $a \in (0, 1)$

$$\underbrace{a \in (-2, 1/3)\ \ \&\ \ a \in (0,1)}_{\cap}$$

$a \in (0, 1/3)$

Q. **TRUE / FALSE**

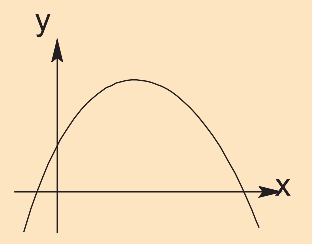
(i) For given graph of $y = ax^2 + bx + c$ we have $a > 0$



A. **False**

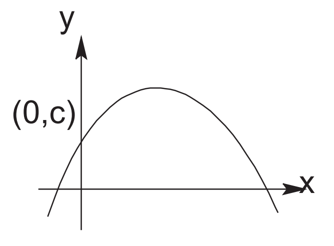
Clearly, for downward parabola, $a < 0$

Q. **(ii)** For given graph of $y = ax^2 + bx + c$ we have $c > 0$

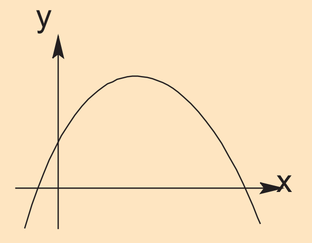


A. **True**

Clearly, y-intercept $> 0 \Rightarrow c > 0$



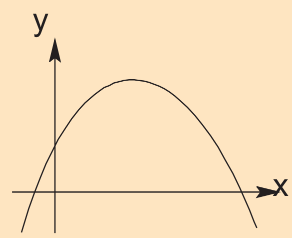
Q. **(iii)** For given graph of $y = ax^2 + bx + c$ we have $D > 0$



A. **True**

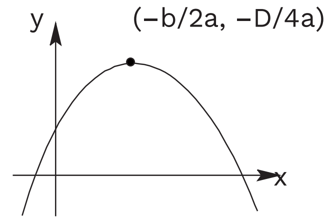
Distinct real roots $\Rightarrow D > 0$

Q. **(iv)** For given graph of $y = ax^2 + bx + c$, we have $-\frac{b}{a} > 0$



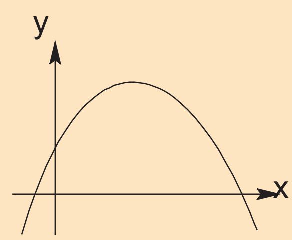
A. **True**

$\because \frac{-b}{2a} > 0 \Rightarrow \frac{-b}{a} > 0$



Q. **(v)** For given graph of $y = ax^2 + bx + c$,

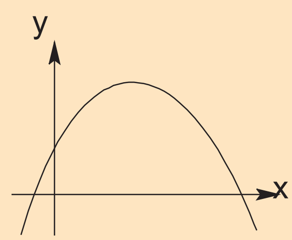
we have $\frac{c}{a} > 0$



A. **Flase**

$\because c > 0$ and $a < 0$

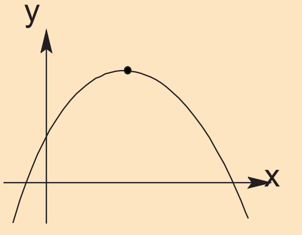
Q. For given graph of $y = ax^2 + bx + c$, we have $b > 0$



A. **True**

$\because \frac{-b}{a} > 0 \Rightarrow \frac{b}{a} < 0$

Q. For given graph of $y = ax^2 + bx + c$, we have $\frac{D}{4a} > 0$



A. **False**

Ordinate of vertex $= -\frac{D}{4a} > 0 \Rightarrow \frac{D}{4a} < 0$

Q. Quadratic Equation $ax^2 + bx + c = 0$ has no real roots then show that $c(a + b + c) > 0$

A. Let $f(x) = ax^2 + bx + c$

now given $D = b^2 - 4ac < 0$ $\because$ roots of $f(x) = 0$ are imaginary

hence $f(x) > 0, \forall x \in R$ or $f(x) < 0, \forall x \in R$

$\therefore f(0).(1) > 0$

$c(a + b + c) > 0$

Q. Let a and b be the roots of equation $x^2 - 10cx - 11d = 0$ and those of $x^2-10ax-11b=0$ are c, d Then find the value of a + b + c + d when $a \neq b \neq c \neq d$.

A. $\because$ $a + b = 10c$ ...(i)

and $c + d = 10a$ ...(ii)

(i) + (ii)

$\Rightarrow a + b + c + d = 10 (a + c)$ ...(iii)

(i)–(ii)

$\Rightarrow (a - c) + (b - d) = 10(c - a)$

$\Rightarrow (b - d) = 11(c - a)$ ...(iv)

Now $\because$ a is root of first equation,

$a^2 - 10ac - 11d = 0$ ...(v)

Also, c is root of second equation,

$c^2 - 10ac - 11b = 0$ ...(vi)

(vi)–(v) gives

$c^2 - a^2 = 11b - 11d$

$(c - a) (c + a) = 11(b - d)$

$(c - a) (c + a) = 11 \times 11 (c - a)$ [from (iv)]

$\Rightarrow (c + a) = 121$

Put in (iii)

$a + b + c + d = 10(121) = 1210$

Q. Let α, β be the roots of the equation $x^2 - px + r = 0$ and $\frac{\alpha}{2}, 2\beta$ be the roots of the equation $x^2 - qx + r = 0$. then the value of r is

(A) $\frac{2}{9}(p-q)(2q-p)$

(B) $\frac{2}{9}(q-p)(2p-q)$

(C) $\frac{2}{9}(q-2p)(2q-p)$

(D) $\frac{2}{9}(2p-q)(2q-p)$

A. **D**

$x^2 - px + r = 0$ ...(1);

$x^2 - qx + r = 0$ ...(2)

$\alpha + \beta = p$, $\alpha\beta = r$ (from equation 1)

$\frac{\alpha}{2} + 2\beta = q$ (from equation 2)

$(2p-q) = \frac{3\alpha}{2} \Rightarrow \alpha = \frac{2}{3}(2p-q)$

$(2q-p) = 3\beta \Rightarrow \beta = \frac{1}{3}(2q-p)$

now $\alpha\beta = r$

hence $\frac{2}{3}(2p-q).\frac{1}{3}(2q-p) = r$

Q. **Fill in the blank**

If $2+i\sqrt{3}$ is a root of the equation $x^2+px+q=0$, where p and q are real then (p, q)..........

A. **(–4, 7)**

If coefficients are real then complex roots are in conjugate pair

$\therefore$ roots $\alpha = 2+i\sqrt{3}$ and $\beta = 2-i\sqrt{3}$

$\alpha + \beta = -p \Rightarrow p = -4$

$\alpha.\beta = q \Rightarrow q = (2)^2 - (i\sqrt{3})^2 = 4+3 = 7$

(p, q) = (–4, 7)

Q. If the product of real roots of the equation $x^2 - 3kx + 2e^{2\log k} - 1 = 0$ is 7, then k = ...

A. **2**

Product of roots = $2e^{\log k^2} - 1 = 7$

$\Rightarrow 2k^2 = 8 \Rightarrow k = \pm 2$

but for k = –2, log k is not defined

Now, for k = 2 equation is $x^2 - 6x + 7 = 0 \Rightarrow D \geq 0$.

So, k = 2

Q. If x, y and z are real and different and $u = x^2 + 4y^2 + 9z^2 - 6yz - 3zx - 2xy$, then u is always

(A) non-negative (B) Zero (C) non-positive (D) positive

A. **A**

$$u = x^2 + (2y)^2 + (3z)^2 - (2y)(3z) - (x)(3z) - (2y)(x)$$

$$u = \frac{1}{2}\left\{(x-2y)^2 + (2y-3z)^2 + (3z-x)^2\right\}$$

$u > 0 \Rightarrow$ u is positive as x, y, z are different.

x = 6, y = 3, z = 2 for these values u = 0

Q. If one root is square of the other root of the equation $x^2 + px + q = 0$ then the relation between p and q is

(A) $p^3 - (3p-1)q + q^2 = 0$ (B) $p^3 - q(3p+1) + q^2 = 0$

(C) $p^3 + q(3p-1) + q^2 = 0$ (D) $p^3 + q(3p+1) + q^2 = 0$

A. **A**

Let root α, α^2

$$\alpha + \alpha^2 = -p, \quad \alpha.\alpha^2 = q \Rightarrow \alpha^3 = q$$

$$(\alpha + \alpha^2)^3 = (-p)^3$$

$$\Rightarrow \alpha^3 + \alpha^6 + 3\alpha.\alpha^2(\alpha + \alpha^2) = -p^3$$

$$\Rightarrow q + q^2 + 3q(-p) = -p^3$$

$$\Rightarrow p^3 - q(3p-1) + q^2 = 0$$

Q. The sum of the all the values of 'm' for which the roots x_1 and x_2 of the quadratic equation $x^2 - 2mx + m = 0$ satisfy the condition $x_1^3 + x_2^3 = x_1^2 + x_2^2$, is

(A) $\frac{3}{4}$ (B) 1 (C) $\frac{9}{4}$ (D) $\frac{5}{4}$

A. **D**

$x_1 + x_2 = 2m, \quad x_1x_2 = m$

$\because x_1^3 + x_2^3 = x_1^2 + x_2^2$

$\therefore (x_1 + x_2)^3 - 3x_1x_2(x_1 + x_2) = (x_1 + x_2)^2 - 2x_1x_2$

$\Rightarrow 8m^3 - 3m(2m) = 4m^2 - 2m$

$\Rightarrow 8m^3 - 10m^2 + 2m = 0$

$\Rightarrow 2m(4m^2 - 5m + 1) = 0$

$\Rightarrow 2m(4m^2 - 4m - m + 1) = 0$

$\Rightarrow 2m(4m - 1)(m - 1) = 0$

$$m = 0, \frac{1}{4}, 1 \Rightarrow \text{sum} = 0 + \frac{1}{4} + 1 = \frac{5}{4}$$

Q. If α, β are the roots of the equation $ax^2 + bx + c = 0$ then the sum of the roots of the equation $a^2x^2 + (b^2 - 2ac)x + (b^2 - 4ac) = 0$ in terms of α and β is given by

(A) $-(\alpha^2 - \beta^2)$ (B) $(\alpha + \beta)^2 - 2\alpha\beta$ (C) $\alpha^2\beta + \beta^2\alpha - 4\alpha\beta$ (D) $-(\alpha^2 + \beta^2)$

A. **D**

$\alpha + \beta = \frac{-b}{a}$, $\alpha\beta = \frac{c}{a}$

Sum of roots of second equation $= -\left(\frac{b^2 - 2ac}{a^2}\right)$

$= -\left(\frac{-b}{a}\right)^2 + 2\left(\frac{c}{a}\right)$

$= -(\alpha + \beta)^2 + 2\alpha\beta = -(\alpha^2 + \beta^2)$

Q. If α and β are the roots of $a(x^2 - 1) + 2bx = 0$ then, which one of the following are the roots of the same equation?

(A) $\alpha + \beta, \alpha - \beta$ (B) $2\alpha + \frac{1}{\beta}, 2\beta + \frac{1}{\alpha}$ (C) $\alpha + \frac{1}{\beta}, \beta + \frac{1}{\alpha}$ (D) $\alpha + \frac{1}{2\beta}, \beta - \frac{1}{2\alpha}$

A. **B**

Given equation $ax^2 + 2bx - a = 0$

$\alpha + \beta = -\frac{2b}{a}$, $\alpha\beta = -1$

$\left(2\alpha + \frac{1}{\beta}\right) + \left(2\beta + \frac{1}{\alpha}\right) = 2(\alpha + \beta) + \frac{\alpha + \beta}{\alpha\beta} = 2(\alpha + \beta) - (\alpha + \beta) = \alpha + \beta$

$\left(2\alpha + \frac{1}{\beta}\right)\left(2\beta + \frac{1}{\alpha}\right) = 4\alpha\beta + 2 + 2 + \frac{1}{\alpha\beta}$

$= -4 + 2 + 2 - 1 = -1$

Q. If $x = 3 + \sqrt{5}$ then find the value of $x^4 - 12x^3 + 44x^2 - 48x + 17$

A. $x = 3 + \sqrt{5} \Rightarrow x - 3 = \sqrt{5}$

$(x - 3)^2 = 5 \Rightarrow x^2 - 6x + 4 = 0$

Now, $x^4 - 12x^3 + 44x^2 - 48x + 17$

$= (x^2 - 6x + 4)(x^2 - 6x + 4) + 1$

$= 0 \times 0 + 1 = 1$

Q. If $p(q-r)x^2 + q(r-p)x + r(p-q) = 0$ has equal roots. Show that $\frac{2}{q} = \frac{1}{p} + \frac{1}{r}$

A. Clearly $x = 1$ satisfies the given eq^n then other root is also 1

Now, Product of roots $= 1 = \frac{r(p-q)}{p(q-r)}$

$p(q-r) = r(p-q)$ $\Rightarrow pq - pr = pr - qr$

$\Rightarrow 2pr = pq + qr$

divide by pqr

$$\frac{2}{q} = \frac{1}{r} + \frac{1}{p}$$

MULTIPLE CORRECT QUESTION

Q. If $x^2 + \frac{1}{x^2} = 14;\ x > 0$ then

(A) $x^3 + x^{-3} = 62$ (B) $x^3 + x^{-3} = 52$ (C) $x^5 + x^{-5} = 624$ (D) $x^5 + x^{-5} = 724$

A. **BD**

$$\left(x + \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} + 2 = 16$$

$$\Rightarrow \left(x + \frac{1}{x}\right) = 4 \qquad (\because x > 0)$$

$$x^3 + \frac{1}{x^3} = \left(x + \frac{1}{x}\right)^3 - 3\left(x + \frac{1}{x}\right)$$

$= 64 - 3(4) = 52$

now $x^5 + \frac{1}{x^5} = \left(x^3 + \frac{1}{x^3}\right)\left(x^2 + \frac{1}{x^2}\right) - \left(x + \frac{1}{x}\right)$

$= 52 \times 14 - 4 = 724$

Q. If ℓ, m are real $\ell \neq m$ then the roots of the equation $(\ell - m)x^2 - 5(\ell + m)x - 2(\ell - m) = 0$ are

(A) Real and equal (B) Complex

(C) Real and unequal (D) None of these

A. **C**

$$D = 25(\ell + m)^2 + 4(\ell - m).2(\ell - m)$$

$$= 25(\ell + m)^2 + 8(\ell - m)^2 > 0$$

$\therefore$ roots real and unequal

Q. Let a, b, c be real numbers with $a \neq 0$ and let α, β be the roots of the equations $ax^2 + bx + c = 0$. Express the roots of $a^3x^2 + abcx + c^3 = 0$ in terms of α, β.

