Wave on a String

DISCLAIMER

Wave on a String

WAVE MOTION

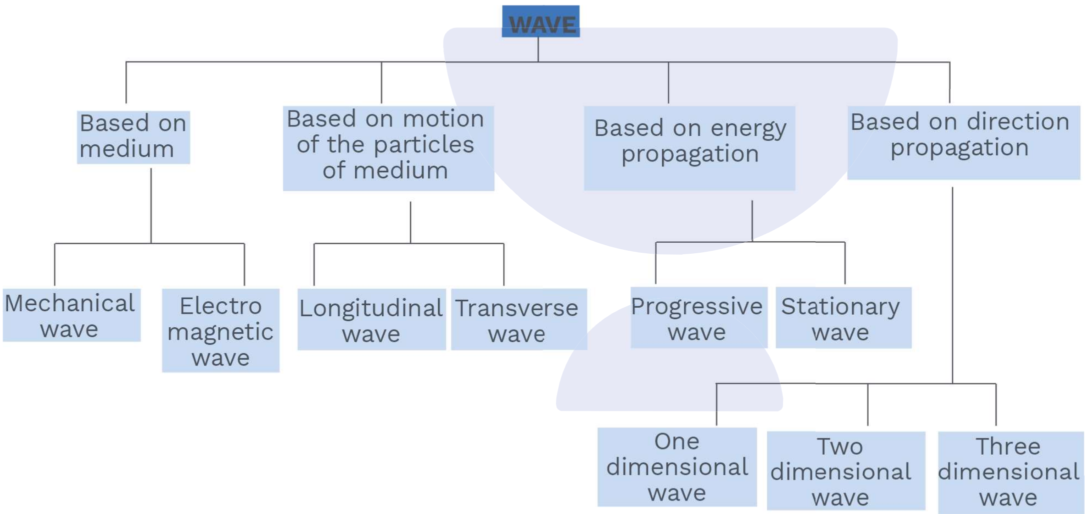
A wave is a disturbance which propagates the energy (and momentum) from one place to the other without the transport of matter. It is well spread over a region of space without clear cut boundaries. It cannot be said to be localized here or there.

Examples : Waves in still pond, light, sound, a pulse travelling on a taut string.

KEY POINTS

- Wave
- Mechanical wave
- Progressive wave
- Stationary wave
- Longitudinal wave
- Transverse wave

Classification of wave



As shown in the above tree, waves can be primarily classified into four categories :-

(A) Based on medium

(B) Based on the motion of particles of the medium

(C) Based on energy propagation

(D) Based on direction of propagation

On The Basis Of Medium.

1. Mechanical or elastic wave.

Waves for which propagation medium is essential are known as mechanical waves. eg. Sound wave, wave on water surface, wave on stretched wire etc.

Definitions

Waves for which propagation medium is essential are known as mechanical waves

For the propagation of mechanical wave a medium should have following 2 properties:-

- Inertia
- Elasticity

2. Non Mechanical or electro-magnetic wave (EMW)

Waves for which propagation medium is not essential are known as non-mechanical wave. eg. Light wave, Radio wave etc.

Definitions

Waves for which propagation medium is not essential are known as non-mechanical wave or electromagnetic wave

On The Basis of Direction of Propagation

1. **1-D wave :**
   Waves in which energy propagate in particular one direction, are known as 1-D waves. eg. Wave on stretched wire, wave in spring etc.
2. **2-D wave :**
   Waves in which energy propagate in a plane is known as 2-D wave. eg. Wave on liquid surface.
3. **3-D wave :**
   Waves in which energy propagate in all possible directions, are known as 3-D waves. eg. Sound wave, light wave, seismic wave.

Concept Reminder

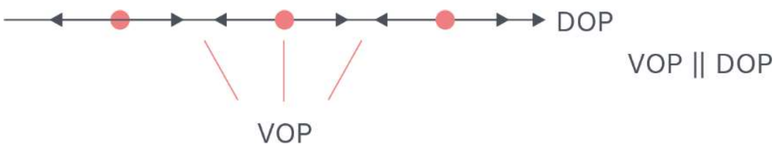
Not all waves require a medium for their propagation. We know that light waves can travel through vacuum. The light emitted by stars, which are hundreds of light years away, reaches us through inter-stellar space, which is practically a vacuum.

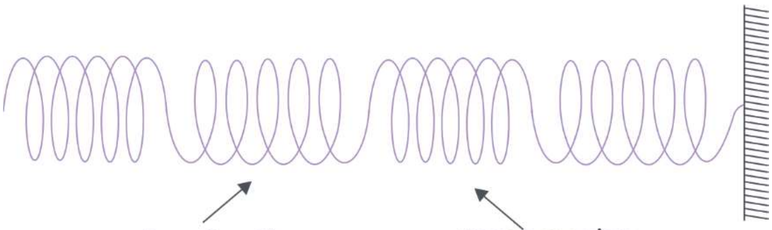
On The Basis of Vibration of Medium Particle:

VOP = Vibration of particles

DOP = Direction of propagation of wave

1. **Longitudinal wave :**
   Waves in which vibration of medium particles are along the direction of propagation of wave are known as longitudinal wave.
   eg. Sound wave, wave in spring etc.





- These wave propagate in form of compression and rarefaction.
- In compression region density of medium particles is maximum and in rarefaction region density of medium particles is minimum.
- These are also known as pressure wave (P-wave because it can only be produced by applying pressure).
- Pressure can be applied on all 3 mediums (Solid, Liquid and Gas). So, it can propagate in all 3 mediums.

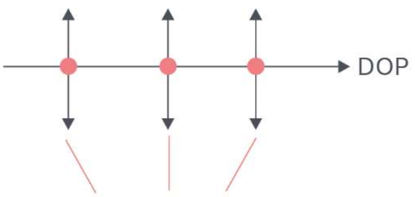
Definitions

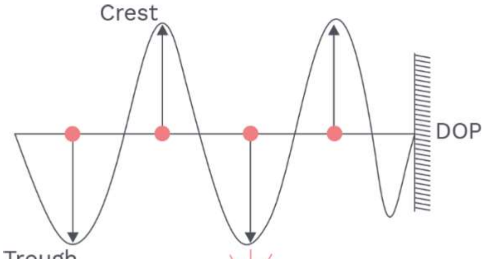
Waves in which vibration of medium particles are along the direction of propagation of wave are known as longitudinal wave

2. Transverse "Mechanical" Wave :

Waves in which vibrations of medium particle is perpendicular to the direction of propagation of wave are known as transverse waves.

eg., Wave on stretched wire (transverse mechanical), light wave (transverse non-mechanical)





Concept Reminder

Waves in which vibrations of medium particle is perpendicular to the direction of propagation of wave are known as transverse waves.

Concept Reminder

A third kind of wave is the so-called Matter waves. They are associated with constituents of matter: electrons, protons, neutrons, atoms and molecules.

These waves can propagate in form of crest and trough.

During wave propagation, medium is subjected to shearing stress and shearing stress can be produces on rigid bodies. So ,these waves can be produced in solid and on liquid surface (due to surface tension).

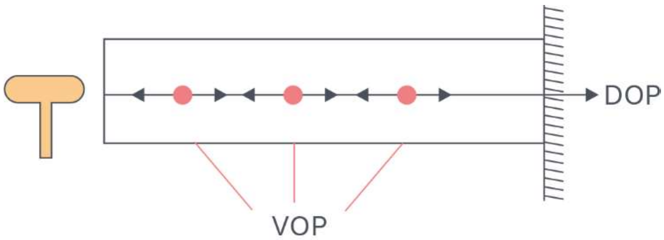
These waves are also known as shear wave (S-wave).

Important points :

- Every longitudinal wave is mechanical in nature but every mechanical wave are not only longitudinal but transverse also.
- All non-mechanical waves are transverse in nature but all transverse waves are not only non-mechanical but mechanical also.

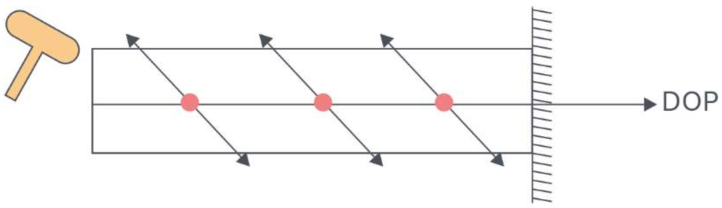
Some Mixed Example of P and S wave

In same solid both P and S wave can be produced. It can only depends on mode of excitation but speed of P-wave > S-wave.



P wave

(∵ VOP || DOP)



(P and S wave together)

Rack your Brain

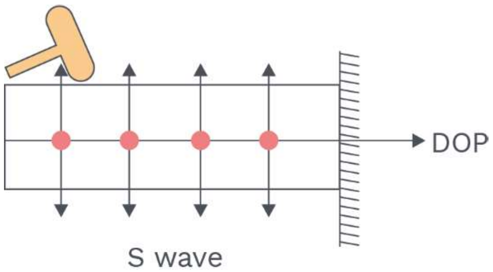
A transverse wave travels along x-axis. The particles of medium move

(1) Along x-axis

(2) Along y-axis

(3) Along z-axis

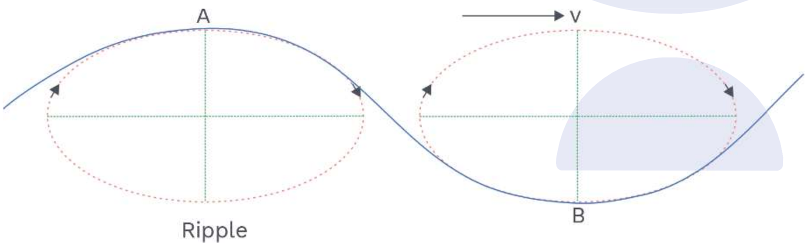
(4) Either along y-axis or z-axis



S wave

($\because$ VOP $\perp$ DOP)

Ripple Wave :
Some waves in the nature are neither transverse nor longitudinal but a combination of the both. These waves are called 'ripple' and waves on the surface of a liquid are of this type. In these waves particles of the medium vibrate up and down and back and forth simultaneously describing ellipses in a vertical plane.



Ripple

Seismic Wave :
Further more in case of the seismic waves produced by Earthquakes both 'S' (shear) and 'P' (pressure) waves are produced simultaneously which travels through the rock in the crust at different speeds
$[v_S \cong 5 km/s$ while $v_P \cong 9 km/s]$ S-waves are transverse while P-waves are longitudinal.

On The Basis of Energy Propagation

1. Progressive or travelling wave.
Waves in which energy propagate in unlimited and unbounded region with finite speed are known as progressive waves.

Definitions

Some waves in nature are neither transverse nor longitudinal but a combination of the two. These waves are called 'ripple'.

Definitions

Waves in which energy propagate in unlimited and unbounded region with finite speed are known as progressive waves eg. Sound wave, Light wave etc.

2. Non-progressive or standing or stationary wave

Waves in which energy exist in a limited bounded region are known as non-progressive wave.