A. $\alpha + \beta = \frac{-b}{a}, \quad \alpha\beta = \frac{c}{a}$

for second equation

Sum of roots $= -\frac{abc}{a^3} = -\frac{bc}{a^2} = \left(-\frac{b}{a}\right)\left(\frac{c}{a}\right) = (\alpha + \beta).\alpha\beta = \alpha^2\beta + \alpha\beta^2$ ...(i)

Product of roots $= \frac{c^3}{a^3} = \left(\frac{c}{a}\right)^3 = (\alpha\beta)^3 = \alpha^2\beta \, . \, \alpha\beta^2$ ...(ii)

Clearly from (i) and (ii)

roots are $\alpha^2\beta, \alpha\beta^2$

Q. If α and β are the roots of $x^2 + px + q = 0$ and γ, δ are the roots of $x^2 + rx + s = 0$ then evaluate $(\alpha-\gamma)(\beta-\gamma)(\alpha-\delta)(\beta-\delta)$ in terms of p, q, r and s.

A. $\alpha + \beta = -p, \qquad \alpha\beta = q$

$\gamma + \delta = -r, \qquad \gamma\delta = s$

now $(\alpha-\gamma)(\beta-\gamma)(\alpha-\delta)(\beta-\delta)$

$= \left(\alpha\beta - (\alpha + \beta)\gamma + \gamma^2\right)\left(\alpha\beta - (\alpha + \beta)\delta + \delta^2\right)$

$= \left(q + p\gamma + \gamma^2\right)\left(q + p\delta + \delta^2\right)$

$= (q + p\gamma - s - r\gamma)(q + p\delta - s - r\delta)$

$= \{(q - s) + (p - r)\gamma\}\{(q - s) + (p - r)\delta\}$

$= (q - s)^2 - r(p - r)(q - s) + s(p - r)^2$

Identity:

If $ax^2 + bx + c = 0$ is identity, then number of roots are infinite and $a = b = c = 0$

Know the facts

3 distinct real roots of quadratic $\Rightarrow$ Infinite roots

Q. Find the value of p for which the equation $(p+2)(p-1)x^2+(p-1)(2p+1)+x(p^2-1)=0$ has infinite roots.

A. It must be an identity

hence $(p+2)(p-1)=0$ and $(p-1)(2p+1)=0$ & $(p^2-1)=0$

$p = -2, 1$ and $p = 1, -\frac{1}{2}$ and $p = 1, -1$

$\therefore$ common value is p = 1

Q. Let a, b, c be different real numbers then prove that

$$\frac{(x-a)(x-b)}{(c-a)(c-b)}+\frac{(x-b)(x-c)}{(a-b)(a-c)}+\frac{(x-c)(x-a)}{(b-c)(b-a)}=1 \text{ is an identity.}$$

A. Put $x=a$ $\quad 0+\frac{(a-b)(a-c)}{(a-b)(a-c)}+0=1$ true

put $x=b$ $\quad 0+0+\frac{(b-c)(b-a)}{(b-c)(b-a)}=1$ true

put $x=c$ $\quad \frac{(c-a)(c-b)}{(c-a)(c-b)}+0+0=1$ true

$\because$ three values $x=a,b,c$ satisfies above

two degree equation has 3 roots

$\Rightarrow$ It is an identity.

Point to Remember!!!

(i) Quadratic with one roots zero

$\Leftrightarrow$ c = 0

$ax^2 + bx + c = 0$

Product of roots $= \frac{c}{a} = 0$

$\Rightarrow c = 0$

(ii) Quadratic with both roots zero

$\Leftrightarrow$ b=0, c=0

$ax^2 + bx + c = 0$

Sum of roots

= Product of roots = 0

$\Rightarrow b = 0, c = 0$

(iii) Quadratic with exactly one root infinite

$\Leftrightarrow$ a = 0, b, c ≠ 0

(iv) Quadratic with both roots infinite $\Leftrightarrow$ a = 0, b = 0, c ≠ 0

Q. If $(2p-q)x^2+(p-1)x+5=0$ has both roots infinite. Find p and q

A. $2p - q = 0$ and $p - 1 = 0$

$q = 2p$ $\quad$ $p = 1$

hence p = 1, q = 2

Symmetric function:

If $f(\alpha, \beta) = f(\beta, \alpha) \ \forall \ \alpha, \beta$
Then $f(\alpha, \beta)$ is called symmetric function of α, β

Q. Check if $f(\alpha,\beta)$ is symmetric or not

(i) $f(\alpha, \beta) = \alpha^2\beta + \alpha\beta^2$
(ii) $f(\alpha, \beta) = \cos(\alpha - \beta)$
(iii) $f(\alpha, \beta) = \sin(\alpha - \beta)$
(iv) $f(\alpha, \beta) = (\alpha^2 - \beta)$

A. **(i)** $f(\beta, \alpha) = \beta^2\alpha + \beta\alpha^2$
$= f(\alpha, \beta)$ Yes

(ii) $f(\beta, \alpha) = \cos(\beta - \alpha)$
$= \cos(\alpha - \beta)$
$= f(\alpha, \beta)$ Yes

(iii) $f(\beta, \alpha) = \sin(\beta - \alpha)$
$= -\sin(\alpha - \beta)$
$\neq f(\alpha, \beta)$ No

(iv) $f(\beta, \alpha) = (\beta^2 - \alpha)$
$\neq \alpha^2 - \beta$
$\neq f(\alpha, \beta)$ No

Condition of common roots:

(I) Condition for both roots common:

$a_1x^2 + b_1x + c_1 = 0$
$a_2x^2 + b_2x + c_2 = 0$
$\mathbf{a_1/a_2 = b_1/b_2 = c_1/c_2}$

(II) Condition for one root common:

$$\frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1} = \frac{b_1c_2 - b_2c_1}{a_2c_1 - a_1c_2}$$

Q. Find k for which equations $x^2 - 3x + 2 = 0$ and $3x^2 + 4kx + 2 = 0$ have a common root.

A. $\left\{-\frac{5}{4}, -\frac{7}{4}\right\}$
$x^2 - 3x + 2 = 0 \Rightarrow (x - 1)(x - 2) = 0$
$x = 1, 2$
If $x = 1$ is common root then
$3(1)^2 + 4k(1) + 2 = 0$

$5 + 4k = 0$

$\Rightarrow k = -5/4$

If $x = 2$ is common root then

$3(2)^2 + 4k(2) + 2 = 0$

$14 + 8k = 0$

$\Rightarrow k = -7/4$

Q. Find p and q if $px^2 + 5x + 2 = 0$ and $3x^2 + 10x + q = 0$ have both roots in common

A. $\frac{p}{3} = \frac{5}{10} = \frac{2}{q}$

$\Rightarrow p = \frac{3}{2}, q = 4$

Q. Find the value of a and b if $x^2 - 4x + 5 = 0$, $x^2 + ax + b = 0$ have a common root where $a, b \in R$

A. $x^2 - 4x + 5 = 0$

$D = 16 - 4(5) < 0$ $\Rightarrow$ Imaginary roots.

Also, coefficients of second equation are real hence only one root cannot be common

$\Rightarrow \frac{1}{1} = \frac{a}{-4} = \frac{b}{5}$

$\Rightarrow a = -4, b = 5$

Q. Let a, b, c be distinct real numbers. If $4x^2\sin^2\theta - (4\sin\theta)x + 1 = 0$ and $a^2(b^2 - c^2)x^2 + b^2(c^2 - a^2)x + c^2(a^2 - b^2) = 0$ have a common root and the second equation has equal roots then find possible value of θ where $\theta \in (0, \pi)$

A. Clearly, $x = 1$ satisfies second equation

hence, second equation has both roots 1.

$\Rightarrow$ 1 is common root of both equations,

Now $4(1)^2\sin^2\theta - (4\sin\theta)1 + 1 = 0$

$\Rightarrow (2\sin\theta - 1)^2 = 0$

$\Rightarrow 2\sin\theta - 1 = 0$

$\Rightarrow \sin\theta = \frac{1}{2}$

$\Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$

Q. If the quadratic equation $x^2+bx+c=0$ and $x^2+cx+b=0$, $b\neq c$ have a common root then prove that their uncommon roots are roots of the equation $x^2+x+bc=0$

A. Let common root be α

$x^2 + bx + c = 0$ (α, β are its roots)

$x^2 + cx + b = 0$ (α, γ are its roots)

now $\alpha+\beta=-b,\ \ \alpha\beta=c$

$\alpha+\gamma=-c,\ \ \alpha\gamma=b$

$\because$ α is common root

$\therefore\ \alpha^2+b\alpha+c=0$ ...(i)

$\underset{-}{-}\ \alpha^2\ \underset{-}{+}\ c\alpha\ \underset{-}{+}\ b=0$ ...(ii)

$0+(b-c)\alpha+(c-b)=0\Rightarrow(b-c)\alpha=(b-c)\Rightarrow\alpha=1$

now $\left.\begin{matrix}1.\beta=c\Rightarrow\beta=c\\1.\gamma=b\Rightarrow\gamma=b\end{matrix}\right\}$ uncommon roots

required equation, $x^2 - (b + c)x + bc = 0$

$x^2 - (-1)x + bc = 0$ {put $\alpha = 1$ in (i)}

$x^2 + x + bc = 0$ Hence proved.

Q. $x^2+ax+12=0, x^2+bx+15=0$ and $x^2+(a+b)x+36=0$ have a common positive root. Find a, b & common root of equation.

A. Let α be the common root.

$\alpha^2+a\alpha+12=0$ ...(i)

$\alpha^2+b\alpha+15=0$ ...(ii)

$\alpha^2+(a+b)\alpha+36=0$ ...(iii)

(i) + (ii) – (iii)

$\alpha^2-9=0\Rightarrow\alpha=3$ $(\because\ \alpha>0)$

from (i): $9+3a+12=0\Rightarrow a=-7$

from (ii): $9+3b+15=0\Rightarrow b=-8$

Q. If one root of quadratic equation $x^2-x+3a=0$ is double of one root of the equation $x^2-x+a=0$ then find a

A. $x^2 - x + a = 0$ (α, β are its roots) $\Rightarrow \alpha^2 - \alpha + a = 0$ ... (i)

$x^2 - x + 3a = 0$ (2α, γ are its roots) $\Rightarrow 4\alpha^2 - 2\alpha + 3a = 0$ ... (ii)

(ii) – 4 × (i):

$$2\alpha - a = 0 \Rightarrow \alpha = \frac{a}{2}$$

from (i): $\frac{a^2}{4} - \frac{a}{2} + a = 0$

$\Rightarrow a^2 - 2a + 4a = 0$ $\Rightarrow a(a + 2) = 0$

$\Rightarrow a = 0, -2$

Q. If $Q_1(x) = x^2 + (k-29)x - k$

$Q_2(x) = 2x^2 + (2k-43)x + k$

both are factors of a cubic polynomial then find k.