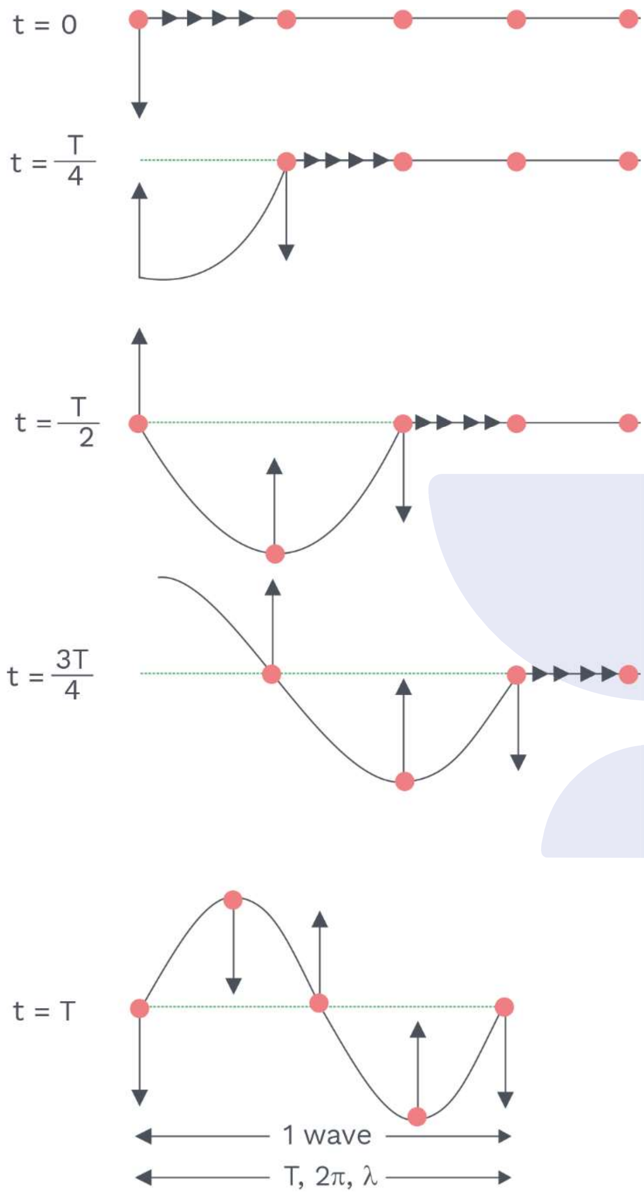
eg. (i) waves in organ pipes,

(ii) sonometer wire.

Definitions

Waves in which energy exist in a limited bounded region are known as non-progressive wave.

Concept of One Dimensional Simple Harmonic Wave

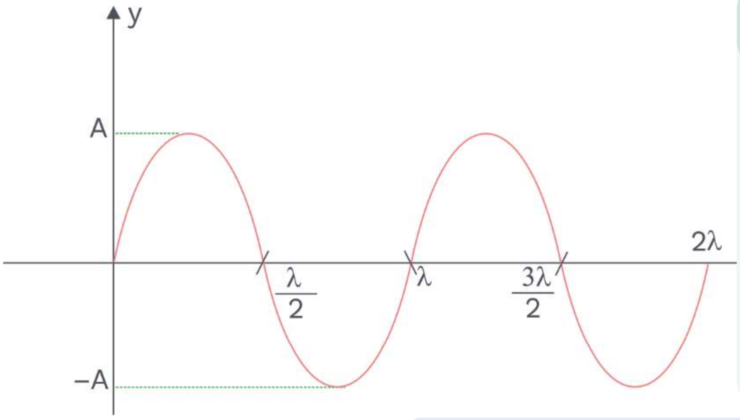


- All the medium particles execute SHM of same type, means all particles have same time period, frequency, angular frequency and amplitude but have different phase angle during oscillation.
- Every particle transfer energy to their neighboring particle but never leave their mean position.
- Direction of energy propagation is also known as direction of wave motion.

Concept Reminder

A harmonic (sinusoidal) wave travelling along a stretched string is an example of a transverse wave. An element of the string in the region of the wave oscillates about its equilibrium position perpendicular to the direction of wave propagation.

Fundamental Elements of Wave Motion:



1. **Amplitude (A) :** Maximum displacement of the vibrating particle from its equilibrium position.
2. **Time period (T) :** Time taken by the wave to travel a distance equal to one wavelength.
3. **Frequency (n/f/ν) :** Number of cycle (number of complete wavelengths) completed by a particle in unit time.
4. **Angular frequency (ω) :** It is defined as

$$\omega = \frac{2\pi}{T} = 2\pi n$$

5. **Wave length (λ):**

Length of 1 wave

Or

Minimum distance between the any two particles which are vibrating in same phase.

Or

Distance covered by wave in 1 time period

Or

Distance between two successive crest or trough in a transverse wave.

Or

KEY POINTS

- Amplitude
- Time period
- Frequency
- Angular frequency
- Wavelength
- Wave number
- Propagation constant

Rack your Brain

Which one of the following represents a wave?

(1) $y = A\sin(\omega t - kx)$

(2) $y = A\cos^2(at - bx + c) + A\sin^2(at - bx + c)$

(3) $y = A\sin kx$

(4) $y = A\sin\omega t$

Concept Reminder

Distance travelled by the wave per unit time is known as wave speed

$$\Rightarrow \quad v = \frac{\omega}{k}$$

Distance between two successive compression or rarefaction in a longitudinal wave is known as wave length.

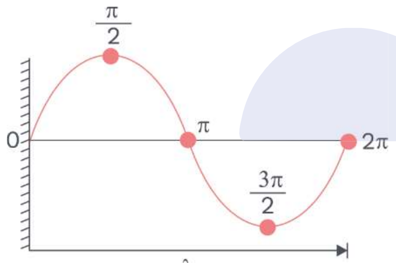
6. Wave number ($\bar{\nu}$)

Number of wave produced per unit distance is known as wave number.

$$\bar{\nu} = \frac{1}{\lambda} \quad (\text{Unit} = m^{-1})$$

7. Angular wave number or wave propagation constant (k) :

The ratio of phase difference and path difference of any two particles in the direction of wave propagation is known as angular wave number.



$$k = \frac{\Delta\theta}{\Delta x} = \frac{2\pi}{\lambda} = \frac{\omega}{v}$$ Unit = rad / m

8. Wave Speed (v)

Distance travelled by a wave per unit time is known as wave speed.

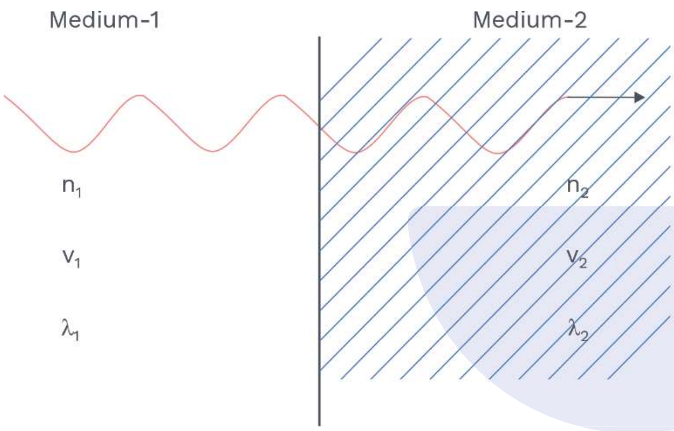
$$v = \frac{\lambda}{T} = n\lambda \text{ and } v = \frac{\omega}{k} \Rightarrow \boxed{v = n\lambda = \frac{\omega}{k}}$$

(I) Medium given
So, v = constant
$\Rightarrow n\lambda$ = constant
$\Rightarrow n \propto \frac{1}{\lambda}$
eg.

(II) Source given
So, n = constant
$\Rightarrow v \propto \lambda$

Concept Reminder

Mechanical waves are related to the elastic properties of the medium. In transverse waves, the constituents of the medium oscillate perpendicular to wave motion causing change its shape. That is, each element of the medium in subject to shearing stress.



If a wave of particular frequency propagate from one medium to another medium then frequency remains unchanged. Velocity of wave is medium dependent quantity, it depends on properties of medium like density, elasticity, inertia etc.
Frequency of wave is a property of source, means it is decided by source.

Concept Reminder

Solids and strings have shear modulus, that is they sustain shearing stress. Fluids have no shape of their own - they yield to shearing stress. This is why transverse waves are possible in solids and strings (under tension) but not in fluids.

Mathematical Description of Waves

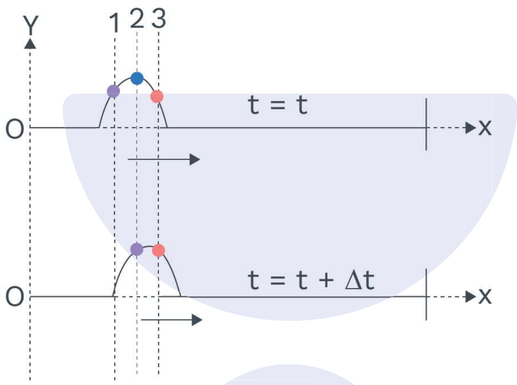
We shall attempt here to evolve mathematical model of the travelling transverse wave. For this, we select a specific set up of the string and associated transverse wave travelling from it. The string is tied to a fixed end, while the disturbance is imparted at the free end of the string by up and down motion. For our purpose, we consider that pulse which is small in dimension; the string is homogeneous, light and elastic. The assumptions are needed as we visualize a small travelling pulse which remains undiminished when it moves

Rack your Brain

In a stretched string,
(1) Only transverse waves can exist
(2) Only longitudinal waves can exist
(3) Both transverse and longitudinal waves can exist
(4) None of these

through the strings. We also consider that the length of the string is long enough so that our observation is not subjected to the pulse reflected at the fixed end.

For understanding purpose, we first take a single pulse as shown blow (irrespective of whether we can see such pulse in practice or not). Our objective here is to find the nature of a mathematical description which will enable us to determine displacement (disturbance) of the string as pulse passes through it. We concieve two snap shots of the travelling pulse at the two close time instants "t" and "t + Δt ". The single pulse is moving towards the right in the positive x-direction.



The vibration and the wave motion are at 90° to each other.

Three position along the x-axis named "1", "2" and "3" are denoted with three vertical dotted lines. At any of two instants as shown, the positions of the string particles have different displacements from the undisturbed position on the horizontal x-axis. We can conclude from the above observation that the displacement in y-direction is a function of the positions of particle in x-axis direction. As such, the displacement of a particle building the string is a function of "x".

Let us now observe the positions of the given particle, say "1". It has fixed positive displacement at time t = t, At next constant at t = t + Δt , the displacement has been reduced to zero. The particle at "2" position has maximum displacement at t = t, but the same has reduced at t = t + Δt . The third particle at '3' position has certain positive displacement at t = t, At t = t + Δt , it acqures extra positive displacement and reaches the position of maximum displacement. From these observations, we can conclude that displacement of a particle at any position along the string is a function of "t".

Combining two observations, we conclude that displacement of a particle is a function of both (i) position of the particle along the string and time.

Concept Reminder

In order for the function to represent a wave travelling at speed v, the quantities x, v and t must appear in the combination (x + vt) or (x – vt).

$$y = f(x, t)$$

Further we can specify the nature of the mathematical function by the association of the speed of the wave in our assumption. Let "v" be the uniform speed with which the wave travels from the left end to the right end. We notice that the wave function for a given position of the string is a function of time only as we are taking displacement at a particular value of "x". Let us assume left hand end of the string as origin of reference (x = 0 and t = 0). The displacement in the y-axis direction (disturbance) at x = 0 is a function of time, "t" only :

$$y = f(t) = A \sin \omega t$$

The disturbance travels to right at constant speed 'v'. Let it reaches a point given as x = x after time "t". If we see to describe the origin of this disturbance at x = 0, then time elapsed for the disturbance to move from the origin (x = 0) to point (x = x) is "x/v". Therefore, if we want to apply the function of displacement at x = 0 as given above, then we require to subtract the time elapsed and set the equation is :

$$y = f\left(t - \frac{x}{v}\right) = A \sin \omega \left(t - \frac{x}{v}\right)$$

This can also be expressed as

$$\Rightarrow \; f\left(\frac{vt - x}{v}\right) \qquad \Rightarrow \; -f\left(\frac{x - vt}{v}\right)$$

$$y(x, t) = g(x - vt)$$

using any certain value of 't' (i.e. at any instant), this shows shape of the string. If the wave is travelling in –x direction, the wave equation is written as

$$y(x, t) = f(t + \frac{x}{v})$$

The quantity '(x – vt)' is known as the phase of the wave function. As phase of the pulse has constant value

$$x - vt = \text{const.}$$

Taking the derivative w.r.t. time $\frac{dx}{dt} = v$

where 'v' is the phase velocity although often known as wave velocity. This is the velocity at which a particular phase of the disturbance travels through the space. In order to the function to represent a wave travelling at a speed 'v', the quantities 'x', 'v' and 't' must appear in the combination is (x + vt) or (x – vt). So $(x - vt)^2$ is acceptable but $x^2 - v^2 t^2$ is not.

Concept Reminder

The wavelength λ of a wave is generally taken as the distance between two successive crests or two successive trough. To be more specific, it is the distance between two consecutive points on the wave which have same phase

Describing Waves :

Two kinds of graph can be drawn the displacement - distance and displacement-time. A displacement-distance graph for the transverse mechanical waves shows the displacement 'y' of the vibrating particles of the transmitting medium at the different distance 'x' from the source at a certain instant i.e. it is as a photograph showing shape of the wave at that certain instant.

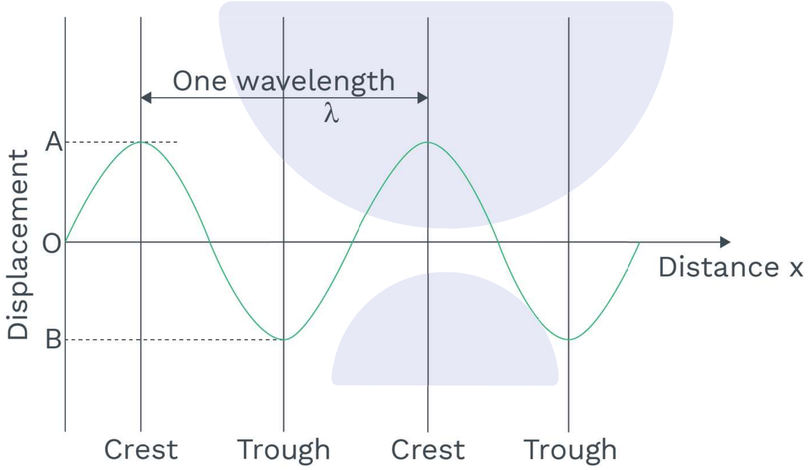
The amplitude of the wave a maximum is displacement of each particle from its undisturbed position. In the figure, it is OA or OB.

Rack your Brain

The equation of a simple harmonic wave is given by $y = 2\sin\frac{\pi}{2}(50t - x)$, where x and y are in metres and t is in seconds. The ratio of maximum particle velocity to the wave velocity is

(1) 2π (2) π

(3) 3π (4) 4π



The wavelength λ of a wave is generally taken as the distance between two successive crests or two successive trough. To be more specific, it is the distance between two consecutive points on the wave which have same phase.

A displacement-time graph may also be drawn for a wave motion, showing how the displacement of one particle at a particular distance from the source varies with time. If this is simple harmonic variation then the graph is a sine curve.

Wave Length, Frequency, Speed

If the source of a wave makes f vibrations per second, so they will the particles of the transmitting medium. That is, the frequency of the waves equals frequency of the source. When the source makes one complete vibration, one wave is generated and the disturbance spreads out a distance λ from the source. If the source continues to vibrate with constant frequency f, then f waves will be produced per second and the wave advances a distance f λ in one second. If v is the wave speed then

$v = f\lambda$

This relationship holds for all wave motions.

- Frequency depends on source (not on medium), v depends on medium (not on source frequency), but wavelength depends on both medium and source.

Initial Phase :

At x = 0 and t = 0, the sine function evaluates to zero and as such y-displacement is zero. However, a wave form can be such that y-displacement is not zero at x =0 and t = 0. In such case, we need to account for the displacement by introducting an angle like :

$$y(x, t) = A\sin(kx - \omega t + \phi)$$

where "ϕ" is initial phase. At x = 0 and t = 0.

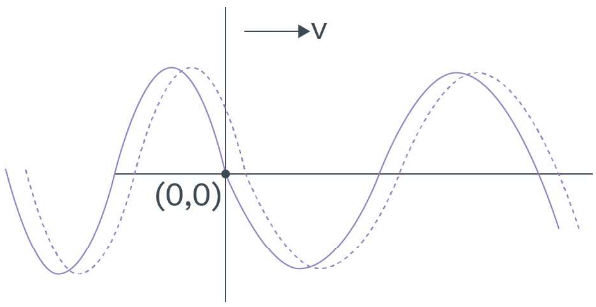
$$y(0, 0) = A\sin(\phi)$$

The measurement of angle determines following two aspects of wave form at x = 0, t = 0 : (i) whether the displacement is positive or negative and (ii) whether wave form has positive or negative slope. For a harmonic wave represented by sine function, there are two values of initial phase angle for which displacement at reference origin (x = 0, t = 0) is positive and has equal magnitude. We know that the sine values of angles in first and second quadrants are positive. A pair of initial phase angles, say $\phi = \pi/3$ and $2\pi/3$, correspond to equal positive sine values are :

$$\sin\theta = \sin(\pi - \theta)$$

$$\sin\frac{\pi}{3} = \sin\left(\pi - \frac{\pi}{3}\right) = \sin\left(\frac{2\pi}{3}\right) = \frac{1}{2}$$

To choose the initial phase in between the two values $\frac{\pi}{3}$ & $\frac{2\pi}{3}$. We can look at a wave motion in yet another way. A wave form at an instant is displaced by a distance Δt in very small time interval Δt then speed of the particle at t = 0 & x = 0 is in upward +ve direction in further time Δt



Concept Reminder

- Equation of wave moving in +ve x-direction is $y = A\sin\omega\left(t - \frac{x}{v}\right)$
- Equation of wave moving in –ve x-direction is $y = A\sin\omega\left(t + \frac{x}{v}\right)$

Ex. Find the expression of the wave equation which in moving is +ve x direction and at $x = 0,\ t = 0, y = \frac{A}{\sqrt{2}}$

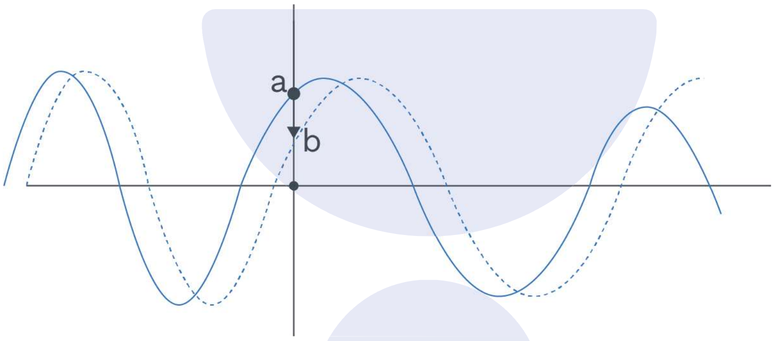
Sol. Let $y = A\sin(\omega t - kx + \phi)$

at $t = 0$ and $x = 0$

$$\frac{A}{\sqrt{2}} = A\sin\phi \Rightarrow \sin\phi = \frac{1}{\sqrt{2}}$$

$$\phi = \frac{\pi}{4}, \frac{3\pi}{4}$$

To choose correct phase angle ϕ we displaced the wave. Slightly in +ve x direction.



In above figure, particle at a move downward towards point b i.e. particle at x = 0 & $y = \frac{A}{\sqrt{2}}$ have negative velocity which gives

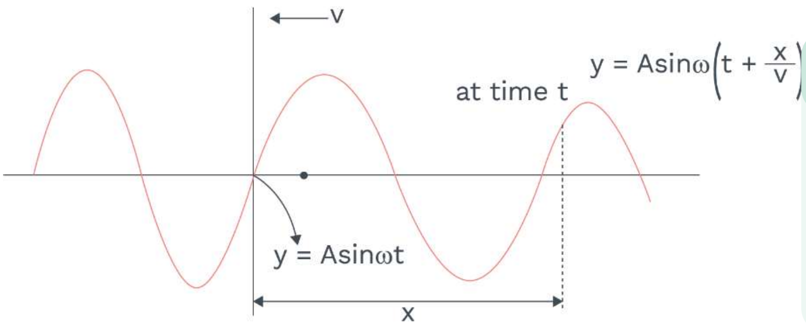
$$\frac{\partial y}{\partial t} = \omega A\cos(\omega t - kx + \phi)$$

at $t = 0, x = 0$

is $\cos\phi = -ve$ (from figure) ... (2)

from above discussion $3\pi/4$ gives $\sin\phi + ve$ and $\cos\phi$ negative i.e. $\phi = \frac{3\pi}{4}$

Note : Equation of wave which is moving in –ve x direction.



KEY POINTS

- Phase
- Phase difference
- Path difference
- Time difference

- If (ωt) & (kx) terms have same sign then the wave move toward –ve x direction and vice versa and with different initial phase.

$$\left.\begin{array}{l} y = A\sin(\omega t - kx) \\ y = A\sin(-kx + \omega t) \end{array}\right\} \text{Wave move toward} + \text{ve x direction}$$

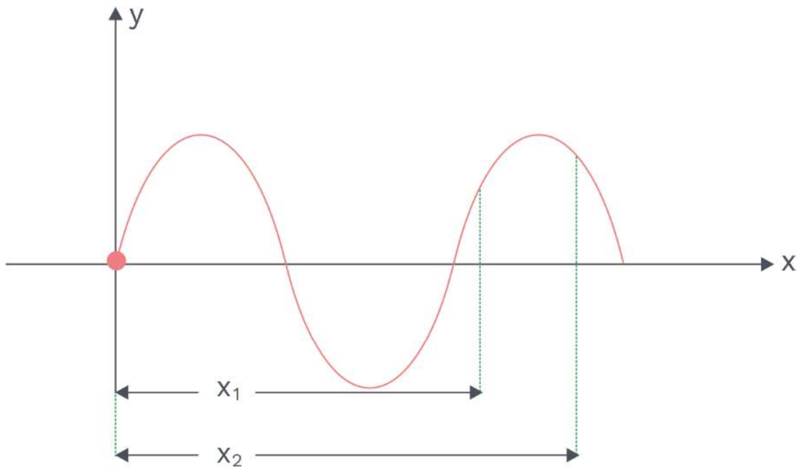
Concept Reminder

Relation between phase difference, path difference and time difference

$$\boxed{\frac{\Delta\phi}{2\pi} = \frac{\Delta x}{\lambda} = \frac{\Delta t}{T}}$$

$$\left.\begin{array}{l} y = A\sin(-kx - \omega t) \\ = A\sin(kx + \omega t + \pi) \\ y = A\sin(kx + \omega t) \end{array}\right\} \text{Wave move toward} - \text{ve x direction}$$

- **Relation between the Phase difference, the Path difference & the Time difference**



$$y_1 = A\sin[\omega t - kx_1 + \phi]$$

$$y_2 = A\sin[\omega t - kx_2 + \phi]$$

$$\Delta\phi = k(x_2 + x_1)$$

$$\boxed{\Delta\phi = \frac{2\pi}{\lambda}\Delta x} \quad \ldots (1)$$

$$t_1 = \frac{x_1}{v}, \qquad t_2 = \frac{x_2}{v}$$

$$\Delta t = t_2 - t_1$$

$$\Delta t = \frac{\Delta x}{v}$$

$$\boxed{\Delta t = \frac{\Delta x}{\lambda}T} \quad \ldots (2)$$

From equation 1 & 2, we can tell relation between phase difference, path difference and time difference which is

$$\boxed{\frac{\Delta\phi}{2\pi} = \frac{\Delta x}{\lambda} = \frac{\Delta t}{T}}$$

Ex. Find phase difference between $y_1 = A\sin\left(\omega t - kx + \frac{\pi}{3}\right)$ and $y_2 = A\cos\left(\omega t - kx\right)$

Sol. $y_1 = A\sin\left(\omega t - kx + \frac{\pi}{3}\right)$ and $y_2 = A\sin\left(\omega t - kx + \frac{\pi}{2}\right)$

$$\phi_2 - \phi_1 = \frac{\pi}{2} - \frac{\pi}{3} = \frac{3\pi - 2\pi}{6} = \frac{\pi}{6}\text{rad.}$$

Ex. Find out phase difference between given waves $y_1 = A\sin(\omega t - kx + 0.57)$ and $y_2 = A\cos(\omega t - kx)$.

Sol. $y_2 = A\sin\left(\omega t - kx + \frac{\pi}{2}\right)$

$$\Delta\phi = \phi_2 - \phi_1 = 1.57 - 0.57 = 1 \text{ rad}$$

Ex. In transverse wave distance of 5th and 9th crest is 35 cm and 63 cm from centre of oscillation of this wave takes 2 s to go from 5th to 9th crest. Find out wave length, velocity, frequency of wave.