A. $Q_1(x) = 0$ and $Q_2(x) = 0$ have atleast one common root but both roots cannot be common ($\because$ coeff. are not in proportion)

hence

$x^2 + (k-29)x - k = 0$

$2x^2 + (2k-43)x + k = 0$ have only one common root (let α)

$\therefore$ $\alpha^2 + (k-29)\alpha - k = 0$ ...(i)

$2\alpha^2 + (2k-43)\alpha + k = 0$ ...(ii)

(ii) – 2 × (i)

$$15\alpha + 3k = 0 \Rightarrow \alpha = \frac{-k}{5}$$

from (i)

$$\frac{k^2}{25} - \frac{k^2}{5} + \frac{29k}{5} - k = 0$$

$$\Rightarrow \frac{4}{25}k^2 = \frac{24}{5}k \qquad \Rightarrow k = 0, 30$$

Q. If $x^2 + abx + c = 0$ & $x^2 + acx + b = 0$ have only one common root then show that quadratic equation with roots as their other uncommon roots is $a(b + c)x^2 + (b + c)x - abc = 0$

A. Let common root = α

$x^2 + abx + c = 0$ (α, β are its roots)

$x^2 + acx + b = 0$ (α, γ are its roots)

now $\alpha + \beta = -ab$, $\alpha\beta = c$

$\alpha + \gamma = -ac$, $\alpha\gamma = b$

$\because$ α is common root

$\therefore$ $\alpha^2 + ab\alpha + c = 0$ ...(i)

$\alpha^2 + ac\alpha + b = 0$ ...(ii)

(i) – (ii)

$$0 + a(b - c)\alpha + (c - b) = 0 \Rightarrow a(b - c)\alpha = (b - c) \Rightarrow \alpha = \frac{1}{a}$$

Now $\left.\begin{array}{l}\frac{1}{a}.\ \beta = c \Rightarrow \beta = ac \\ \frac{1}{a}.\ \gamma = b \Rightarrow \gamma = ab\end{array}\right\}$ uncommon roots

required equation $x^2 - a(b + c)x + a^2bc = 0$

$$\frac{1}{a}x^2 - (b+c)x + abc = 0 \text{ ...(iii)}$$

put $\alpha = \frac{1}{a}$ in (i)

$$\frac{1}{a^2} + b + c = 0 \Rightarrow \frac{1}{a} = -a(b+c)$$

$$\Rightarrow a(b+c)x^2 + (b+c)x - abc = 0$$

Hence Proved.

Q. A value of b for which the equations $x^2 + bx - 1 = 0$, $x^2 + x + b = 0$ have one root in common, is

(A) $-\sqrt{2}$ (B) $-i\sqrt{3}$ (C) $i\sqrt{5}$ (D) $\sqrt{2}$

A. **B**

Let α be the common root then

$$\alpha^2 + b\alpha - 1 = 0 \quad \text{...(i)}$$

$$\alpha^2 + \alpha + b = 0 \quad \text{...(ii)}$$

(i) – (ii)

$$(b-1)\alpha = (1+b) \Rightarrow \alpha = \frac{b+1}{b-1}$$

from (ii) $\left(\frac{b+1}{b-1}\right)^2 + \left(\frac{b+1}{b-1}\right) + b = 0$

$$(b+1)^2 + (b+1)(b-1) + b(b-1)^2 = 0$$

$$\Rightarrow (2b^2 + 2b) + (b^3 - 2b^2 + b) = 0$$

$$\Rightarrow b^3 + 3b = 0$$

$$\Rightarrow b = 0, \pm\sqrt{3}i$$

Q. **Fill in the blank:**

If the quadratic equations $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ $(a \neq b)$ have a common root, then the numerical value of a + b is ....

A. **–1**

Let common root be α

$x^2 + ax + b = 0$ (α, β are its roots)

$x^2 + bx + a = 0$ (α, γ are its roots)

now $\alpha + \beta = -a$, $\alpha\beta = b$

$\alpha + \gamma = -b$, $\alpha\gamma = a$

$\because \alpha$ is common root

$\therefore$ $\alpha^2 + a\alpha + b = 0$ ...(i)

$\alpha^2 + b\alpha + a = 0$ ...(ii)

(i) – (ii)

$0 + (a - b)\alpha + (b - a) = 0 \Rightarrow (a - b)\alpha = (a - b) \Rightarrow \alpha = 1$

put $\alpha = 1$ in (i) gives.

$a + b = -1$

Q. If every solution of the equation $3\cos^2 x - \cos x - 1 = 0$ is a solution of the equation $a\cos^2 2x + b\cos 2x - 1 = 0$. Then the value of (a + b) is equal to

(A) 5 (B) 9 (C) 13 (D) 14

A. **C**

$\because 3\cos^2 x - 1 = \cos x$

$\therefore (3\cos^2 x - 1)^2 = \cos^2 x$

$$\Rightarrow \left\{3\frac{(1+\cos 2x)}{2} - 1\right\}^2 = \frac{1+\cos 2x}{2}$$

$$\Rightarrow \frac{(3\cos 2x + 1)^2}{4} = \frac{1+\cos 2x}{2}$$

$\Rightarrow 9\cos^2 2x + 6\cos 2x + 1 = 2(1+\cos 2x)$

$\Rightarrow 9\cos^2 2x + 4\cos 2x - 1 = 0$

from comparison of given equation, we get

$a = 9, b = 4 \Rightarrow a + b = 13$

Quadratic Equations

Q. If $x^2 + 3x + 5 = 0$ and $ax^2 + bx + c = 0$ have common root/roots and a, b, $c \in N$ then find minimum value of a + b + c

A. $x^2 + 3x + 5 = 0$

$D = 9 - 4(5) < 0 \Rightarrow$ imaginary roots also coefficient of equation are real hence only one root cannot be common

$\therefore$ both roots will be common

$$\Rightarrow \frac{a}{1} = \frac{b}{3} = \frac{c}{5}$$

for minimum a = 1, b = 3, c = 5

$\therefore (a + b + c)_{min} = 9$

Q. Determine the value of m for which the equation $3x^2 + 4mx + 2 = 0$ and $2x^2 - 3x - 2 = 0$ may have a common root.

A. $$2x^2 - 3x - 2 = 0 \Rightarrow (2x+1)(x-2) = 0 \quad x = -\frac{1}{2}, 2$$

If $x = -\frac{1}{2}$ is common root, then

$$3\left(-\frac{1}{2}\right)^2 + 4m\left(-\frac{1}{2}\right) + 2 = 0$$

$$-8m + 11 = 0 \Rightarrow m = \frac{11}{8}$$

If x = 2 is common root, then

$$14 + 8m = 0 \Rightarrow m = -\frac{7}{4}$$

$$\Rightarrow m = \left\{\frac{11}{8}, \frac{-7}{4}\right\}$$

Q. For what value of a is the difference between the roots of the equation $(a-2)x^2 - (a-4)x - 2 = 0$ equal to 3?

A. Let roots α, β

$$\alpha + \beta = \frac{a-4}{a-2}, \alpha\beta = \frac{-2}{a-2}$$

now $|\alpha - \beta| = 3$

$$\Rightarrow (\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = 9$$

$$\Rightarrow \frac{(a-4)^2}{(a-2)^2} + \frac{8}{(a-2)} = 9$$

$$\Rightarrow 9(a-2)^2 - 8(a-2) - (a-4)^2 = 0$$

$$\Rightarrow 8a^2 - 36a + 36 = 0$$

$$\Rightarrow 4\left(2a^2 - 9a + 9\right) = 0$$

$$\Rightarrow 2a^2 - 6a - 3a + 9 = 0$$

$$\Rightarrow (2a-3)(a-3) = 0$$

$$\Rightarrow a = \frac{3}{2}, a = 3$$

Q. Find all values of a for which the sum of the roots of the equation $x^2 - 2a(x-1) - 1 = 0$ is equal to the sum of squares of its roots.

A. $x^2 - 2ax + 2a - 1 = 0$ (α, β are its roots)

$\alpha + \beta = 2a,\ \alpha\beta = 2a - 1$

given $\alpha + \beta = \alpha^2 + \beta^2$

$$\alpha + \beta = (\alpha + \beta)^2 - 2\alpha\beta$$

$$\Rightarrow 2a = 4a^2 - 2(2a-1)$$

$$\Rightarrow 2a = 4a^2 - 4a + 2$$

$$\Rightarrow 4a^2 - 6a + 2 = 0$$

$$\Rightarrow 2a^2 - 3a + 1 = 0$$

$$\Rightarrow (a-1)(2a-1) = 0$$

$$\Rightarrow a = 1, \frac{1}{2}$$

Q. For what values of 'a' equations $x^2 + ax + 1 = 0$ and $x^2 + x + a = 0$ have a root in common.

A. Let α be the common root then

$$\alpha^2 + a\alpha + 1 = 0 \qquad \text{...(i)}$$

$$\alpha^2 + \alpha + a = 0 \qquad \text{...(ii)}$$

(i) – (ii):

$$(a-1)\alpha = (a-1) \Rightarrow a = 1, \alpha = 1$$

if $\alpha = 1$, then from (ii):

$$1 + 1 + a = 0 \Rightarrow a = -2$$

hence, a = 1, – 2

Maximum and minimum value of Quadratic Expression:

$y = ax^2 + bx + c$ attains its maximum or minimum at point where $x = -\frac{b}{2a}$ as $a < 0$ or $a > 0$, respectively.

Maximum and Minimum value can be obtained by making a perfect square.

Q. $p(x) = ax^2 + bx + 8$ is quadratic. If the minimum value of p(x) is 6 when x = 2. Find a and b

A. **$b = -2,\ a = \frac{1}{2}$**

Clearly $\frac{-b}{2a} = 2 \Rightarrow b = -4a$

now, $p(2) = 6 \Rightarrow 4a + 2b + 8 = 6$

$-b + 2b + 8 = 6 \Rightarrow b = -2$

$a = \frac{-b}{4} = \frac{2}{4} = \frac{1}{2}$

Q. $y = 2x^2 - 3x + 1$, find minimum value of y

A. $\frac{-b}{2a} = \frac{-(-3)}{2(2)} = \frac{3}{4}$

y is minimum if $x = \frac{3}{4}$

hence $y_{min} = 2\left(\frac{3}{4}\right)^2 - 3\left(\frac{3}{4}\right) + 1 = 2 \cdot \frac{9}{16} - \frac{9}{4} + 1 = \frac{-1}{8}$

Q. $y = 7 + 5x - 2x^2$, find maximum value of y

A. $\frac{-b}{2a} = \frac{-5}{2(-2)} = \frac{5}{4}$

y is max. if $x = \frac{5}{4}$

hence $y_{max} = 7 + 5\left(\frac{5}{4}\right) - 2\left(\frac{5}{4}\right)^2$

$= 7 + \frac{25}{4} - \frac{25}{8} = \frac{81}{8}$

Q. For $x \geq 2$ smallest possible value of $\log_{10}\left(x^3 - 4x^2 + x + 26\right) - \log_{10}(x+2)$

A. $x^3 - 4x^2 + x + 26 = (x+2)\left(x^2 - 6x + 13\right)$

hence, given expression is

$\log_{10}(x+2) + \log_{10}\left(x^2 - 6x + 13\right) - \log_{10}(x+2)$

$= \log_{10}\left(x^2 - 6x + 13\right) = y\,(\text{let})$

now $y_{min} = \log_{10}\left(x^2 - 6x + 13\right)_{min} = \log_{10}\left\{(x-3)^2 + 4\right\}$

$y_{min} = \log_{10} 4$

Range of Linear:

$\mathbf{y = ax + b; a \neq 0}$ is $\mathbf{y \in R}$

Q. y = f(x) = x + 1

A. $\because$ f(x) is linear

$\therefore$ $y \in R$

Range of $\mathbf{\dfrac{linear}{linear}}$**:** $\mathbf{y = \dfrac{ax + b}{cx + d}}$ is $\mathbf{R - \left\{\dfrac{a}{c}\right\}}$

Q. $y = \dfrac{2x+3}{x+1}$, Find range of y ?

A. $\because\ y = \dfrac{\text{Linear}}{\text{Linear}}$

hence, $y \in R - \{2\}$

Q. $y = \dfrac{1}{3x-1}$,Find range of y ?

A. $y = \dfrac{(0)x + 1}{3x - 1}$

$y \in R - \left\{\dfrac{0}{3}\right\} \Rightarrow y \in R - \{0\}$

Q. $y = \dfrac{(x-1)(x-2)(x-3)}{(x-2)(x-3)}$, Find range of y ?

A. $y = (x-1); x \neq 2, 3$

At $x = 2, x - 1 = 1$

At $x = 3, x - 1 = 2$

Hence, $y \in R - \{1, 2\}$

$$\frac{\textbf{Linear}}{\textbf{Quadratic}}, \frac{\textbf{Quadratic}}{\textbf{Quadratic}}, \frac{\textbf{Quadratic}}{\textbf{Linear}}$$

Type 1: If common factors are there

If common factors are there then solve by range of $\frac{\text{Linear}}{\text{Linear}}$ by taking care of domain

Type 2: If common factors are not there

Step I: Cross multiply and make quadratic in x

Step II: Apply D ≥ 0 (since x is real)

Step III: Solve inequality in y and hence find the range

Note:

Always Cross check for coefficient of x^2 equal to zero.