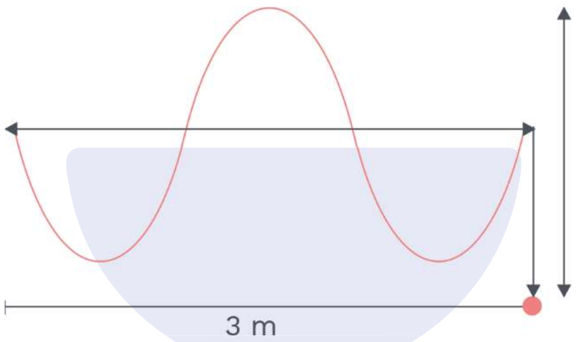
Sol. (i) $9\,\text{crest} - 5\,\text{crest} = 4\lambda$ (ii) $v = \frac{D}{t} = \frac{28}{2}$ (iii) $v = n\lambda$

$\Rightarrow 4\lambda = 28\,cm$ $\quad\Rightarrow v = 14\,cm/s$ $\quad\Rightarrow 14 = n \times 7$

$\Rightarrow \lambda = 7\,cm$ $\quad\Rightarrow n = 2\,Hz$

Ex. A man generate a wave in a string by moving his hand up and down. At t = 0 point in his hand moves downward. The wave travels with speed of 3 m/s on string and his hand passes 6 time in each second from mean position. Find out time taken by particle which is situated at 3 m from his hand to reach at upper extreme.

Sol.



$n = 3\,Hz \Rightarrow T = \frac{1}{3}\,s \Rightarrow v = \frac{\lambda}{T} \Rightarrow \lambda = vT$

$\Rightarrow \lambda = 3 \times \frac{1}{3} = 1\,m$

$t_1 = 3T \times 3 \times \frac{1}{3} = 1\,s$

$t_2 = \frac{3T}{4} = \frac{3}{4} \times \frac{1}{3} = \frac{1}{4}\,s$

$\text{Time} = t_1 + t_2 = 1 + \frac{1}{4} = 1.25\,s$

Ex. In a stretched wire speed of wave 20 m/s and n = 50 Hz. Find phase difference between particles separated by 10 cm from each other.

Sol. $v = n\lambda$

$\lambda = \frac{2}{5}\,m$ $\qquad \because \frac{\Delta\phi}{2\pi} = \frac{\Delta x}{\lambda}$

$\Rightarrow \frac{\Delta\phi}{2\pi} = \frac{10 \times 5}{100 \times 2}$ $\qquad \Rightarrow \Delta\phi = \frac{\pi}{2}$

Ex. For a sound wave, path difference of 40 cm is ec

If speed = 330 m/s. Find n.

Concept Reminder

Relation between wave speed and maximum velocity of particle in a medium is

$$\frac{v}{(v_P)_{max}} = \frac{k}{A}$$

Sol. $\frac{\Delta\phi}{2\pi} = \frac{\Delta x}{\lambda}$

$$\Rightarrow \frac{1.6}{2} = \frac{40}{\lambda \times 100} \Rightarrow \lambda = \frac{40 \times 2 \times 10}{100 \times 16} \Rightarrow \lambda = \frac{1}{2}$$

$$\because \quad v = n\lambda \quad \Rightarrow n = 330 \times 2 = 660\,\text{Hz}$$

Ex. If equation of plane progressive wave is given by $y = 10\sin\left[200\pi t - \frac{\pi x}{9}\right]$. Find out

(a) DOP (b) Nature
(c) Amplitude (d)
Angular frequency
(e) Time period (f) Wave
speed
(g) Wave length (h)
Maximum velocity

Sol. $y = 10\sin\left[200\pi t - \frac{\pi x}{9}\right]$

(a) (+x) direction
(b) Transverse
(c) 10 cm
(d) $\omega = 200\,\pi$

(e) $T = \frac{2\pi}{\omega} = \frac{2\pi}{200\pi} = \frac{1}{100} = 0.01\text{s}$

$\therefore n = 100\,\text{Hz}$

(f) $v = \frac{\omega}{k} = \frac{200\pi}{\pi / 9} = 1800\,\text{cm / s}$

(g) $k = \frac{\pi}{9} \Rightarrow \frac{2\pi}{\lambda} = \frac{\pi}{9} \Rightarrow \lambda = 18\,\text{cm}$

(h) $v_{max} = A\omega = 10 \times 200\,\pi = 2\pi \times 10^3\ \text{cm / s}$

Velocity and Acceleration of a Particle in a Wave

- **Velocity of in a medium particle (v_p)**

As we know that $y = A\sin(\omega t \pm kx)$

$$\Rightarrow \frac{\delta y}{\delta t} = A[\cos(\omega t \pm kx)(\omega \times 1 \pm 0)]$$

$$\Rightarrow v_p = A\omega t\cos(\omega t \pm kx)$$

Concept Reminder

velocity of particle at a given point = (–) wave velocity × slope of wave at that point. (Slope of wave is also called as wave strain)

- **Acceleration of particle in a medium (a_p)**

Ex. If equation of plane progressive wave $y = y_0 \sin(wt-kx)$ and here wave velocity = maximum particle velocity. Find out .

Sol.

Relation Between wave Velocity and Particle Velocity

Let ... (1)

and .... (2)

From equation 1 & 2

Concept Reminder

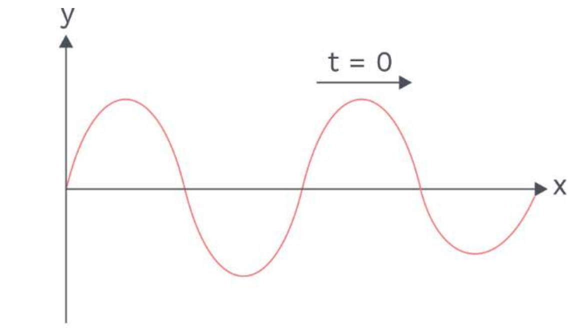
If represent travelling wave function then-

velocity of particle at a given point = (–) wave velocity × slope of wave at that point. (Slope of wave is also called as wave strain)

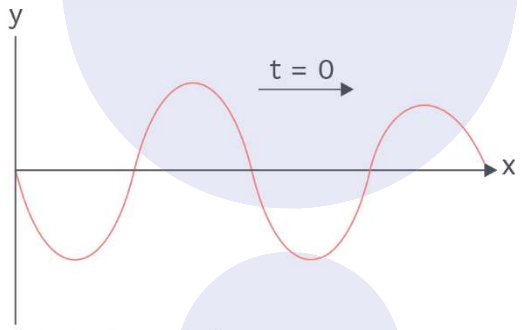
From equation 1 & 2

This is the differential equation of harmonic progressive waves.

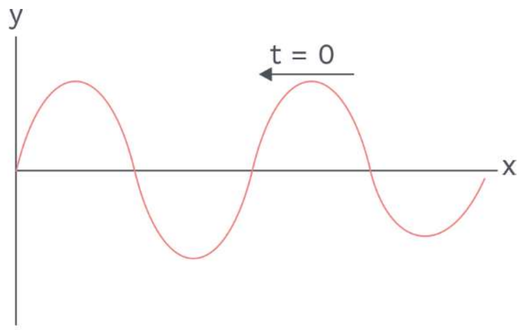
Note :



y = A sin [kx – ωt]



y = A sin [ωt – kx]



y = A sin [ωt + kx]

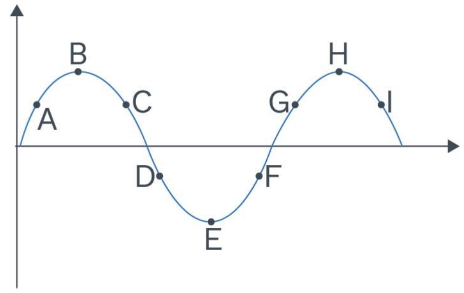
Note :-

If $y = f(ax \pm bt)$ represent travelling wave function then-

(i) y = satisfy $\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$

(ii) For any value of x & t y should be finite.

Ex. If given wave is moving along (+x) direction. Find out which of the particle have



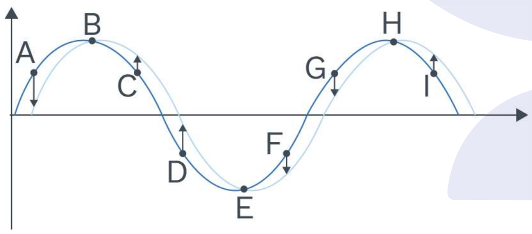
(a) upward velocity
(b) downward velocity
(c) zero velocity

Sol. $v_A = -v(+) = -$ $\quad$ $v_B = v(0) = 0$ $\quad$ $v_C = -v(-) = +$

$v_D = -v(-) = +$ $\quad$ $v_E = -v(0) = 0$ $\quad$ $v_F = -v(+) = -$

$v_G = -v(+) = -$ $\quad$ $v_H = -v(+) = -$ $\quad$ $v_I = -v(-) = +$

Short trick



Ex. If a wave of n = 500 Hz move from point x to y and travel a distance 600 m in 2 s. Find number of waves between x and y.

Sol. n = 500 Hz, d = 600 m, t = 2 s

$$v = \frac{d}{t} = \frac{600}{2} = 300\,m/s$$

$\because \quad v = n\lambda$

$\therefore \quad 300 = 500 \times \lambda$

$\therefore \quad \lambda = \frac{3}{5} m$

Number of waves between x and y

$$= \frac{600}{\left(\frac{3}{5}\right)} = 1000$$

Definitions

The maximum amount of energy passes from unit area in unit time in direction of wave propagation is known as intensity.

Concept Reminder

- $I = \frac{E_{max}}{A \times t}\left(\frac{J}{m^2/s}\right)$
- $I = \frac{P}{A}\left(\frac{Watt}{m^2}\right)$

Alternate method

no. of waves in 1 seconds = 500

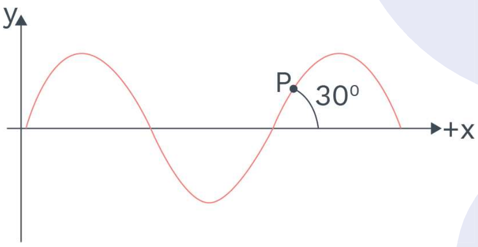
no. of waves in 2 seconds = 500 × 2 = 1000

Ex. Equation of wave is given by $y = 8\sin(0.5\,\pi x - 4t + \frac{\pi}{4})$. Find out velocity of wave

Sol. $y = 8\sin(0.5\,\pi x - 4\pi t + \frac{\pi}{4})$

So, $v = \frac{\omega}{k} = \frac{4\pi}{0.5\pi} = 8\,m/s$

Ex. If the given wave moves along +x direction with speed of 330 m/s. Find out velocity of particle p.



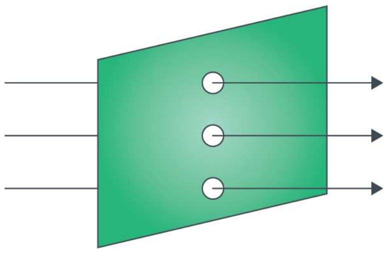
Sol. v = 330 m/s

$\tan 30° = \frac{1}{\sqrt{3}}$

$v_p = -330 \times \frac{1}{\sqrt{3}} = -110\sqrt{3}\,m/s$ (Downward)

Intensity :

The maximum amount of energy passes from unit area in unit time in direction of wave propagation is known as intensity.



$I = \frac{E_{max}}{A \times t}\left(\frac{J}{m^2/s}\right)$

Concept Reminder

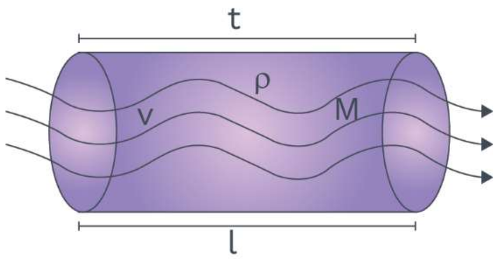
Intensity of mechanical wave is

$I = 2\pi^2 a^2 n^2 \rho v$

$$I = \frac{P}{A}\left(\frac{\text{Watt}}{m^2}\right)$$

Intensity of mechanical wave

$$M = \text{vol} \times \rho = A \times l \times \rho = A \times v \times t \times \rho$$



$$I = \frac{E_{max}}{A \times t} = \frac{\frac{1}{2}Mv_{p_{max}}^2}{A \times t} \Rightarrow I = \frac{1}{2} \times \frac{A \times v \times T \times \rho \times v_{p_{max}}^2}{A \times t}$$

$\because \quad v_{p_{max}} = a\omega = a(2\pi n) \;\Rightarrow\; I = \frac{1}{2} \times v \times \rho \times (2\pi n a)^2$

$\Rightarrow \quad I = 2\pi^2 a^2 n^2 \rho v$

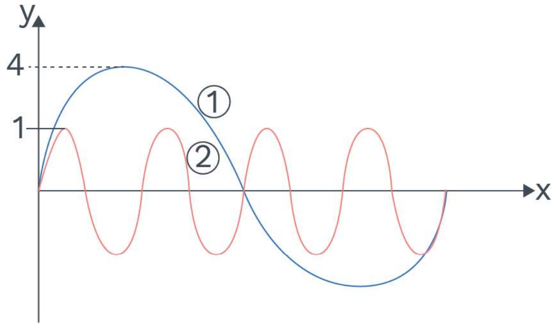
Note :

(i) If medium is same then v and ρ are constant so $I \propto a^2 n^2$

(ii) If medium and source are same then v, ρ,

n = constant so $I \propto a^2$

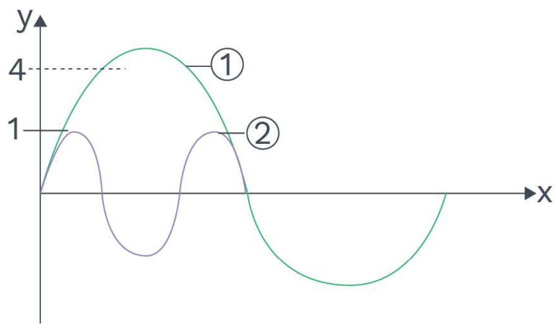
Ex. Find $\frac{I_1}{I_2}$ for waves shown in figure



Sol. $I \propto a^2 n^2$

$$\frac{I_1}{I_2} = \left(\frac{a_1}{a_2} \times \frac{n_1}{n_2}\right)^2 = \frac{1}{1}$$

Ex. Find $\frac{I_1}{I_2}$ for waves shown in figure



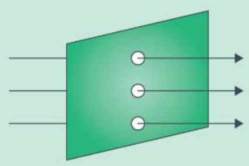
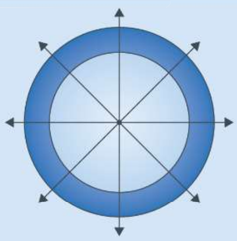
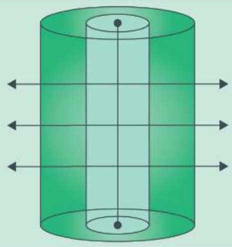
Concept Reminder

The speed of a mechanical wave is determined by the inertial (linear mass density for strings, mass density in general) and elastic properties (Young's modulus for linear media/ shear modulus, bulk modulus) of the medium.

Sol. $I \propto a^2n^2$

$$\frac{I_1}{I_2} = \left(\frac{a_1}{a_2} \times \frac{n_1}{n_2}\right)^2 = \left(\frac{4}{1} \times \frac{1}{2} \times \frac{2}{3}\right)^2 = \left(\frac{4}{3}\right)^2 = \frac{16}{9}$$

Wave front → An imaginary surface on which waves incident perpendicular & in same phase.

Wave front	Plane	Spherical	Cylindrical
			
Source	Source at infinite distance (Sun, Torch, Loud Speaker)	Point source (Bulb, small siren)	Linear source (Tubelight)
Area of wavefront	$A = l \times b$ A = const.	$A = 4\pi r^2$ $A \propto r^2$	$A = 2\pi rl$ $A \propto r$
$I \propto \frac{1}{\text{Area}} \propto a^2$	I = constant	$I \propto \frac{1}{r^2} \propto a^2$	$I \propto \frac{1}{r} \propto a^2$
Variation during propagation	a = constant	$a \propto \frac{1}{r}$	$a \propto \frac{1}{\sqrt{r}}$
Equation	$y = a \sin(\omega t - kx)$ Plane progressive wave	$y = \frac{c}{r} \sin(\omega t - kx)$ Spherical progressive wave	$y = \frac{c}{\sqrt{r}} \sin(\omega t - kx)$ Cylindrical progressive wave

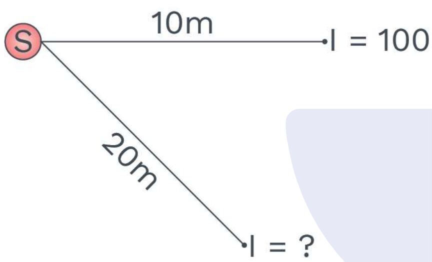
Ex. A source is placed at a distance of 10 m. The power of source is 100 watt. Determine intensity.

Sol.

P = 100 watt

$$I = \frac{P}{A} = \frac{100}{4\pi(10)^2} = \frac{1}{4\pi} \text{watt} / m^2$$

Ex.



Sol. $I \propto \frac{1}{A}$

$A \propto r^2$

$$\frac{I_1}{I_2} = \left(\frac{r_2}{r_1}\right)^2 \Rightarrow \frac{100}{I_2} = \left(\frac{20}{3}\right)^2$$

$$\Rightarrow I_2 = 25 \text{ watt} / m^2$$

Rack your Brain

A string is stretched between fixed points separated by 75.0 cm. It is observed to have resonant frequencies of 420 Hz and 315 Hz. There are no other resonant frequencies between these two. The lowest resonant frequency for this string is

(1) 105 Hz (2) 155 Hz

(3) 205 Hz (4) 10.5 Hz

The Linear Wave Equation

By using wave function $y = A\sin(\omega t - kx + \phi)$, we can describe the motion of any point on string. Any point on string moves only vertically, and so its 'x'-coordinate remains constant. The transverse velocity v_y of a point and its transverse acceleration a_y are therefore.

$$v_y = \left[\frac{dy}{dt}\right]_{x=\text{constant}} \quad \text{... (1)}$$

$$\Rightarrow \frac{\partial y}{\partial t} = \omega A\cos(\omega t - kx + \phi)$$

$$a_y = \left[\frac{dv_y}{dt}\right]_{x=\text{constant}} \quad \text{.... (2)}$$

$$\Rightarrow \frac{\partial v_y}{\partial t} = \frac{\partial^2 y}{\partial t^2} = -\omega^2 A\sin(\omega t - kx + \phi)$$

The transverse velocity and the transverse acceleration of any point on the string don't reach their maximum value simultaneously. In fact, the transverse velocity reaches its maximum value (ωA) when the displacement y = 0, whereas the transverse acceleration reaches its maximum magnitude $(\omega^2 A)$ when $y = \pm A$

further

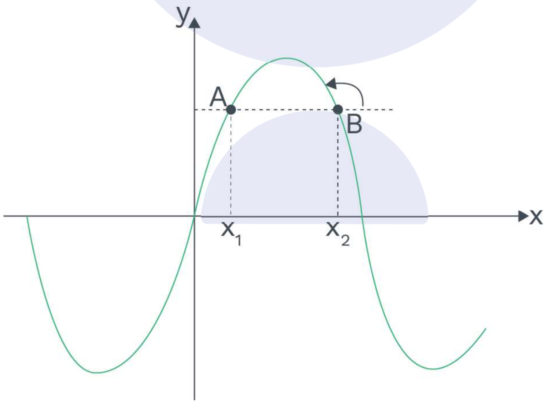
$$\left[\frac{dy}{dt}\right]_{x=\text{constant}} \Rightarrow \frac{\partial y}{\partial x} = -kA\cos(\omega t - kx + \phi) \quad .... (3)$$

$$= \frac{\partial^2 y}{\partial x^2} = -k^2 A \sin(\omega t - kx + \phi) \quad .... (4)$$

From (1) & (3) $\frac{\partial y}{\partial t} = -\frac{\omega}{k}\frac{\partial y}{\partial x}$

$\Rightarrow v_p = -v_w \times \text{slope}$

i.e. if the slope at the any point is negative, the particle velocity for a wave moving along the positive 'x'-axis i.e. v_w is positive and vice versa.



For example, assume two points 'A' and 'B' on the y-curve for a wave, as shown above. The wave is moving along the positive x-axis.

Slope at 'A' is positive therefore at the given moment, its velocity will be negative. That means it is coming downward. Reverse is the situation for the particle at point B.

Now using equation (2) & (4)

$$\frac{\partial^2 y}{\partial x^2} = \frac{k^2}{\omega^2}\frac{\partial^2 y}{\partial t^2} \Rightarrow \frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

This is called the linear wave equation or the differential equation representation of travelling wave model. We have created the linear wave equation from the sinusoidal mechanical wave travelling across a medium. But it is much more general. The linear wave equation successfully defines waves on strings, sound waves and also electromagnetic waves.

Thus, the above equation can be given as,

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2} \qquad ...(a)$$

The general solution of this equation is of the form

$$y(x, t) = f(ax \pm bt) \qquad ... (b)$$

Thus, any function of 'x' and 't' which satisfies the Eq. (a) or which can be written as the Eq. (b) represents a wave. The only condition is that it must be finite everywhere and at all the times. Further, if these conditions are fulfilled, then speed of wave (v) is given by, $y(x, t) = f(ax \pm bt)$

Thus plus (+) sign between 'ax' and 'bt' shows that the wave is travelling along negative x-direction and minus (–) sign shows that it is travelling along positive x-direction.

Ex. Verify that the wave function

$$y = \frac{2}{(x-3t)^2 + 1}$$

is a solution to linear wave equation 'x' and 'y' are in cm.

Sol. By taking the partial derivatives of this function w.r.t. x and to t.

$$\frac{\partial^2 y}{\partial x^2} = \frac{12(x-3t)^2 - 4}{[(x-3t)^2 + 1]^3}, \text{ and}$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{108(x-3t)^2 - 36}{[(x-3t)^2 + 1]^3}$$

or $$\frac{\partial^2 y}{\partial t^2} = \frac{1}{9}\frac{\partial^2 x}{\partial t^2}$$

Comparing with the linear wave equation, we see that the wave function is solution to the linear wave equation if speed at which pulse moves is 3 cm/s. It is apparent from the wave function therefore it is a solution to the linear wave equation.

Ex. A wave pulse is travelling on the string at 2 m/s and displacement y is given as

$$y = \frac{2}{t^2 + 1}$$

Find

(i) Equation of a function y = (x, t) i.e., displacement of a particle position 'x' and time 't'.

(ii) Shape of the pulse at time t = 0 and t = 1 s.

Sol. **(i)** By replacing t by $\left(t - \frac{x}{v}\right)$, we can get the desired wave function i.e.,

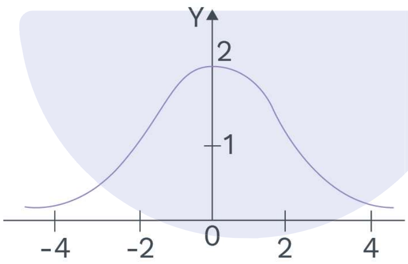
$$y = \frac{2}{\left(t - \frac{x}{2}\right)^2 + 1}$$

(ii) We can use the wave function at a particular instant, say t = 0, to find the shape of the wave pulse using the different values of x.

at t = 0 $y = \dfrac{2}{\dfrac{x^2}{4} + 1}$

at	x = 0	y = 2
	x = 2	y = 1
	x = −2	y = 1
	x = 4	y = 0.4
	x = −4	y = 0.4

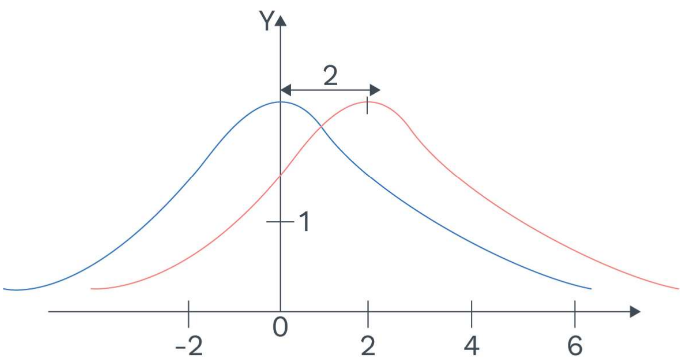
Using these value, shape can be drawn.