Q. Find range of $\frac{x^2 - x + 1}{x^2 + x + 1}$

A. Let $y = \frac{x^2 - x + 1}{x^2 + x + 1}$

$y(x^2 + x + 1) = x^2 - x + 1$

$(y-1)x^2 + (y+1)x + (y-1) = 0$ ...(1)

Case–I: $y - 1 \neq 0 \Rightarrow y \neq 1$

(1) is a quadratic and $x \in R$

$\Rightarrow$ it has real roots

$\therefore$ D ≥ 0

$(y + 1 + 2y - 2)(y + 1 - 2y + 2) \geq 0$

$(3y-1)(3-y) \geq 0 \Rightarrow (3y-1)(y-3) \leq 0$

$y \in \left[\frac{1}{3}, 3\right] \cap y \neq 1$

Hence, $y \in \left[\frac{1}{3}, 3\right] - \{1\}$ ...(2)

Case–II: y – 1 = 0 $\Rightarrow$ y = 1

in (1) put y = 1

$(0)x^2 + 2x + 0 = 0 \Rightarrow x = 0,$ which is real

$\therefore$ y can be 1 ...(3)

(2) ∪ (3)

$y \in \left[\frac{1}{3}, 3\right]$

Q. Find range of $\frac{x^2+2x-11}{2(x-3)}$

A. Let $y=\frac{x^2+2x-11}{2(x-3)}$

$\Rightarrow 2yx - 6y = x^2 + 2x - 11$
$\Rightarrow x^2 + 2(1 - y)x + (6y - 11) = 0$
$\because x \in R$
$\therefore$ roots of above equation are real hence $D \geq 0$
$4(1-y)^2 - 4(6y-11) \geq 0$

$\Rightarrow\ 4\{y^2-8y+12\} \geq 0$

$\Rightarrow\ (y^2-8y+12) \geq 0$

$\Rightarrow (y-6)(y-2) \geq 0$

hence $y \in (-\infty, 2] \cup [6, \infty)$

Find range of following $\frac{x^2-3x+4}{x^2+3x+4}$

Let $y=\frac{x^2-3x+4}{x^2+3x+4}$

$\Rightarrow (y-1)x^2+3(y+1)x+4(y-1)=0$ ...(a)

Case–I: $y - 1 \neq 0 \Rightarrow y \neq 1$
then $D \geq 0$
$9(y + 1)^2 - 4 \times (y - 1) \times 4(y - 1) \geq 0$
$\Rightarrow (3y+3+4y-4)(3y+3-4y+4) \geq 0$

$\Rightarrow (7y-1)(7-y) \geq 0$

$\Rightarrow (7y-1)(y-7) \leq 0$

$\Rightarrow y \in \left[\frac{1}{7}, 7\right] \cap y \neq 1$

hence $y \in \left[\frac{1}{7}, 7\right] - \{1\}$ ...(1)

Case–II: $y-1=0 \Rightarrow y=1$

Put in (a)
$(0)x^2 + 6x + 0 = 0 \Rightarrow x = 0 \in R$
hence y can be 1 ...(2)
(1) $\cup$ (2)

$y \in \left[\frac{1}{7}, 7\right]$

Q. Find range of $\dfrac{(x+1)(x-2)}{x(x+3)}$

A. Let $y = \dfrac{x^2 - x - 2}{x^2 + 3x}$

$\Rightarrow (y-1)x^2 + (3y+1)x + 2 = 0$ ...(a)

Case–I: $y - 1 \neq 0 \Rightarrow y \neq 1$

then $D \geq 0$

$(3y+1)^2 - 8(y-1) \geq 0$

$\Rightarrow (9y^2 + 6y + 1) - 8y + 8 \geq 0$

$\Rightarrow 9y^2 - 2y + 9 \geq 0$ (a > 0, D < 0)

$\Rightarrow y \in R - \{1\}$ ... (1)

Case–II: If $y - 1 = 0 \Rightarrow y=1$

from (a)

$(0)x^2 + 4x + 2 = 0 \Rightarrow x = \dfrac{-1}{2} \in R$

$\therefore$ y can be 1 ...(2)

(1) $\cup$ (2)

$y \in R$

Q. Find range of $\dfrac{x^2 + 2x - 2}{x^2 + 2x + 1}$

A. Let $y = \dfrac{x^2 + 2x - 2}{x^2 + 2x + 1}$

$\Rightarrow (y-1)x^2 + 2(y-1)x + (y+2) = 0$ ...(a)

Case–I: $(y-1) \neq 0 \Rightarrow y \neq 1$

then $D \geq 0$

$4(y-1)^2 - 4(y-1)(y+2) \geq 0$

$\Rightarrow 4(y-1)\{(y-1) - (y+2)\} \geq 0$

$\Rightarrow (y-1) \leq 0$

$\Rightarrow y \in (-\infty, 1)$ ...(1)

Case–II : If $y - 1 = 0 \Rightarrow y = 1$

from (a)

$(0)x^2 + (0)x + 3 = 0$ (not possible)

$\therefore$ y cannot be 1 ...(2)

hence, $y \in (-\infty, 1)$

Q. Find range of following $\frac{x^2+14x+9}{x^2+2x+3}$

A. Let $y = \frac{x^2+14x+9}{x^2+2x+3}$

$\Rightarrow (y-1)x^2 + 2(y-7)x + 3(y-3) = 0$ ...(a)

Case–I: $y-1 \neq 0 \Rightarrow y \neq 1$

then $D \geq 0$

$4(y-7)^2 - 4(y-1).3(y-3) \geq 0$

$\Rightarrow 4\{y^2 - 14y + 49 - 3y^2 + 12y - 9\} \geq 0$

$\Rightarrow 2y^2 + 2y - 40 \leq 0$

$\Rightarrow y^2 + y - 20 \leq 0 \Rightarrow (y+5)(y-4) \leq 0$

$\Rightarrow y \in [-5,4] \cap y \neq 1$

$\Rightarrow y \in [-5,4] - \{1\}$ ... (1)

Case–II: $y - 1 = 0 \Rightarrow y = 1$

Put in (a)

$(0)x^2 - 12x - 6 = 0 \Rightarrow x = -½ \in R$

hence, y can be 1 ...(2)

$(1) \cup (2)$

$y \in [-5, 4]$

Q. Find range of $\frac{x^2-5x+6}{x^2-4x+3}$

A. $y = \frac{(x-2)(x-3)}{(x-3)(x-1)}$

$= \frac{x-2}{x-1}, x \neq 3$

Range = R − {1, y(3)}

Range = R − $\left\{1, \frac{1}{2}\right\}$

Q. Find the least value of $\frac{6x^2-22x+21}{5x^2-18x+17} = y \ \forall \ x \in R$

A. $(5y-6)x^2 + 2(11-9y)x + (17y-21) = 0$ ...(a)

Case–I: $5y - 6 \neq 0 \Rightarrow y \neq 6/5$

then $D \geq 0$

$4(11-9y)^2 - 4(5y-6)(17y-21) \geq 0$

$\Rightarrow \{121 + 81y^2 - 198y - 85y^2 + 207y - 126\} \geq 0$

$\Rightarrow 4y^2 - 9y + 5 \leq 0$

$\Rightarrow (4y-5)(y-1) \leq 0$

$\Rightarrow y \in \left[1, \frac{5}{4}\right] \cap y \neq 6/5$

$\Rightarrow y \in \left[1, \frac{5}{4}\right] - \left\{\frac{6}{5}\right\}$ ...(1)

Case–II: $5y - 6 = 0 \Rightarrow y = 6/5$

Put in (a) we get

$2\left(11 - \frac{54}{5}\right)x + \frac{102}{5} - 21 = 0$

$\Rightarrow \frac{2x}{5} - \frac{3}{5} = 0$

$\Rightarrow x = \frac{3}{2} \in R$

$\therefore$ y can be 6/5 ...(2)

$(1) \cup (2)$

$y \in \left[1, \frac{5}{4}\right]$

hence $y_{min} = 1$

Q. Find all possible values of 'a' for which the expression $\frac{ax^2 - 7x + 5}{5x^2 - 7x + a}$ may be capable of taking all values where x being any real quantity.

A. Let $y = \frac{ax^2 - 7x + 5}{5x^2 - 7x + a}$

$\Rightarrow (5y-a)x^2 - 7(y-1)x + (ay-5) = 0$

$\because x \in R \Rightarrow D \geq 0$

$49(y-1)^2 - 4(5y-a)(ay-5) \geq 0 \ \forall \ y \in R$

$49y^2 - 98y + 49 - 20ay^2 + 100y + 4a^2y - 20a \geq 0 \ \forall \ y \in R$

$\Rightarrow (49 - 20a)y^2 + 2(2a^2 + 1)y + (49 - 20a) \geq 0 \ \forall \ y \in R$

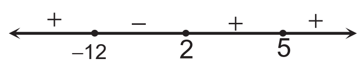
Which implies

$D \le 0$ and $49 - 20a > 0$

$4(2a^2 + 1)^2 - 4(49 - 20a)^2 \le 0$ and $a < \dfrac{49}{20}$

$\left(2a^2 + 1 + 49 - 20a\right)\left(2a^2 + 1 - 49 + 20a\right) \le 0$ and $a < \dfrac{49}{20}$

$\left(2a^2 - 20a + 50\right)\left(2a^2 + 20a - 48\right) \le 0$ and $a < \dfrac{49}{20}$



$\left(a^2 - 10a + 25\right)\left(a^2 + 10a - 24\right) \le 0$ and $a < \dfrac{49}{20}$

$(a-5)^2 (a+12)(a-2) \le 0$ and $a < \dfrac{49}{20}$

$a \in [-12, 2] \cup \{5\}$ and $a < \dfrac{49}{20}$

$a \in [-12, 2]$

but for $a = -12$ and $a = 2$, N^r and D^r have common factors which implies $y \notin R$

hence, $a \in (-12, 2)$

Q. Find the domain and Range of $f(x) = \sqrt{x^2 - 3x + 2}$

A. Domain:

$$x^2 - 3x + 2 \ge 0$$

$$(x-1)(x-2) \ge 0$$

$$x \in (-\infty, 1] \cup [2, \infty) = D_f$$

Range:

$$\left(x^2 - 3x + 2\right)_{min} = \frac{-D}{4a}$$

$$= \frac{-[9 - 4(2)]}{4} = \frac{-1}{4}$$

$\therefore \quad x^2 - 3x + 2 \in \left[-\dfrac{1}{4}, \infty\right)$

hence, $\sqrt{x^2 - 3x + 2} \in [0, \infty)$ = Range

General second degree in x and y

$f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$

> Condition of general second degree in x & y to be resolved into two linear factors
>
> $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

Steps for factorization:

Step 1: Factorize second degree homogeneous part.

Step 2: Add constant to both the linear.

Step 3: Compare coefficient of x & coefficient of y and absolute term if needed.

Q. Prove that expression $2x^2 + 3xy + y^2 + 2y + 3x + 1$ can be factorized into two linear factors and find them

A. $a = 2, 2h = 3, b = 1, 2g = 3, 2f = 2, c = 1$

Condition for factorization, $abc + 2fgh - af^2 - bg^2 - ch^2$

$$= 2(1)(1) + 2(1)\left(\frac{3}{2}\right)\left(\frac{3}{2}\right) - 2(1)^2 - 1\left(\frac{3}{2}\right)^2 - 1\left(\frac{3}{2}\right)^2 = 2 + \frac{9}{2} - 2 - \frac{9}{4} - \frac{9}{4}$$

$= 0$ **Hence Proved.**

For factorization, $2x^2 + 3xy + y^2 = (2x + y)(x + y)$

$(2x + y + a)(x + y + b) = 2x^2 + 3xy + y^2 + 2y + 3x + 1$

$$\left.\begin{array}{l}\text{comparing coefficient of } x \Rightarrow a + 2b = 3\\ \text{comparing coefficient of } y \Rightarrow a + b = 2\end{array}\right\} \to a = b = 1$$

Factors are $2x + y + 1, x + y + 1$

Q. Prove that the expression $x^2-3xy+2y^2-2x-3y-35$ can be factorized into two linear factors & find them

A. $a = 1, 2h = -3, b = 2, 2g = -2, 2f = -3, c = -35$

Condition for factorization, $abc + 2fgh - af^2 - bg^2 - ch^2$

$$=1(2)(-35) + 2\left(-\frac{3}{2}\right)(-1)\left(-\frac{3}{2}\right) - 1\left(-\frac{3}{2}\right)^2 - 2(-1)^2 - (-35)\left(-\frac{3}{2}\right)^2$$

$$= -70 - \frac{9}{2} - \frac{9}{4} - 2 + \frac{315}{4} = -72 - \frac{27}{4} + \frac{315}{4}$$

$= 0$ **Hence Proved.**

For factorization, $x^2 - 3xy + 2y^2 = (x - 2y)(x - y)$

$(x - 2y + a)(x - y + b) = x^2 - 3xy + 2y^2 - 2x - 3y - 35$

$$\left.\begin{array}{l}\text{comparing coefficient of } x \Rightarrow a + b = -2 \\ \text{comparing coefficient of } y \Rightarrow -a - 2b = -3\end{array}\right\} \rightarrow b = 5, a = -7$$

Factors are $x - 2y - 7,\ x - y + 5$

Q. If the equation $x^2 + 16y^2 - 3x + 2 = 0$ is satisfied by real values of x and y then show that $x \in [1,\ 2]$ and $y \in \left[-\frac{1}{8}, \frac{1}{8}\right]$

A. Given equation can be expressed as

$$x^2 - 3x + (16y^2 + 2) = 0$$

$\because \quad x \in R \Rightarrow D \geq 0$

$$9 - 4 \times (16y^2 + 2) \geq 0$$

$$(9 - 64y^2 - 8) \geq 0$$

$\Rightarrow \quad 64y^2 - 1 \leq 0$

$$(8y + 1)(8y - 1) \leq 0 \Rightarrow y \in \left[-\frac{1}{8}, \frac{1}{8}\right]$$

Now, given equation,

$$16y^2 = -(x^2 - 3x + 2)$$

$\because \quad LHS \geq 0 \Rightarrow -(x^2 - 3x + 2) \geq 0$

$$x^2 - 3x + 2 \leq 0$$

$$(x - 1)(x - 2) \leq 0 \Rightarrow x \in [1,\ 2]$$

Hence Proved.