Similarly for the time t = 1s, shape can be drawn.
Here is the procedure.

$y = \dfrac{2}{\left(1 - \dfrac{x}{2}\right)^2 + 1}$ at t = 1 s

at	x = 2	y = 2 (maximum value)
at	x = 0	y = 1
at	x = 4	y = 1



The pulse has moved to right by 2 units 1 s interval.

Also as $t - \frac{x}{2} = \text{constt.}$

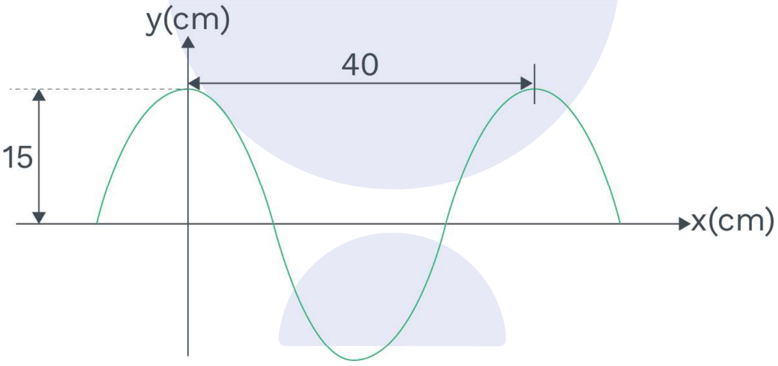
Differentiating w.r.t. time

$$1 - \frac{1}{2} \cdot \frac{dx}{dt} = 0 \Rightarrow \frac{dx}{dt} = 2$$

Concept Reminder

For waves on a string, the restoring force is provided by the tension T in the string. The inertial property will in this case be linear mass density μ, which is mass m of the string divided by its length L.

Ex. A sinusoidal wave travelling in the positive x-axis direction has an amplitude of 15 cm, wavelength of 40 cm and frequency of 8 Hz. The vertical displacement of medium at t = 0 and x = 0 is also 15 cm, as shown



(a) Calculate the angular wave number, period angular frequency and speed of the wave.

(b) Find the phase constant ϕ, and write a general equation for the wave function.

Sol. **(a)** $k = \frac{2\pi}{\lambda} = \frac{2\pi \text{rad}}{40 \text{ cm}} = \frac{\pi}{20} \text{rad/cm}$

$T = \frac{1}{f} = \frac{1}{8} s, \quad \omega = 2\pi f = 16 s^{-1}$

$v = f\lambda = 320 \text{ cm/s}$

(b) It is given that A = 15 cm

and also y = 15 cm at x = 0 and t = 0

then using $y = A\sin(\omega t - kx + \phi)$

$15 = 15 \sin\phi \Rightarrow \sin\phi = 1$

Therefore, the wave function is

$$y = A\sin(\omega t - kx + \frac{\pi}{2})$$

$$= (15)\sin\left[(16\pi)t - \left(\frac{\pi}{20}\right)x + \frac{\pi}{2}\right]$$

Concept Reminder

Speed of wave on stretched string is $v = \sqrt{\frac{T}{\mu}}$

Speed of A Transverse Wave On the String

Consider a pulse travelling along the string with a speed 'v' to the right. If the amplitude of the pulse is small compared to length of the string, the tension 'T' will be approximately constant along the string. In the reference frame moving with the speed 'v' to the right, the pulse in stationary and the string moves with the speed 'v' to the left. Diagram shows a small segment of the string of length Δl. This segment makes part of a circular arc of radius 'R'. Instantaneously the segment is moving with a speed of 'v' in a circular path, so it has centripetal acceleration v^2/R. The forces acting on segment are tension 'T' at each end. The horizontal component of these forces are same and opposite and show cancel. The vertical component of these forces point radically inward towards the centre of the circular arc. These radial forces provide centripetal acceleration. Let the angle subtended by the segment at centre be 2θ. The net radial force acting on segment is

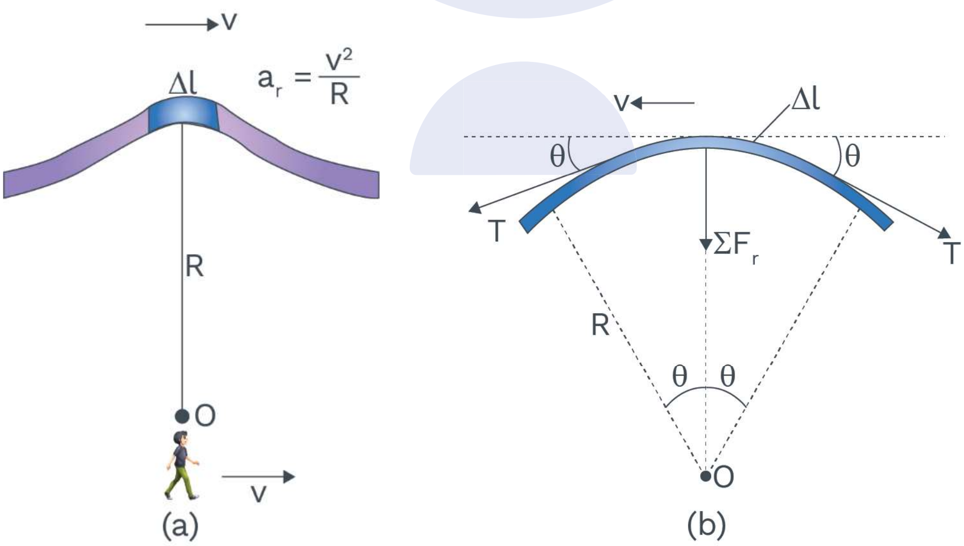


Fig. (a) To obtain the speed 'v' of the wave on a stretched string. It is convenient to characterize the motion of small segment of the string in a moving frame of reference.

Fig. (b) In the moving reference frame, the small segment of length Δl moves to the left with the speed of 'v'. The net force on segment is in radial direction because the horizontal components of tension force cancel.

$$\sum F_r = 2T\sin\theta = 2T\theta$$

Where we have used approximation $\sin\theta \approx \theta$ for small θ.

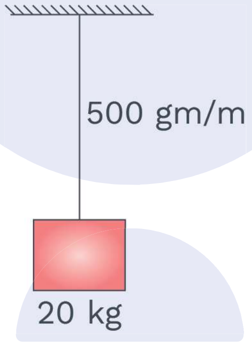
$m = \mu \Delta l = 2\mu R\theta \qquad (\text{as } \Delta l = 2R\theta)$

From Newton's second law $\sum F_r = ma = \frac{mv^2}{R}$

or $\quad 2T\theta = (2\mu R\theta)\left(\frac{v^2}{R}\right)$

$\therefore \quad v = \sqrt{\frac{T}{\mu}}$

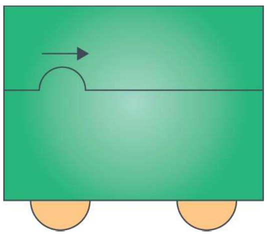
Ex. Find the speed of the wave generated in a string as in the situation shown. Suppose that the tension is not affected by the mass of the cord.



Sol. $T = 20 \times 10 = 200$ N

$v = \sqrt{\frac{200}{0.5}} = 20\,m/s$

Ex. A taut string having the tension 100 N and the linear mass density 0.25 kg/m is used inside a cart to produce a wave pulse starting at the left end, as shown below. Find the velocity of the cart so that pulse stays stationary w.r.t ground.



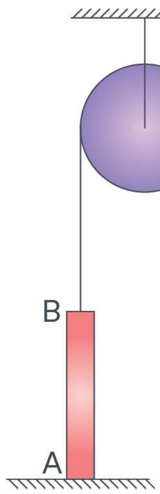
Sol. Velocity of pulse $= \sqrt{\frac{T}{\mu}} = 20\,m/s$

Now $\vec{v}_{PG} = \vec{v}_{PC} + \vec{v}_{CG}$

$0 = 20\hat{i} + \vec{v}_{CG}$

$\vec{v}_{CG} = -20\hat{i}\,m/s$

Ex. One end of the 12.0 m long rubber tube with a total mass of 0.9 kg is fastened to the fixed support. A cord attached to the other end passes over a pulley and supports an object of 5.0 kg. The tube is struck, creating a transverse blow at the one end. Find the time needed for the pulse to reach other end ($g = 9.8\ m/s^2$)



Sol. Tension in rubber tube AB, T = mg

$T = (5.0)(9.8) = 49\,N$

or

Mass per unit length of rubber tube,

$$= \frac{0.9}{12} = 0.075\,kg/m$$

∴ Speed of wave on the tube,

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{49}{0.075}} = 25.56\,m/s$$

∴ The required time is,

$$t = \frac{AB}{v} = \frac{12}{25.26} = 0.47\,s$$

Ex. A uniform rope of the mass 0.1 kg and length 2.45 m hangs from a ceiling

(a) Calculate the speed of transverse wave in the rope at a point 0.5 m distant from lower end.

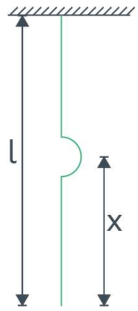
(b) Calculate the time taken by the transverse wave to travel the full length of the rope.

Concept Reminder

Speed of transverse mechanical wave is greater in hollow wire than the solid wire of same material, same radius under same tension.

$$\left(v \propto \frac{1}{\sqrt{\mu}}\right)$$

Sol. (a) As string has mass and it is suspended vertically, the tension in it will be different at the different points. For a point at a distance 'x' from the free end, the tension will be due to weight of the string below it. So, if 'm' is the mass of string of length 'l', the mass of length 'x' of the string will be, $\left(\frac{m}{l}\right)x$.



$\therefore \quad T = \left(\frac{m}{l}\right)xg = \mu xg \qquad \left(\frac{m}{l} = \mu\right)$

$\therefore \quad \frac{T}{\mu} = xg$

or $\quad v = \sqrt{\frac{T}{\mu}} = \sqrt{xg} \quad$ ... (i)

At $x = 0.5$ m, $v = \sqrt{0.5 \times 9.8} = 2.21$ m / s

(b) From the Eq. (i) we see that the velocity of the wave is different at the different points. So, if at point 'x' the wave travels a distance dx in time dt, then

$$dt = \frac{dx}{v} = \frac{dx}{\sqrt{gx}}$$

$$\therefore \quad \int_0^t dt = \int_0^l \frac{dx}{\sqrt{gx}}$$

or $\quad t = 2\sqrt{\frac{l}{g}} = 2\sqrt{\frac{2.45}{9.8}} = 1.0$ s

Concept Reminder

Speed of transverse mechanical wave is greater in thin wire than the thick wire of same material, same density under same tension. $\left(v \propto \frac{1}{r}\right)$

Note :-

$\boxed{v_T = \sqrt{\frac{T}{\mu}}}$ this is the velocity of wave with respect to string

where T = tension in wire

$\mu \rightarrow$ Mass per unit length of wire / linear mass density / linear density

$$\boxed{\mu = \frac{M_{wire}}{L}}\left(\frac{kg}{M}\right)$$

$$\mu = \frac{M_{wire}}{L} = \frac{Volume \times density}{L} = \frac{(AL)\rho}{L}$$

$$\boxed{\mu = A\rho = \pi r^2 \rho} \text{ So, } \boxed{v_T = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{T}{A\rho}} = \sqrt{\frac{T}{\pi r^2 \rho}}}$$

where A = cross-section area of wire
r = radius of wire
ρ = density of wire

Concept Reminder

$$\boxed{v_T = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{T}{A\rho}} = \sqrt{\frac{T}{\pi r^2 \rho}}}$$

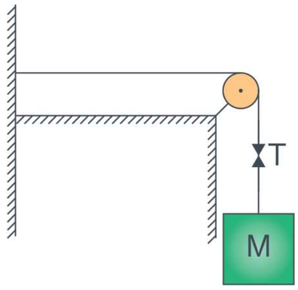
Note :-

(1) Speed of transverse mechanical wave is greater in hollow wire than the solid wire of same material, same radius under same tension. $\left(v \propto \frac{1}{\sqrt{\mu}}\right)$

(2) Speed of transverse mechanical wave is greater in thin wire than the thick wire of same material, same density under same tension. $\left(v \propto \frac{1}{r}\right)$

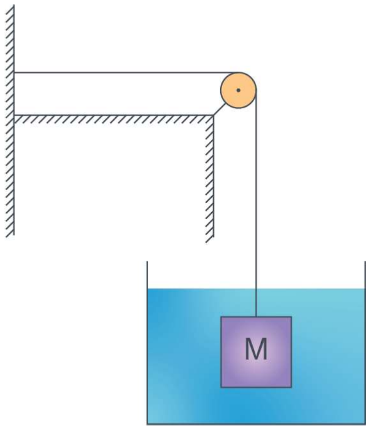
Results :-

(1)



$$v_{T_1} = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{Mg}{\mu}}$$

(2)



$$v_{T_2} = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{Mg - V\rho g}{\mu}}$$

(1) From (1) & (2)

$$\frac{v_1}{v_2} = \sqrt{\frac{Mg}{Mg - V\rho g}} = \sqrt{\frac{V\sigma g}{V\sigma g - V\rho g}} = \sqrt{\frac{1}{1 - \frac{\rho}{\sigma}}}$$

$\Rightarrow$ $\boxed{\frac{v_1}{v_2} = \sqrt{\frac{1}{1 - \frac{\rho}{\sigma}}}}$ $\rho \rightarrow$ density of liquid

$\sigma \rightarrow$ density of solid

(2) In solid wire

$$v_T = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{T}{A\rho}} = \sqrt{\frac{Stress}{\rho}} = \sqrt{\frac{y \times strain}{\rho}}$$

means $\boxed{v_T = v_L \sqrt{strain}}$

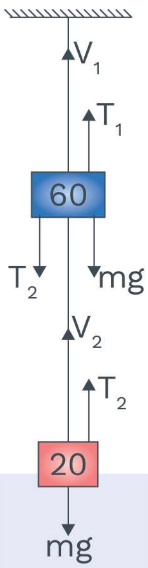
(3) Strain $= \left(\frac{\Delta l}{l}\right)$ $\boxed{V_T = \sqrt{\frac{Y}{\rho} \times \frac{\Delta l}{l}}}$

(4) Thermal strain:- $(\alpha \Delta \theta)$

$\boxed{v_T = \sqrt{\frac{Y}{\rho} \times \propto \Delta\theta}}$ α = coefficient of thermal expansion

Ex. Find out the ratio of v_T in both identical wire

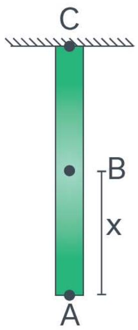
Sol. $\frac{v_{T_1}}{v_{T_2}} = \frac{\sqrt{T_1}}{\sqrt{T_2}}$



$$\Rightarrow \quad \frac{v_{T_1}}{v_{T_2}} = \sqrt{\frac{80\,g}{20\,g}} = \sqrt{4} = 2:1$$

Ex. Find out v_T in given thick rope at point A, B, C. If length of rope is (L) and Mass is (M)

Sol. At A, $v_A = \sqrt{\frac{T_A}{\mu}}$



$\because \quad T_A = 0$

$\therefore \quad \boxed{v_A = 0}$

At B

$v_B = \sqrt{\frac{T_B}{\mu}}$

$T_B = M_x g$

Mass of Length $L \rightarrow M$

Mass of unit Length $\rightarrow \frac{M}{L}$

Mass of length $x \rightarrow \frac{M}{L} \times x$

At C

$v_C = \sqrt{\frac{T_C}{\mu}}$

$v_C = \sqrt{\frac{Mg}{\frac{M}{L}}}$

$\boxed{v_C = \sqrt{L.g}}$

$$\therefore \quad v_B = \sqrt{\frac{\left(\frac{M}{L}\times x\right)\times g}{\frac{M}{L}}}$$

$$\boxed{v_B = \sqrt{xg}}$$

Ex. Find out time taken by transverse wave to reach from point A to C in previous ques.

Sol. Acceleration of wave $= v\dfrac{dv}{dx} = \dfrac{g}{2} = \text{const.}$

$$S = \left(\frac{u+v}{2}\right)\times t$$

$$L = \frac{0+\sqrt{gL}}{2}\times t \Rightarrow t = \frac{2L}{\sqrt{g}\sqrt{L}} = \boxed{2\sqrt{\frac{L}{g}}}$$

Alternate Method :

$$\because \quad v = \sqrt{gx} \Rightarrow \frac{dx}{dt} = \sqrt{gx} \Rightarrow \int_0^t dt = \int_0^L \frac{1}{\sqrt{gx}}dx$$

$$\Rightarrow \quad t = \frac{1}{\sqrt{g}}\int_0^L x^{-1/2}dx \Rightarrow t = \frac{1}{\sqrt{g}}2[x^{1/2}]_0^L \Rightarrow t = 2\sqrt{\frac{L}{g}}$$

Ex. Two wires of different densities but same area of cross section are joined together at their one end and other ends of wire are stretched to a tension 'T'. If speed of transverse wave in one wire is double of other, find out the ratio of densities.

Sol. $v = \sqrt{\dfrac{T}{\mu}} = \sqrt{\dfrac{T}{A\rho}} \Rightarrow v \propto \sqrt{\dfrac{1}{\rho}}$

$$\frac{v_1}{v_2} = \sqrt{\frac{\rho_2}{\rho_1}} \Rightarrow \frac{2v}{v} = \sqrt{\frac{\rho_2}{\rho_1}} \Rightarrow 4:1 = \frac{\rho_2}{\rho_1}$$

$$\rho_1 : \rho_2 = 1:4$$

Ex. A 4 kg block is attached to one end of

Rack your Brain

If n_1, n_2 and n_3 are the fundamental frequencies of three segments into which a string is divided, then the original fundamental frequency n of the string is given by

(1) $\dfrac{1}{n} = \dfrac{1}{n_1} + \dfrac{1}{n_2} + \dfrac{1}{n_3}$

(2) $\dfrac{1}{\sqrt{n}} = \dfrac{1}{\sqrt{n_1}} + \dfrac{1}{\sqrt{n_2}} + \dfrac{1}{\sqrt{n_3}}$

(3) $\sqrt{n_1} = \sqrt{n_1} + \sqrt{n_2} + \sqrt{n_3}$

(4) $n = n_1 + n_2 + n_3$

string and other end of string is attached with ceiling of a lift moving upward with acceleration of $2 m/s^2$. If linear mass density of wire is 19.2×10^{-3} kg/m, find speed with which wave travel along string.

Sol. $T = mg + ma$

$\Rightarrow T = m(g+a) = 4 \times (10+2) = 4 \times 12 = 48$ N

$$\because \quad v_T = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{48}{19.2 \times 10^{-3}}} = \frac{10^2}{2} = 50 \, m/s$$

Ex. A string of 2.5 kg is under some tension. The length of stretched string is 20 m. If transverse wave at one end of string taken 0.5 s to reach at the other end. Find T

Sol. d = 20 m

t = 0.5 s

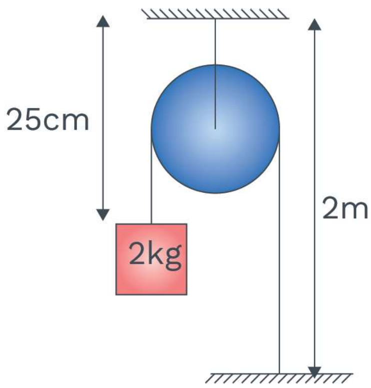
$$v = \frac{d}{t} = \frac{20}{5} \times 10 = 40 \, m/s$$

$$v = \sqrt{\frac{T \times L}{M}} \Rightarrow 40 \times 40 = \frac{200}{25} \times T$$

$$T = \frac{40 \times 40 \times 25}{200} = 200 \, N$$

Ex. In the diagram, string has M = 4.5 g. Find time taken by a transverse wave produced at floor to reach the pulley.

Sol. $v_T = \sqrt{\frac{T}{m}} = \sqrt{\frac{2 \times 10 \times 2.25}{4.5 \times 10^{-3}}}$



Concept Reminder

Average rate of transmission of kinetic energy is

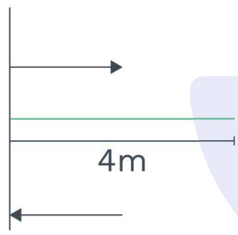
$$\frac{dk_{avg}}{dt} = \frac{1}{4} \mu v \omega^2 A^2$$

Wave on a String

$$\Rightarrow v_T = \sqrt{10 \times 10^{+3}} \Rightarrow v_T = 100\,m/s$$

Ex. The length of a wire is 4 m long and has a mass of 0.2 kg. The wire is kept horizontally. A wave is generated by plucking one end of wire. If the wave pulse makes 4 trips back and forth in 0.4 sec. Find tension in wire.

Sol. $v = \dfrac{D}{t} = \dfrac{4 \times 8 \times 10}{4} = 80\,m/s$



$$v = \sqrt{\frac{T}{\mu}} \Rightarrow 80 = \sqrt{\frac{T \times 24 \times 10}{2}} \Rightarrow T = 320\,N$$

Energy Calculation In Waves :

(a) Kinetic energy per unit length

The velocity of the string element in the transverse direction is greatest at the mean position and zero at extreme positions of the waveform. We can find expression of transverse velocity by

$$y = A\sin(kx - \omega t)$$

Differentiating partially w.r.t. time, the expression of particle velocity is :

$$v_p = \frac{\partial y}{\partial t} = -\omega A\cos(kx - \omega t)$$

In order to calculate the kinetic energy, we consider a small string element of length "dx" of having mass per unit length "μ". The K.E. of the element is given by

$$dK = \frac{1}{2}dmv_p^2 = \frac{1}{2}\mu dx\omega^2 A^2\cos^2(kx - \omega t)$$

This is the K.E. associated with the element in the motion. Since it involves squared of cos

Concept Reminder

The elastic potential energy of the string element results as string element is stretched during its oscillation.

Concept Reminder

Greater extension of string element corresponds to greater elastic energy.

function, its value is the greatest for a phase of zero (mean position) and zero for the phase of $\frac{\pi}{2}$ (maximum displacement)

Now, we get K.E. per unit length, "KL", by dividing this expression with the length of small string considered :

$$K_L = \frac{dK}{dx} = \frac{1}{2}\mu\omega^2 A^2 \cos^2(kx - \omega t)$$

- The rate, at which K.E. is transmitted, is obtained by dividing expression of kinetic energy by small time element, "dt" :

$$\frac{dK}{dx} = \frac{1}{2}\mu\frac{dx}{dt}\omega^2 A^2 \cos^2(kx - \omega t)$$

But, the wave or the phase speed, 'v', is time rate of position i.e. $\frac{dx}{dt}$. Hence,

$$\frac{dK}{dt} = \frac{1}{2}\mu v\omega^2 A^2 \cos^2(kx - \omega t)$$

Here K.E. is a periodic function. We can obtain the average rate of transmission of K.E. by integrating the expression for integral wavelengths. Since only $\cos^2(kx - \omega t)$ is the varying entity, we need to find average of this quantity only. Its integration over integral wavelengths give a value of $\frac{1}{2}$. Hence, average rate of transmission of K.E. is :

$$\boxed{\frac{dK_{avg}}{dt} = \frac{1}{2}\times\frac{1}{2}\mu v\omega^2 A^2 = \frac{1}{4}\mu v\omega^2 A^2}$$

(b) Elastic potential energy:-

The elastic potential energy of string element results as the string element is stretched during its oscillation. The extension or stretching is maximum at the mean position. We can see in the diagram that the length of the string element of equal x-length "dx" is greater at the mean

Rack your Brain

Two identical piano wires, kept under the same tension T have a fundamental frequency of 600 Hz. The fractional increase in the tension of one of the wires which will lead to occurrence of 6 beats/s when both the wires oscillate together would be

(1) 0.04 (2) 0.01
(3) 0.02 (4) 0.03

position than at the extreme. As a matter of the fact, the elongation depends on slope of the curve. Greater slope, greater is elongation. The string has the least length when the slope is zero. For illustration purpose, the curve is deliberately drawn in such a way that the elongation of string element at mean position is highlighted.