Theory of Equation

Cubic:

$$\mathbf{ax^3 + bx^2 + cx + d = a(x - \alpha)(x - \beta)(x - \gamma)}$$

Sum of roots taken one at a time = $\alpha + \beta + \gamma = \frac{-b}{a}$

Product of roots = $\alpha\beta\gamma = \frac{-d}{a}$

Sum of product of roots taken two at a time = $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$

Bi Quadratic:

$$\mathbf{ax^4 + bx^3 + cx^2 + dx + e = a(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)}$$

Sum of product of roots taken two at a time = $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$

Sum of product of roots taken three at a time = $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = \frac{-d}{a}$

In General:

If $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are the roots of the equation:

$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0$, where $a_0 \neq 0$

$$\sum \alpha_1 = -\frac{a_1}{a_0}$$

$$\sum \alpha_1\alpha_2 = +\frac{a_2}{a_0}$$

$$\sum \alpha_1\alpha_2\alpha_3 = -\frac{a_3}{a_0}$$

$$\vdots \qquad \vdots$$

$$\alpha_1\alpha_2\alpha_3\cdots\alpha_n = (-1)^n\frac{a_n}{a_0}$$

Note:

$$(a+b+c)^2 = \sum a^2 + 2\sum ab$$

Q. Find sum of squares of roots and sum of cubes of roots of the cubic equation $x^3 - px^2 + qx - r = 0$

A. Let roots are α, β, γ then

$\alpha + \beta + \gamma = p$

$\alpha\beta + \beta\gamma + \gamma\alpha = q$, $\alpha\beta\gamma = r$

sum of squares of roots = $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

$$= p^2 - 2q$$

sum of cubes of roots

$$\alpha^3 + \beta^3 + \gamma^3 = (\alpha+\beta+\gamma)\{\alpha^2 + \beta^2 + \gamma^2 - (\alpha\beta + \beta\gamma + \gamma\alpha)\} + 3\alpha\beta\gamma$$

$$= p\{(p^2 - 2q) - q\} + 3r$$

$$= p(p^2 - 3q) + 3r$$

$$= p^3 - 3pq + 3r$$

Q. Solve the cubic $4x^3 + 16x^2 - 9x - 36 = 0$ where sum of two roots is zero.

A. Let roots $\alpha, -\alpha, \beta$

now $\alpha + (-\alpha) + \beta = \frac{-16}{4} \Rightarrow \beta = -4$

also $\alpha(-\alpha)(\beta) = -\frac{(-36)}{4} \Rightarrow -\alpha^2(-4) = 9$

$\alpha^2 = \frac{9}{4} \Rightarrow \alpha = \pm\frac{3}{2}$

$\therefore$ roots $= -4, -\frac{3}{2}, \frac{3}{2}$

Q. If a, b, c are roots of cubic $x^3 - x^2 + 1 = 0$ find $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$

A. $a + b + c = 1$

$ab + bc + ca = 0, abc = -1$

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{a^2b^2 + b^2c^2 + c^2a^2}{(abc)^2} = \frac{\Sigma(ab)^2 - 2abc\,\Sigma a}{(abc)^2}$$

$$= \frac{0 - 2(1)(-1)}{(-1)^2}$$

$= 2$

Q. If $\alpha, \beta, \gamma, \delta$ are roots of the equation $\tan\left(\frac{\pi}{4} + x\right) = 3\tan 3x$ then find the value of $\tan\alpha + \tan\beta + \tan\gamma + \tan\delta$

A. $\frac{1 + \tan x}{1 - \tan x} = \frac{3(3\tan x - \tan^3 x)}{1 - 3\tan^2 x}$

let $\tan x = t$ then

$$\frac{1 + t}{1 - t} = \frac{9t - 3t^3}{1 - 3t^2}$$

$\Rightarrow 1 - 3t^2 + t - 3t^3 = 9t - 3t^3 - 9t^2 + 3t^4$

$\Rightarrow 3t^4 - 6t^2 + 8t - 1 = 0$

above equation roots $\tan\alpha, \tan\beta, \tan\gamma, \tan\delta$

$\Rightarrow$ sum of roots = 0

Q. If roots of $x^3 - 5x^2 + 6x - 3 = 0$ are α, β, γ. Tell equation whose roots are $\alpha + 1, \beta + 1, \gamma + 1$

A. Let $\alpha + 1 = x \Rightarrow \alpha = x - 1$

also α is root of given equation

$\therefore\ \alpha^3 - 5\alpha^2 + 6\alpha - 3 = 0$

$$\Rightarrow (x-1)^3 - 5(x-1)^2 + 6(x-1) - 3 = 0$$

$$\Rightarrow (x^3 - 3x^2 + 3x - 1) - 5(x^2 - 2x + 1) + 6(x-1) - 3 = 0$$

$$\Rightarrow x^3 - 8x^2 + 19x - 15 = 0$$

which is required equation.

Q. Find the cubic whose roots are cubes of the roots of $x^3 + 3x^2 + 2 = 0$

A. If roots of given equation α, β, γ then required equation will have roots $\alpha^3, \beta^3, \gamma^3$

now let $\alpha^3 = x \Rightarrow \alpha = x^{\frac{1}{3}}$

$$\Rightarrow \left(x^{\frac{1}{3}}\right)^3 + 3\left(x^{\frac{1}{3}}\right)^2 + 2 = 0$$

$$\Rightarrow -3x^{\frac{2}{3}} = x + 2$$

$$\Rightarrow \left(-3x^{\frac{2}{3}}\right)^3 = (x+2)^3$$

$$\Rightarrow -27x^2 = x^3 + 3x^2(2) + 3x(2)^2 + 2^3$$

$$\Rightarrow x^3 + 33x^2 + 12x + 8 = 0$$

which is required equation.

Q. The length of sides of a triangle are roots of the equation $x^3 - 12x^2 + 47x - 60 = 0$ and Δ is area of the triangle then find Δ^2

A. Let sides of triangle are a, b, c

then a + b + c = 12

ab + bc + ca = 47

abc = 60

Now, $\Delta^2 = s(s - a)(s - b)(s - c)$

$$= 6(6-a)(6-b)(6-c) \qquad \because s = \frac{a+b+c}{2} = 6$$

$$= 6\left\{6^3 - 12(6)^2 + 47(6) - 60\right\}$$

$$= 6(216 - 432 + 282 - 60)$$

$$= 36$$

Location of roots:

Type–1:

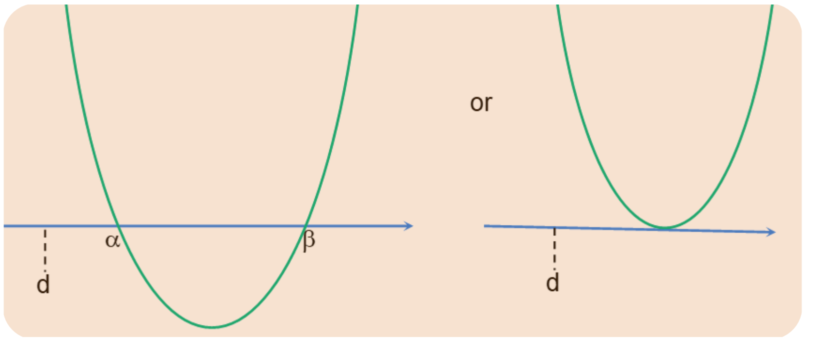
Both roots of a quadratic equation are greater than a specified number, i.e., $\alpha, \beta > d$

Step–1: Make leading coefficient positive
Step–2: Apply conditions

(i) $D \geq 0$

(ii) $\frac{-b}{2a} > d$

(iii) $f(d) > 0$



Q. Find the value of d for which both roots of the equation $x^2 - 6dx + 2 - 2d + 9d^2 = 0$ are greater than 3

A. Step–1: $a > 0 =>$ Let $f(x) = x^2 - 6dx + 2 - 2d + 9d^2$

Step–2:

(i) $D \geq 0 \Rightarrow 36d^2 - 4\left(2 - 2d + 9d^2\right) \geq 0 \Rightarrow 8d - 8 \geq 0 \Rightarrow d \geq 1$

(ii) $\frac{-b}{2a} > 3 \Rightarrow \frac{6d}{2} > 3 \Rightarrow d > 1$

(iii) $f(3) > 0 \Rightarrow 9 - 18d + 2 - 2d + 9d^2 > 0 \Rightarrow 9d^2 - 20d + 11 > 0$

$\Rightarrow 9d^2 - 9d - 11d + 11 > 0 \Rightarrow 9d(d-1) - 11(d-1) > 0$

$\Rightarrow (9d - 11)(d - 1) > 0 \Rightarrow d \in (-\infty, 1) \cup \left(\frac{11}{9}, \infty\right)$

(i) ∩ (ii) ∩ (iii) gives

$d \in \left(\frac{11}{9}, \infty\right)$

Q. Find all the values of 'a' for which both roots of the equation $x^2 + x + a = 0$ exceed the quantity 'a'.

A. Step–1: $A > 0 \Rightarrow$ Let $f(x) = x^2 + x + a$
Step–2:

(i) $D \geq 0 \Rightarrow 1^2 - 4a \geq 0 \Rightarrow 4a \leq 1 \Rightarrow a \in (-\infty, 1/4]$

(ii) $\dfrac{-B}{2A} > a \Rightarrow \dfrac{-1}{2} > a \Rightarrow a \in \left(-\infty, \dfrac{-1}{2}\right)$

(iii) $f(a) > 0 \Rightarrow a^2 + a + a > 0 \Rightarrow a^2 + 2a > 0 \Rightarrow a(a+2) > 0 \Rightarrow a \in (-\infty, -2) \cup (0, \infty)$

(i) ∩ (ii) ∩ (iii) gives

$a \in (-\infty, -2)$

Q. Determine the values of 'a' for which both roots of the quadratic equation $(a^2 + a - 2)x^2 - (a+5)x - 2 = 0$ exceed the number minus one.

A. Step–1: Divide by $(a^2 + a - 2)$

$$A > 0 \Rightarrow \text{Let } f(x) = x^2 - \frac{(a+5)}{(a^2+a-2)}x - \frac{2}{(a^2+a-2)}$$

Step–2:

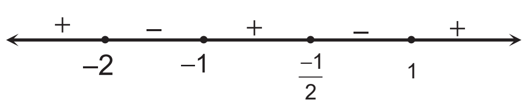
(i) $D \ge 0 \Rightarrow \dfrac{(a+5)^2}{(a^2+a-2)^2} + \dfrac{8}{(a^2+a-2)} \ge 0 \quad \dfrac{(a+5)^2 + 8(a^2+a-2)}{(a^2+a-2)^2} \ge 0$

$$\Rightarrow \frac{9a^2 + 18a + 9}{(a-1)^2(a+2)^2} \ge 0 \Rightarrow \frac{9(a+1)^2}{(a-1)^2(a+2)^2} \ge 0 \Rightarrow a \in R - \{-2, 1\}$$

(ii) $\dfrac{-B}{2A} > -1 \Rightarrow \dfrac{(a+5)}{2(a^2+a-2)} > -1 \Rightarrow \dfrac{a+5}{2(a^2+a-2)} + 1 > 0$

$$\Rightarrow \frac{2a^2 + 3a + 1}{2(a+2)(a-1)} > 0 \Rightarrow \frac{(2a+1)(a+1)}{2(a+2)(a-1)} > 0$$

$$\Rightarrow a \in (-\infty, -2) \cup \left(-1, -\frac{1}{2}\right) \cup (1, \infty)$$



(iii) $f(-1) > 0 \Rightarrow 1 + \dfrac{(a+5)}{(a^2+a-2)} - \dfrac{2}{(a^2+a-2)} > 0$

$$\Rightarrow \frac{a^2 + 2a + 1}{a^2 + a - 2} > 0 \Rightarrow \frac{(a+1)^2}{(a-1)(a+2)} > 0$$

$$\Rightarrow a \in (-\infty, -2) \cup (1, \infty)$$

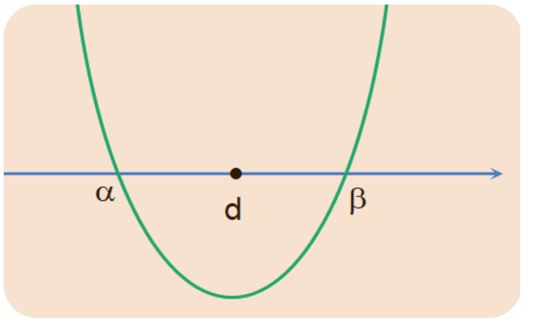
Now, (i) ∩ (ii) ∩ (iii) gives
$a \in (-\infty, -2) \cup (1, \infty)$

Type–2:

Both root lies on either side of a fixed number, say d, i.e., $\alpha < d < \beta$

Step–1: Make leading coefficient positive

Step–2: Apply condition f(d) < 0



Q. Find k for which one root of the equation $x^2 - (k + 1)x + k^2 + k - 8 = 0$ is greater than 2 and other is less than 2

A. Step–1: a > 0

Let $f(x) = x^2 - (k+1)x + (k^2 + k - 8)$

Step–2:

$$f(2) < 0$$

$$\Rightarrow \quad 4 - (k+1)2 + (k^2 + k - 8) < 0$$

$$\Rightarrow \quad k^2 - 3k + 2k - 6 < 0$$

$$\Rightarrow \quad k(k-2) + 2(k-3) < 0$$

$$\Rightarrow \quad (k+2)(k-3) < 0$$

$$\Rightarrow \quad k \in (-2, 3)$$

Q. Find the set of values of ‘a’ for which zeros of the quadratic polynomial $(a^2 + a + 1)x^2 + (a-1)x + a^2$ are located on either side of 3.

A. Step–1: Making leading coefficient positive

Let $f(x) = x^2 + \dfrac{(a-1)}{a^2 + a + 1}x + \dfrac{a^2}{a^2 + a + 1}$

Step–2: f(3) < 0

$$\Rightarrow \quad 9 + \frac{3(a-1)}{a^2 + a + 1} + \frac{a^2}{a^2 + a + 1} < 0$$

$$\Rightarrow \quad \frac{9(a^2 + a + 1) + 3(a-1) + a^2}{(a^2 + a + 1)} < 0$$

$$\Rightarrow \frac{10a^2+12a+6}{a^2+a+1}<0$$

$$\Rightarrow \frac{2\left(5a^2+6a+3\right)}{\left(a^2+a+1\right)}<0$$

$$\Rightarrow a \in \phi$$

Q. Find a for which one root is positive and other root is negative for $-x^2-(3a-2)x+a^2+1=0$

A. Step–1: Making leading coefficient positive

Let $f(x)=x^2+(3a-2)x-\left(a^2+1\right)$ then

Step–2: $f(0)<0$

$$\Rightarrow -\left(a^2+1\right)<0$$

$$\Rightarrow a \in R$$

Q. Find a for which both root lie on either side of – 1 of quadratic equation $\left(a^2-5a+6\right)x^2-(a-3)x+7=0$

A. Step– 1: Making leading coefficient positive

Let $f(x)=x^2-\frac{(a-3)x}{a^2-5a+6}+\frac{7}{a^2-5a+6}$

Step– 2: $f(-1)<0$

$$\Rightarrow 1+\frac{(a-3)}{\left(a^2-5a+6\right)}+\frac{7}{\left(a^2-5a+6\right)}<0$$

$$\Rightarrow \frac{\left(a^2-5a+6\right)+(a-3)+7}{\left(a^2-5a+6\right)}<0$$

$$\Rightarrow \frac{\left(a^2-4a+10\right)}{(a-2)(a-3)}<0$$

$$\Rightarrow a \in (2,3)$$

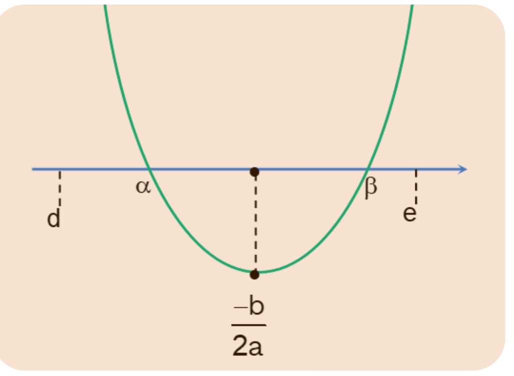
Type–3:

Both roots lies between two fixed number, i.e., $d < \alpha < \beta < e$

Step–1: Make leading coefficient positive

Step–2: Apply conditions

(i) $D \geq 0$
(ii) $f(d) > 0$
(iii) $f(e) > 0$
(iv) $d < -\dfrac{b}{2a} < e$



Q. If $\alpha, \beta \in (-6, 1)$ then find k for which $x^2 + 2(k-3)x + 9 = 0$ has roots α, β.

A. Step–1: As leading coefficient is positive

Let $f(x) = x^2 + 2(k - 3)x + 9$

(i) $D \geq 0 \Rightarrow 4(k-3)^2 - 4 \times 9 \geq 0 \Rightarrow (k-3+3)(k-3-3) \geq 0$

$\Rightarrow k(k-6) \geq 0 \Rightarrow k \in (-\infty, 0] \cup [6, \infty)$

(ii) $f(-6) > 0 \Rightarrow 36 - 12(k-3) + 9 > 0 \Rightarrow 81 - 12k > 0 \Rightarrow k \in \left(-\infty, \dfrac{27}{4}\right)$

(iii) $f(1) > 0 \Rightarrow 1 + 2(k - 3) + 9 > 0 \Rightarrow 2k + 4 > 0 \Rightarrow k > -2$

(iv) $-6 < \dfrac{-2(k-3)}{2} < 1 \Rightarrow -12- < 2(k-3) < 2 \Rightarrow 6 > (k-3) > -1 \Rightarrow k \in (2, 9)$

(i) ∩ (ii) ∩ (iii) ∩ (iv)

$k \in \left[6, \dfrac{27}{4}\right)$

Q. At what value of 'a' do all the zeroes of the function $(a-2)x^2 + 2ax + a + 3$ lies in the interval (–2, 1)

A. Step–1: Making leading coefficient positive

Let $f(x) = x^2 + \dfrac{2ax}{(a-2)} + \dfrac{(a+3)}{(a-2)}$

Step–2:

(i) $D \geq 0 \Rightarrow \dfrac{4a^2}{(a-2)^2} - \dfrac{4(a+3)}{(a-2)} \geq 0$

$$\frac{a^2-(a+3)(a-2)}{(a-2)^2} \geq 0 \Rightarrow \frac{-(a-6)}{(a-2)^2} \geq 0 \Rightarrow a \in (-\infty, 6] - \{2\}$$

(ii) $f(-2) > 0 \Rightarrow 4 - \frac{4a}{(a-2)} + \frac{(a+3)}{(a-2)} > 0$

$$\frac{4(a-2) - 4a + (a+3)}{(a-2)} > 0 \Rightarrow \frac{a-5}{a-2} > 0$$

$$\Rightarrow a \in (-\infty, 2) \cup (5, \infty)$$

(iii) $f(1) > 0 \Rightarrow 1 + \frac{2a}{(a-2)} + \frac{(a+3)}{(a-2)} > 0$

$$\frac{(a-2) + 2a + (a+3)}{(a-2)} > 0 \Rightarrow \frac{4a+1}{a-2} > 0$$

$$\Rightarrow a \in \left(-\infty, -\frac{1}{4}\right) \cup (2, \infty)$$

(iv) $-2 < \frac{-2a}{2(a-2)} < 1 \Rightarrow -2 < \frac{-a}{a-2} < 1$

$$-2 < \frac{-a}{a-2} \Rightarrow \frac{a}{a-2} - 2 < 0 \Rightarrow \frac{a-4}{a-2} > 0$$

$\therefore a \in (-\infty, 2) \cup (4, \infty)$

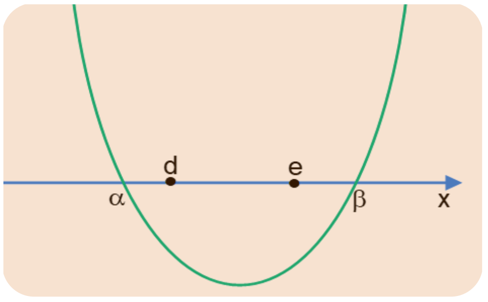
$$\frac{-a}{a-2} < 1 \Rightarrow \frac{a}{a-2} + 1 > 0 \Rightarrow \frac{2(a-1)}{a-2} > 0$$

$\therefore \ a \in (-\infty, 1) \cup (2, \infty)$

Intersection of all conditions is $x \in (-\infty - 1/4) \ \cup (5,6]$

Type–4:

Both roots lies on either side of two fixed numbers,
i.e., $\alpha < d < e < \beta$



Step–1: Make leading coefficient positive

(i) f(d)<0

(ii) f(e) < 0

Step–2: Apply conditions

Q. Find k for which one root of the equation $(k-5)x^2 - 2kx + (k-4) = 0$ is smaller than 1 and the other root is greater than 2.

A. Step–1: Making leading coefficient positive.

Let $f(x) = x^2 - \frac{2kx}{(k-5)} + \frac{(k-4)}{(k-5)}$

Step–2:

(i) $f(2) < 0$

$$4 - \frac{4k}{(k-5)} + \frac{k-4}{k-5} < 0 \Rightarrow \frac{4(k-5) - 4k + (k-4)}{(k-5)} < 0$$

$$\Rightarrow \frac{k-24}{k-5} < 0 \Rightarrow k \in (5, 24)$$

(ii) $f(1) < 0$

$$1 - \frac{2k}{k-5} + \frac{k-4}{k-5} < 0 \Rightarrow \frac{(k-5) - 2k + (k-4)}{(k-5)} < 0$$

$$\frac{-9}{k-5} < 0 \Rightarrow k \in (5, \infty)$$

(i) $\cap$ (ii)

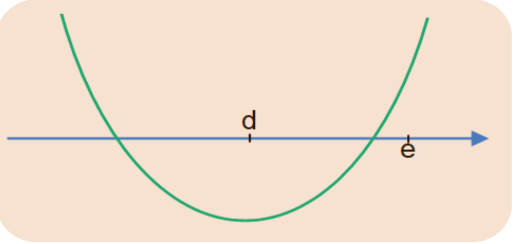
$\Rightarrow k \in (5, 24)$

Types–5:

Exactly one root lies in the interval (d, e)

Step–1: Make leading coefficient positive.

Step–2: Apply condition $f(d)f(e) < 0$



Q. Find the set of values of m for which exactly one root of the equation $x^2 + mx + (m^2 + 6m) = 0$ lie in (– 2, 0)

A. Step–1: Making leading coefficient positive.

Let $f(x) = x^2 + mx + m^2 + 6m$

Step–2:

Case–I: When no root is – 2 or 0

$f(-2)\, f(0) < 0$

$$\Rightarrow (4 - 2m + m^2 + 6m)(m^2 + 6m) < 0 \Rightarrow (m^2 + 4m + 4)(m^2 + 6m) < 0$$

$$\Rightarrow (m+2)^2 m(m+6) < 0 \Rightarrow m \in (-6, 0) - \{-2\}$$

Case–II: When one of the root is – 2 or 0

(i) if $f(-2) = 0 \Rightarrow 4 - 2m + m^2 + 6m = 0$

$\Rightarrow m^2 + 4m + 4 = 0 \Rightarrow m = -2$

For m= – 2 equation: $x^2 - 2x - 8 = 0 \Rightarrow x = 4, -2 \Rightarrow$ No root in (–2, 0)

(ii) If $f(0) = 0 \Rightarrow m^2 + 6m = 0 \Rightarrow m = 0, -6$

For m = 0 equation: $x^2 = 0 \Rightarrow x = 0, 0 \Rightarrow$ No root in (–2, 0)

Hence, $m \in (-6, 0) - \{-2\}$

Q. Find a for which exactly one root of the equation $x^2 - (a+1)x + 2a = 0$ lies in (0, 3)

A. Step–1: Making leading coefficient positive

Let $f(x) = x^2 - (a+1)x + 2a$

Step–2:

Case–I: When no root is 0 or 3

$f(0).f(3) < 0$

$\Rightarrow (2a)(9 - 3(a+1) + 2a) < 0$

$\Rightarrow 2a(6 - a) < 0 \Rightarrow a(a - 6) > 0$

$a \in (-\infty, 0) \cup (6, \infty)$ ...(i)

Case– II: when one of the root is 0 or 3

(a) If $f(0) = 0 \Rightarrow 2a = 0 \Rightarrow a = 0$

For a=0 equation : $x^2 - x = 0 \Rightarrow x = 0, 1 \Rightarrow$ one root in (0,3)

hence, a can be 0 ...(ii)

(b) If $f(3) = 0 \Rightarrow 9 - (a + 1)3 + 2a = 0 \Rightarrow a = 6$

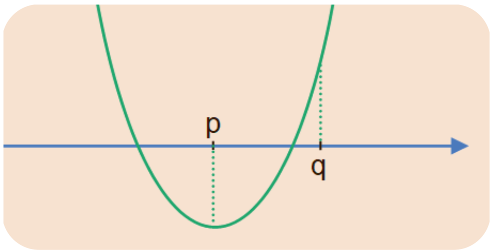
For a = 6 equation: $x^2 - 7x + 12 = 0 \Rightarrow x = 3, 4 \Rightarrow$ No root in (0,3)

(i) U (ii)

hence, $a \in (-\infty, 0] \cup (6, \infty)$

Note: If f(p) f(q) < 0, then

Exactly one root lies between (p, q)



Miscellaneous Examples

Q. If $a < b < c < d$ show that quadratic $(x - a)(x - c) + \lambda(x - b)(x - d) = 0$ has real root for all real values of λ except -1.

A. Let $f(x) = (x - a)(x - c) + \lambda(x - b)(x - d)$

now

$$f(a) = \lambda(a - b)(a - d)$$

$$f(b) = (b - a)(b - c) < 0$$

$$f(c) = \lambda(c - b)(c - d)$$

$$f(d) = (d - a)(d - c) > 0$$

now $f(b)f(d)<0 \Rightarrow$ exactly one root in interval (b, d)

$\because$ coefficient real $\Rightarrow$ other root must be real for $\forall\ \lambda \in R-\{-1\}$

Q. Find p for which the expression $x^2 - 2px + 3p + 4 < 0$ is satisfied for atleast one real x.

A. Clearly $D > 0$

$(-2p)^2 - 4(3p + 4) > 0$

$\Rightarrow p^2 - 3p - 4 > 0$

$\Rightarrow (p - 4)(p + 1) > 0$

$\Rightarrow p \in (-\infty, -1) \cup (4, \infty)$

Q. Find the value of m for which $x^2 - 4x + 3m + 1 > 0$ is satisfied for all positive x.

A. $x^2 - 4x + 4 > 3 - 3m$

for $x > 0$, $(x - 2)^2 \in [0,\infty)$

hence $3 - 3m < 0 \Rightarrow m > 1$

Q. Show that $(a^2 + 3)x^2 + (a + 2)x - 5 < 0$ is true for at least one negative x for any real value of a .

A. Let $f(x) = (a^2 + 3)x^2 + (a + 2)x - 5$

clearly f(x) is upward parabola and $f(0) < 0$

$\Rightarrow$ it is negative for atleast one negative x, $\forall\ a \in R$

Quadratic Equations

Q. If $f(x) = 4x^2 + ax + (a-3)$ is negative for atleast one negative x, find all values of a.

A. f(x) = 0 have atleast one root negative

Case–I: Both roots < 0 (distinct roots)

(i) D > 0

$\Rightarrow a^2 - 16(a-3) > 0 \Rightarrow a^2 - 16a + 48 > 0 \Rightarrow (a-12)(a-4) > 0$

$\Rightarrow a \in (-\infty, 4) \cup (12, \infty)$

(ii) $-\frac{B}{2A} < 0$

$\Rightarrow \frac{-a}{8} < 0 \Rightarrow a > 0$

(iii) f(0) > 0

$\Rightarrow a - 3 > 0 \Rightarrow a > 3$

(i) ∩ (ii) ∩ (iii)

$a \in (3, 4) \cup (12, \infty)$ ...(1)

Case–II: One root > 0 and other root < 0

f(0) < 0

$\Rightarrow a - 3 < 0 \Rightarrow a < 3$ ...(2)

Case–III: One root < 0 and other root = 0

$f(0) = 0 \Rightarrow a = 3$

for a = 3 equation is $4x^2 + 3x = 0$

$x = 0,\ x = -\frac{3}{4}$ (negative)

$\therefore a = 3$ ...(3)

(1) ∪ (2) ∪ (3)

$a \in (-\infty, 4) \cup (12, \infty)$

Q. Find a for which $x^2 + 2(a-1)x + a + 5 = 0$ has at least one positive root.

A. Let $f(x) = x^2 + 2(a-1)x + a + 5$

Case–I: Both roots > 0

(i) $D \geq 0$

$\Rightarrow 4(a-1)^2 - 4(a+5) \geq 0 \Rightarrow (a^2 - 2a + 1) - (a+5) \geq 0$

$\Rightarrow (a^2 - 3a - 4) \geq 0 \Rightarrow (a-4)(a+1) \geq 0$

$\Rightarrow a \in (-\infty, -1] \cup [4, \infty)$

(ii) $\frac{-B}{2A} > 0$

$\Rightarrow \frac{-2(a-1)}{2} > 0 \Rightarrow a-1 < 0 \Rightarrow a < 1$

(iii) f(0)>0

$a+5 > 0 \Rightarrow a > -5$

(i) ∩ (ii) ∩ (iii)

$\Rightarrow a \in (-5, -1)$ ...(1)

Case–II: one root < 0, other root > 0

f(0) < 0

$a + 5 < 0 \Rightarrow a < -5$ ...(2)

Case–III: one root=0, other root > 0

f(0)=0

$a + 5 = 0 \Rightarrow a = -5$

for $a = -5$ equation is $x^2 - 12x = 0 \Rightarrow x = 0, x = 12$ (positive)

$\therefore a = -5$ ...(3)

(1) ∪ (2) ∪ (3)

$a \in (-\infty, -1)$

Q. Find p for which the least value of $4x^2 - 4px + p^2 - 2p + 2$ for $x \in [0,2]$ is equal to 3

A. Let $f(x) = 4x^2 - 4px + p^2 - 2p + 2$

Case–I: $-\frac{b}{2a} \le 0 \Rightarrow \frac{4p}{8} \le 0 \Rightarrow p \le 0$

then f(0) is minimum in $x \in [0,2]$, hence,

$\Rightarrow f(0)=3 \Rightarrow p^2 - 2p - 1=0 \Rightarrow p = \frac{2 \pm \sqrt{8}}{2}$

$p = 1-\sqrt{2}$, $p = 1+\sqrt{2}$ (rejected, as $p \le 0$)

$\Rightarrow p = \left(1-\sqrt{2}\right)$ ... (1)

Case–II: $0 < \frac{-b}{2a} < 2 \Rightarrow 0 < \frac{4p}{8} < 2 \Rightarrow p \in (0,4)$

In this case f(x) is minimum at vertex

hence, $\frac{-D}{4a} = 3$

$$\frac{-\left\{16p^2 - 16\left(p^2 - 2p + 2\right)\right\}}{16} = 3$$

$-2p + 2 = 3$

$\Rightarrow p = -\frac{1}{2}$ (rejected, as $p \in (0,4)$)

$\Rightarrow p \in \phi$ ... (2)

Case–III: $\frac{-b}{2a} \geq 2 \Rightarrow \frac{4p}{8} \geq 2 \Rightarrow p \geq 4$

In this case f(x) is min at x=2 hence, f(2) = 3

$\Rightarrow 16 - 8p + p^2 - 2p + 2 = 3 \Rightarrow p^2 - 10p + 15 = 0$

$\Rightarrow p = \frac{10 \pm \sqrt{100-60}}{2} = \frac{10 \pm 2\sqrt{10}}{2} = 5 \pm \sqrt{10}$

$\Rightarrow p = 5 + \sqrt{10}$, $\quad p = 5 - \sqrt{10}$ (rejected, as $p \geq 4$)

$\Rightarrow p = 5 + \sqrt{10}$ ... (3)

(1) ∪ (2) ∪ (3)

hence, $p = \{1 - \sqrt{2}, 5 + \sqrt{10}\}$

Q. Find k for which the equation $x^4 + x^2(1-2k) + k^2 - 1 = 0$ has

(i) No real solution
(ii) one real solution
(iii) Two real solutions
(iv) Three real solutions
(v) Four real solutions.

A. Let f(x) = $x^4 + x^2(1-2k) + k^2 - 1 = 0$, $x^2 = t$

Also, let $g(t) = t^2 + (1-2k)t + (k^2 - 1)$

(i) For equation f(x) = 0 to have no real solution, equation g(t)=0 must have either both roots < 0 or imaginary roots.

Case–I: both roots < 0

(i) $D \geq 0 \Rightarrow (1-2k)^2 - 4(k^2-1) \geq 0$

$\Rightarrow (4k^2 - 4k + 1) - 4k^2 + 4 \geq 0$

$\Rightarrow 4k \leq 5 \Rightarrow k \in \left(-\infty, \frac{5}{4}\right]$

(ii) $\frac{-b}{2a} < 0 \Rightarrow \frac{(2k-1)}{2} < 0 \Rightarrow k \in \left(-\infty, \frac{1}{2}\right)$

(iii) $g(0) > 0 \Rightarrow k^2 - 1 > 0 \Rightarrow k \in (-\infty, -1) \cup (1, \infty)$

(i) ∩ (ii) ∩ (iii)

$\Rightarrow k \in (-\infty, -1)$ ...(1)

Case–II: Imaginary roots

$$D < 0 \Rightarrow k \in \left(\frac{5}{4}, \infty\right) \qquad ...(2)$$

(1) ∪ (2)

$$k \in (-\infty, -1) \cup \left(\frac{5}{4}, \infty\right)$$

(ii) For equation $f(x) = 0$ to have one real solution equation g(t) = 0 must have one root = 0 and other root < 0

hence, g(0) = 0 ⇒ k = ± 1

Put k = 1 in f(x) = 0: $x^2 [x^2 - 1] = 0$ giving three solutions so not possible

put k = –1 in f(x) = 0: $x^2[x^2 + 3] = 0$ giving one solution so possible.

k = {–1}

(iii) For equation $f(x) = 0$ to have two real solution equation g(t) = 0 can be equal and positive roots or have one root > 0 and other root < 0

Case–I:

$$D = 0 \Rightarrow k = \frac{5}{4}$$

$$(x^2)^2 - \frac{3}{2}x^2 + \frac{9}{16} = 0$$

$$\Rightarrow x^2 = \frac{9}{16} \Rightarrow x = \pm\frac{3}{4}$$ giving exactly 2 solutions.

$$\Rightarrow k = \frac{5}{4} \qquad ... (1)$$

Case–II:

g(0)< 0

$k^2 - 1 < 0 \Rightarrow k \in (-1, 1)$ ... (2)

(1) ∪ (2)

$$\Rightarrow k \in (-1, 1) \cup \left\{\frac{5}{4}\right\}$$

(iv) For equation $f(x) = 0$ to have 3 real solution equation g(t) = 0 must have one root = 0 and other root > 0

hence $g(0) = 0 \Rightarrow k = \pm 1$

If k=1 then equation: $t^2 - t = 0 \Rightarrow t = 0, 1$ gives 3 solutions for f(x) = 0.

So, accepted

If k= – 1 then equation: $t^2 + 3t = 0 \Rightarrow t = 0, -3$ gives only 1 solution for f(x) = 0.

So, Rejected

Hence k = 1

(v) For equation $f(x)=0$ to have 4 real solution equation g(t) = 0 must have unequal and positive roots. Hence,

(i) $D>0 \Rightarrow (1-2k)^2-4\left(k^2-1\right)>0$

$\Rightarrow \left(4k^2-4k+1\right)-4k^2+4>0$

$\Rightarrow 4k<5 \Rightarrow k \in \left(-\infty, \frac{5}{4}\right)$

(ii) $g(0)>0 \Rightarrow k^2-1>0$

$\Rightarrow k \in (-\infty,-1) \cup (1,\infty)$

(iii) $-\frac{b}{2a}>0 \Rightarrow \frac{2k-1}{2}>0 \Rightarrow k>\frac{1}{2}$

(i) ∩ (ii) ∩ (iii)

$\Rightarrow k \in \left(1, \frac{5}{4}\right)$

Q. Find all values of the parameter 'a' for which the inequality $4^x - a.2^x - a + 3 \le 0$ which must be satisfied for atleast one real x.

A. Let $f(t)=t^2-at-a+3$, $t>0$ then at least one root of $f(t)=0$ must be positive

Case–I: both roots > 0 (smaller can be 0)

(i) $D \ge 0 \Rightarrow a^2-4(-a+3) \ge 0$

$a^2+4a-12 \ge 0$

$(a+6)(a-2) \ge 0 \Rightarrow a \in (-\infty,-6] \cup [2,\infty)$

(ii) $\frac{-B}{2A}>0 \Rightarrow \frac{a}{2}>0 \Rightarrow a>0$

(iii) $f(0) \ge 0 \Rightarrow (-a+3) \ge 0 \Rightarrow a \le 3$

(i) ∩ (ii) ∩ (iii)

$a \in [2, 3]$ ...(1)

Case– II: one root > 0, other root < 0

$f(0)<0$ $\Rightarrow a>3$ ...(2)

(1) ∪ (2)

$a \in [2, \infty)$

Q. If $\left(x^2+x+2\right)^2-(a-3)\left(x^2+x+1\right)\left(x^2+x+2\right)+(a-4)\left(x^2+x+1\right)^2=0$ has atleast one real root, then find the complete set of value of a.

A. Let $x^2+x+1=t\in\left[\frac{3}{4},\infty\right)$

Given equation becomes:

$t(2 - a + 3) + 1 = 0$

$\Rightarrow t=\frac{1}{a-5}\geq\frac{3}{4}$

$\Rightarrow \frac{1}{a-5}-\frac{3}{4}\geq 0$

$\Rightarrow \frac{4-3a+15}{4(a-5)}\geq 0$

$\Rightarrow \frac{19-3a}{a-5}\geq 0$

$\Rightarrow a\in\left(5,\frac{19}{3}\right]$

Modulus Inequality

For $\alpha,\beta>0$

(i) $|x|<\alpha\Rightarrow x\in(-\alpha,\alpha)$

(ii) $|x|>\beta\Rightarrow x\in(-\infty,-\beta)\cup(\beta,\infty)$

Q. $(|x-1|-3)(|x+2|-5)<0$

A. **Case–I:**

$|x-1|-3<0$ and $|x+2|-5>0$

$|x-1|<3$ and $|x+2|>5$

$x-1\in(-3,3)$ and $x+2\in(-\infty,-5)\cup(5,\infty)$

$x\in(-2,4)$ and $x\in(-\infty,-7)\cup(3,\infty)$

$x\in(3,4)$ ...(1)

Case–II:

$|x-1|-3>0$ & $|x+2|-5<0$

$|x-1|>3$ & $|x+2|<5$

$$x \in (-\infty, -2) \cup (4, \infty) \ \& \ x \in (-7, 3)$$

$$x \in (-7, -2) \qquad \qquad ...(2)$$

$$(1) \cup (2) \Rightarrow x \in (-7, -2) \cup (3, 4)$$

Q. $\left|x^2 + 4x + 2\right| = \dfrac{5x + 16}{3}$

A. **Case–I:** $\dfrac{5x + 16}{3} \geq 0$

$$x^2 + 4x + 2 = \frac{5x + 16}{3}$$

$$3x^2 + 12x + 6 = 5x + 16,$$

$$3x^2 + 7x - 10 = 0$$

$$(3x + 10)(x - 1) = 0$$

$$x = \frac{-10}{3} \ \left(\text{rejected as } x > -\frac{16}{5}\right), \ x = 1$$

$$x = 1 \qquad \qquad ...(1)$$

Case–II: $\dfrac{5x + 16}{3} < 0$

$$x^2 + 4x + 2 = -\frac{(5x + 16)}{3}$$

$$3x^2 + 12x + 6 = -5x - 16$$

$$3x^2 + 17x + 22 = 0$$

$$(3x + 11)(x + 2) = 0$$

$$x = \frac{-11}{3}, x = -2 \ \left(\text{rejected as } x < -\frac{16}{5}\right) \qquad \qquad ...(2)$$

$(1) \cup (2)$

$$x = \left\{1, \frac{-11}{3}\right\}$$

Q. $|x - 5| > \left|x^2 - 5x + 9\right|$

A. Squaring both sides

$$(x - 5)^2 > \left(x^2 - 5x + 9\right)^2$$

$$\Rightarrow \left(x^2 - 5x + 9 + x - 5\right)\left(x^2 - 5x + 9 - x + 5\right) < 0$$

$$\Rightarrow \left(x^2-4x+4\right)\left(x^2-6x+14\right)<0$$

$$\Rightarrow (x-2)^2<0$$

$$\Rightarrow x \in \phi$$

Q. $\left|\dfrac{x^2-5x+4}{x^2-4}\right| \le 1$

A. $\left|x^2-5x+4\right| \le \left|x^2-4\right|, (x \ne \pm 2)$

$$\left(x^2-5x+4\right)^2 \le \left(x^2-4\right)^2 \ (x \ne \pm 2)$$

$$\left(x^2-5x+4+x^2-4\right)\left(x^2-5x+4-x^2+4\right) \le 0, (x \ne \pm 2)$$

$$\Rightarrow \left(2x^2-5x\right)(8-5x) \le 0, (x \ne \pm 2)$$

$$\Rightarrow x(2x-5)(5x-8) \ge 0, x \ne \pm 2$$

$$x \in \left[0, \frac{8}{5}\right] \cup \left[\frac{5}{2}, \infty\right)$$

Q. $|x^2-3x-1| < 3|x^2+x+1|$

A. $|x^2-3x-1| < 3|x^2+x+1|$

Squaring both sides

$$\left(x^2-3x-1\right)^2 < \left(3x^2+3x+3\right)^2$$

$$\Rightarrow \left(4x^2+2\right)\left(2x^2+6x+4\right) > 0$$

$$\Rightarrow 4\left(2x^2+1\right)\left(x^2+3x+2\right) > 0$$

$$\Rightarrow (x+1)(x+2) > 0$$

$$\Rightarrow x \in (-\infty, -2) \cup (-1, \infty)$$

Q. $\left|\left|x-1\right|-1\right| \le 1$

A. $-1 \le |x-1|-1 \le 1$

$$0 \le |x-1| \le 2$$

$$\Rightarrow -2 \le x-1 \le 2$$

$$\Rightarrow -1 \le x \le 3 \Rightarrow x \in [-1, 3]$$

Q. $||x-1|-2| = 1$

A. $|x-1|-2 = 1,-1$

$(x-1) = 3,-3,1,-1$

$x = 4, -2, 2, 0$

Q. $||3x-9|+2| > 2$

A. $|3x-9|+2 < -2$ or $|3x-9|+2 > 2$

$\Rightarrow |3x-9| < -4$ or $|3x-9| > 0$

$\Rightarrow x \in \phi$ or $x \in R - \{3\}$

$\Rightarrow x \in R - \{3\}$

Q. Find the set of solutions of the equation $2^{|y|} - |2^{y-1} - 1| = 2^{y-1} + 1$

A. Zeroes of modulus:

(i) for $|y|$ is $y = 0$

(ii) for $|2^{y-1} - 1| = 0$ is $y = 1$

Case–I: $y \geq 1 \Rightarrow y \in [1, \infty)$

$\Rightarrow 2^y - (2^{y-1} - 1) = 2^{y-1} + 1$

$\Rightarrow y \in R$

$\Rightarrow R \cap [1, \infty) = [1, \infty)$ ... (1)

Case–II: $0 \leq y < 1$

$2^y + (2^{y-1} - 1) = 2^{y-1} + 1$

$\Rightarrow 2^y = 2 \Rightarrow y = 1$

$\Rightarrow y = 1 \cap y \in [0, 1)$

$\Rightarrow y \in \phi$ ...(2)

Case–III: $y < 0$

$$2^{-y} + \left(2^{y-1} - 1\right) = 2^{y-1} + 1$$

$$2^{-y} = 2$$

$$\Rightarrow y = -1$$

$y = -1 \cap (-\infty, 0)$

$\Rightarrow y = -1$ ...(3)

$(1) \cup (2) \cup (3)$

$y \in [1, \infty) \cup \{-1\}$

Log–Inequality

(i) $\log_a x > \log_a y \Rightarrow x > y > 0$, If $a > 1$

(ii) $\log_a x > \log_a y \Rightarrow 0 < x < y$, If $0 < a < 1$

Q. $\log_7\left(\dfrac{2x-6}{2x-1}\right) > 0$

A. $\log_7\left(\dfrac{2x-6}{2x-1}\right) > \log_7 1$

$$\frac{2x-6}{2x-1} > 1 \Rightarrow \frac{2x-6}{2x-1} - 1 > 0$$

$$\Rightarrow \frac{-5}{2x-1} > 0 \Rightarrow 2x - 1 < 0$$

hence, $x < \dfrac{1}{2}$

Q. $\log_3(2x-1) < 2$

A. $0 < (2x-1) < 3^2$

$0 < 2x - 1 < 9$

$1 < 2x < 10 \Rightarrow x \in \left(\dfrac{1}{2}, 5\right)$

Q. $\log_3|2x-1| > 2$

A. $|2x-1| > 3^2$

$|2x-1| > 9$

$2x-1 < -9$ or $2x-1 > 9$

$2x < -8$ or $2x > 10$

$x < -4$ or $x > 5$

$\Rightarrow x \in (-\infty, -4) \cup (5, \infty)$

Q. $\log_3 \dfrac{|x^2-4x|+3}{x^2+|x-5|} \geq 0$

A. $\dfrac{|x^2-4x|+3}{x^2+|x-5|} \geq 1$

$\Rightarrow |x^2-4x|+3 \geq x^2+|x-5|$

zeroes of modulus are $0, 4, 5$

Case–I: $x \geq 5$ ...(i)

$\Rightarrow x^2 - 4x + 3 \geq x^2 + x - 5$

$\Rightarrow 5x \leq 8$

$\Rightarrow x \leq 8/5$ ...(ii)

From (i) and (ii) $x \in \phi$ ...(1)

Case–II: $4 \leq x < 5$ ...(iii)

$\Rightarrow x^2 - 4x + 3 \geq x^2 + 5 - x$

$\Rightarrow 3x \leq -2$

$\Rightarrow x \leq -2/3$ ...(iv)

From (iii) and (iv) $x \in \phi$ ...(2)

Case–III: $0 \leq x < 4$ ...(v)

$\Rightarrow 4x - x^2 + 3 \geq x^2 - x + 5$

$\Rightarrow 2x^2 - 5x + 2 \leq 0$

$\Rightarrow (2x-1)(x-2) \leq 0$

$\Rightarrow x \in \left[\dfrac{1}{2}, 2\right]$ ...(vi)

From (v) and (vi) $x \in \left[\dfrac{1}{2}, 2\right]$ ...(3)

Case–IV: $x \leq 0$ ...(vii)

$\Rightarrow x^2 - 4x + 3 \geq x^2 - x + 5$

$\Rightarrow 3x \leq -2$

$\Rightarrow x \le -2/3$ ...(viii)

From (vii) and (viii) $x \in \left(-\infty, -\frac{2}{3}\right]$ ...(4)

(1) ∪ (2) ∪ (3) ∪ (4)

$$x \in \left(-\infty, \frac{-2}{3}\right] \cup \left[\frac{1}{2}, 2\right]$$

Q. $\log_{0.2}\left(x^2 - x - 2\right) < \log_{0.2}\left(-x^2 + 2x + 3\right)$

A. $0 < -x^2 + 2x + 3 < x^2 - x - 2$

$\Rightarrow -x^2 + 2x + 3 > 0$ and $x^2 - x - 2 > -x^2 + 2x + 3$

$\Rightarrow (x+1)(x-3) < 0$ and $2x^2 - 3x - 5 > 0$

$\Rightarrow x \in (-1, 3)$ and $(2x - 5)(x + 1) > 0$

$\Rightarrow x \in (-1, 3)$ and $(-\infty, -1) \cup \left(\frac{5}{2}, \infty\right)$

$$\Rightarrow x \in \left(\frac{5}{2}, 3\right)$$

Q. $(0.3)^{\log_{\frac{1}{3}}\left(\log_2 \frac{3x+6}{x^2+2}\right)} > 1$

A. $\log_{\frac{1}{3}}\left(\log_2 \frac{3x+6}{x^2+2}\right) < 0$

$\Rightarrow \log_2\left(\frac{3x+6}{x^2+2}\right) > 1$

$\Rightarrow \frac{3x+6}{x^2+2} > 2 \Rightarrow 3x + 6 > 2x^2 + 4$

$\Rightarrow 2x^2 - 3x - 2 < 0$

$\Rightarrow (x-2)(2x+1) < 0$

$\Rightarrow x \in \left(\frac{-1}{2}, 2\right)$

Q. $\left(2\left(\log_3 x\right)^2 - 3\log_3 x - 8\right)\left(2\log_3^2 x - 3\log_3 x - 6\right) \ge 3$

A. Let $\log_3 x = t$, then

$\left(2t^2 - 3t - 8\right)\left(2t^2 - 3t - 6\right) \ge 3$

Again let $2t^2 - 3t = \alpha$

$\Rightarrow (\alpha - 8)(\alpha - 6) \geq 3$

$\Rightarrow \alpha^2 - 14\alpha + 48 - 3 \geq 0$

$\Rightarrow \alpha^2 - 14\alpha + 45 \geq 0$

$\Rightarrow (\alpha - 9)(\alpha - 5) \geq 0$

$\Rightarrow \alpha \leq 5$ or $\alpha \geq 9$

$\Rightarrow 2t^2 - 3t - 5 \leq 0$ or $2t^2 - 3t - 9 \geq 0$

$\Rightarrow (2t - 5)(t + 1) \leq 0$ or $(2t + 3)(t - 3) \geq 0$

$\Rightarrow t \in \left[-1, \frac{5}{2}\right]$ or $t \in \left(-\infty, \frac{-3}{2}\right] \cup [3, \infty)$

$\Rightarrow t \in \left(-\infty, \frac{-3}{2}\right] \cup \left[-1, \frac{5}{2}\right] \cup [3, \infty)$

$\Rightarrow x \in \left(3^{-\infty}, 3^{\frac{-3}{2}}\right] \cup \left[3^{-1}, 3^{\frac{5}{2}}\right] \cup \left[3^3, 3^{\infty}\right)$

hence, $x \in \left(0, \frac{1}{3\sqrt{3}}\right] \cup \left[\frac{1}{3}, 9\sqrt{3}\right] \cup [27, \infty)$

Q. $\log_{2x+3} x^2 < \log_{2x+3}(2x+3)$

A. **Case–I:** $2x + 3 > 1 \Rightarrow x > -1$

So, given equation: $0 < x^2 < 2x + 3$

$0 < x^2$ & $x^2 < 2x + 3$

$x \neq 0$ & $x^2 - 2x - 3 < 0$

$x \neq 0$ & $(x - 3)(x + 1) < 0$

$x \neq 0$ & $x \in (-1, 3)$

$x \in (-1, 0) \cup (0, 3)$

Now, $x \in (-1, 0) \cup (0, 3) \cap (-1, \infty)$

$x \in (-1, 0) \cup (0, 3)$ ...(1)

Case–II: $0 < 2x + 3 < 1 \Rightarrow -3 < 2x < -2 \Rightarrow x \in \left(\frac{-3}{2}, -1\right)$

So, given equation: $x^2 > 2x + 3$

$x^2 - 2x - 3 > 0$

$(x + 1)(x - 3) > 0$

$\Rightarrow x \in (-\infty, -1) \cup (3, \infty)$

Now, $x \in (-\infty, -1) \cup (3, \infty) \cap \left(\frac{-3}{2}, -1\right)$

$$\Rightarrow x \in \left(-\frac{3}{2}, -1\right) \qquad \text{...(2)}$$

(1) ∪ (2)

$$x \in \left(-\frac{3}{2}, -1\right) \cup (-1, 0) \cup (0, 3)$$