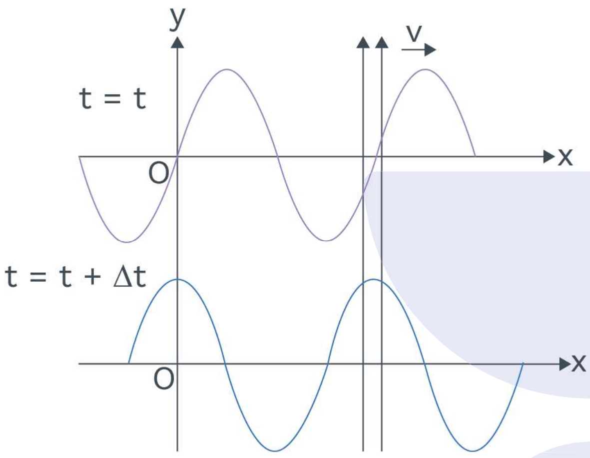


Fig : The string element stretched most at the equilibrium position

Greater extension of the string element corresponds to the greater elastic energy. As such, it is greatest at the mean position and zero at extreme position. This deduction in contrary to case of the SHM in which the potential energy is greatest at the extreme position and zero at the mean position.

- **Potential energy per unit length:-**
When string segment is stretched from the length dx to the length ds the amount of work = T (ds – dx) is done. This is equal to potential energy stored in stretched string segment. So, the P.E. in this case is :

Concept Reminder

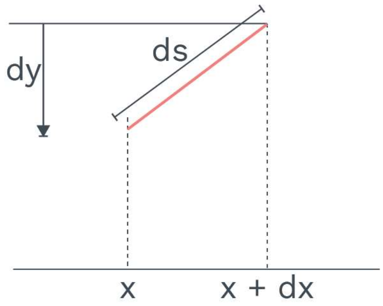
Average rate of transmission of elastic potential energy is

$$\frac{dU_{avg}}{dx} = \frac{1}{4}\mu v\omega^2 A^2$$

Concept Reminder

The average power transmitted by wave is equal to time rate of transmission of mechanical energy over integral wavelengths. It is equal to :

$$P_{avg} = \frac{dE_{avg}}{dx} = 2x\frac{1}{2}\mu v\omega^2 A^2 = \frac{1}{2}\mu v\omega^2 A^2$$



$$U = T(ds - dx)$$

Now $\quad ds = \sqrt{(dx^2 + dy^2)}$

$$= dx\sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}$$

from the binomial expansion

so $\quad ds \approx dx + \frac{1}{2}\left(\frac{dy}{dx}\right)^2 dx$

$$U = T(ds - dx) \approx \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 dx$$

or the potential energy density

$$\frac{dU}{dx} = \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 \quad \text{... (i)}$$

$$\frac{dy}{dx} = kA\cos(kx - \omega t)$$

and $\quad T = v^2\mu$

Put above value in the equation (i) then we get

$$\boxed{\frac{dU}{dx} = \frac{1}{2}\mu\omega^2A^2\cos^2(kx - \omega t)}$$

- **Rate of transmission of the elastic potential energy**

The rate, at which the elastic potential energy is transmitted, is obtained by dividing the expression of kinetic energy by the small time element, “dt”. This expression is same as that for kinetic energy.

$$\frac{dU}{dx} = \frac{1}{2}\mu\omega^2A^2\cos^2(kx - \omega t)$$

Concept Reminder

The average energy per unit volume in travelling wave on string is $\frac{1}{2}\rho v\omega^2A^2$

and average rate of the transmission of elastic potential energy is

$$\frac{dU_{avg.}}{dx} = \frac{1}{2} \times \frac{1}{2} \mu v \omega^2 A^2 = \frac{1}{4} \mu v \omega^2 A^2$$

(c) Mechanical energy per unit length:-

Since the expression for elastic potential energy is same as that of K.E., we get mechanical energy expression by multiplying the expression of kinetic energy by "2". The mechanical energy associated with small string element, "dx", is :

$$dE = 2x\, dK = 2x\, \frac{1}{2} dm v_p^2 = \mu dx \omega^2 A^2 \cos^2(kx - \omega t)$$

Similarly, the mechanical energy per unit length is :

$$E_L = \frac{dE}{dx} = 2x\, \frac{1}{2} \mu \omega^2 A^2 \cos^2(kx - \omega t)$$

$$= \mu \omega^2 A^2 \cos^2(kx - \omega t)$$

(d) Average power transmitted:-

The average power transmitted by the wave is equal to time rate of transmission of mechanical energy over integral wavelengths. It is equal to :

$$P_{avg} = \frac{dE_{avg}}{dx} = 2x \frac{1}{2} \mu v \omega^2 A^2 = \frac{1}{2} \mu v \omega^2 A^2$$

If mass of string is given in terms of the mass per unit volume, "ρ", then we make appropriate change in the derivation. We exchange μ by "ρs" where "s" is the cross section of the string :

$$P_{avg} = \frac{1}{2} \rho s v \omega^2 A^2$$

(e) Energy density:-

Since there is no loss of energy associated, it is expected that the energy per unit length is uniform throughout string. As much energy enters that much energy goes out for the given length of string. This average value along the unit length of the string length is equal to the average rate at which the energy is being transferred.

The average mechanical energy per unit length is equal to the integration of the expression over integral wavelength

$$E_L|_{avg} = 2x \frac{1}{4} \mu v \omega^2 A^2 = \frac{1}{2} \mu v \omega^2 A^2$$

Definitions

Superposition Principle

When two or more waves superpose on a medium particle than the resultant displacement of that medium particle is given by the vector sum of the individual displacements produced by the component waves at that medium particle independently".

We have derived this expression for the harmonic wave along a string. The concept, however, can be extended to 2-D or 3-D transverse waves. In the case of three dimensional transverse waves, we consider the small volumetric element. We, then, use density, ρ, in place of the mass per unit length, μ. The corresponding average energy per unit volume is referred as the energy density (u) :

$$\boxed{\frac{1}{2}\rho v\omega^2 A^2}$$

(f) Intensity:-

Intensity of wave (I) is defined as the power transmitted per unit cross section area of the medium :

$$\boxed{I = \rho s v\omega^2 \frac{A^2}{2s} = \frac{1}{2}\rho v\omega^2 A^2}$$

Intensity of the wave (I) is a very useful concept for 3-D waves radiating in all the direction from the source. This quantity is usually referred in the context of the light waves, which is transverse harmonic wave in 3-D. Intensity is defined as power transmitted per unit cross sectional area. Since the light spreads uniformly all around, intensity is equal to power transmitted, divided by the spherical surface drawn at that point with the source at its centre.

- **Phase difference between the two particles in the same wave :**

The general expression for the sinusoidal wave travelling in the positive x direction is

$$y(x,t) = \sin(\omega t - kx)$$

E^{qn} of particle at x_1 is given by $y_1 = A\sin(\omega t - kx_1)$

E^{qn} of particle which is at x_2 from the origin

$$y_2 = A\sin(\omega t - kx_2)$$

Phase difference between particles is $k(x_2 - x_1) = \Delta\phi$

$$k\Delta x = \Delta\phi \Rightarrow \Delta x \Rightarrow \frac{\Delta\phi}{k}$$

Principle of Superposition:

This principle defines displacement of a medium particle when it is oscillating under influence of two or more than two waves. The principle of the superposition is stated as :

"When the two or more waves superpose on a medium particle than the resultant displacement of that the medium particle is given by vector sum of the individual displacements produced by component waves at that medium particle independently".

Let $\vec{y}_1, \vec{y}_2, ... \vec{y}_N$ are the displacements produced by N independent waves at a medium particle in absence of others then displacement of that medium, when all waves are superposed at that point, is given as

$$\vec{y} = \vec{y}_1 + \vec{y}_2 + \vec{y}_3 + \ldots\ldots. \vec{y}_N$$

If all waves are producing oscillations at that point are collinear then the displacement of medium particle where superposition is taking place may be simply given by algebraic sum of the individual displacement. Thus we have,

$$y = y_1 + y_2 + \ldots\ldots\ldots + y_N$$

The above eqn. is valid only if all the individual displacements y_1, y_2 .... y_N are along the same straight line. A simple example of superposition can be understood by the figure shown. Suppose that the two wave pulses are travelling simultaneously in the opposite directions as shown. When they overlap on each other, then the displacement of the particle on the string is the algebraic sum of the two displacement as displacements of the two pulses are in same direction. Figure also shows the similar situation when the wave pulses are in the opposite side.

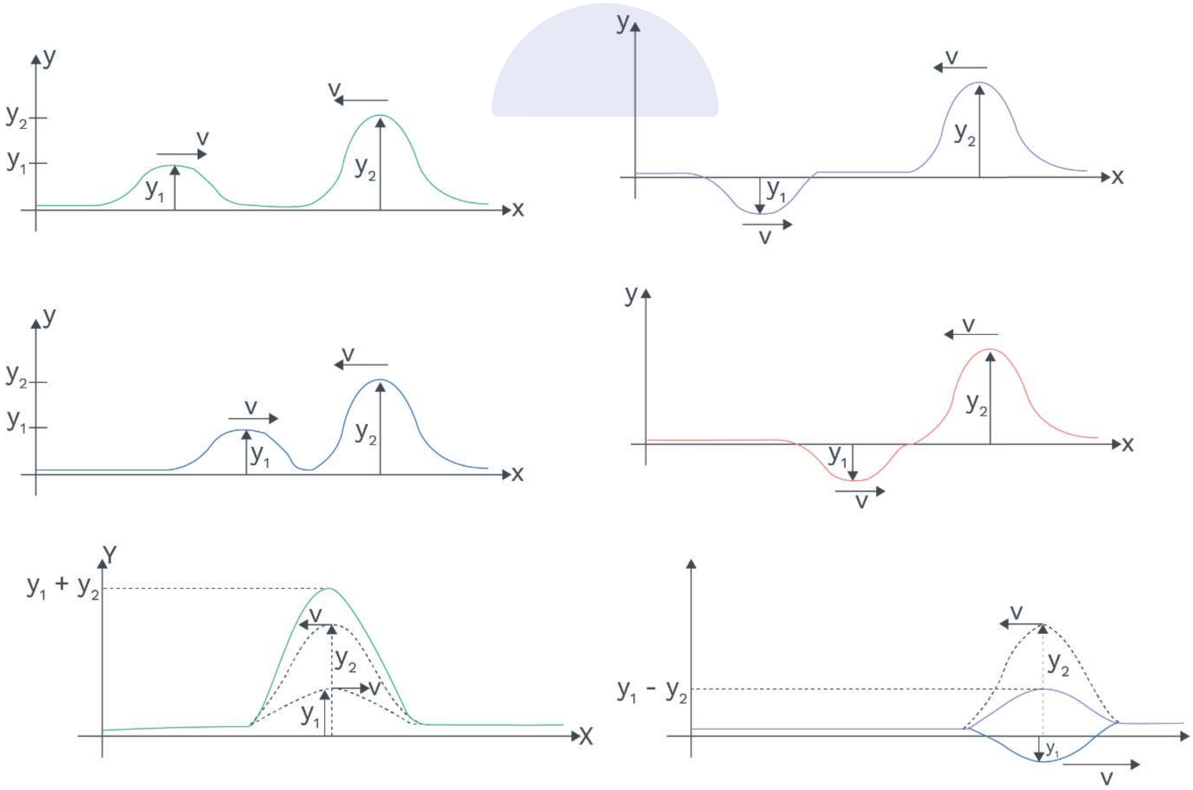
Concept Reminder

Applications of Principle of Superposition of Waves

(1) Interference of Wave
(2) Stationary Waves
(3) Beats

Concept Reminder

Interference is the phenomenon of superposition of two coherent waves travelling in same direction.



(a) Applications of Principle of Superposition of Waves:-

There are many different phenomenon which takes the place during superposition of two or more wave depending on wave characteristics which are being superposed. We'll discuss some standard phenomenon's, and these are :

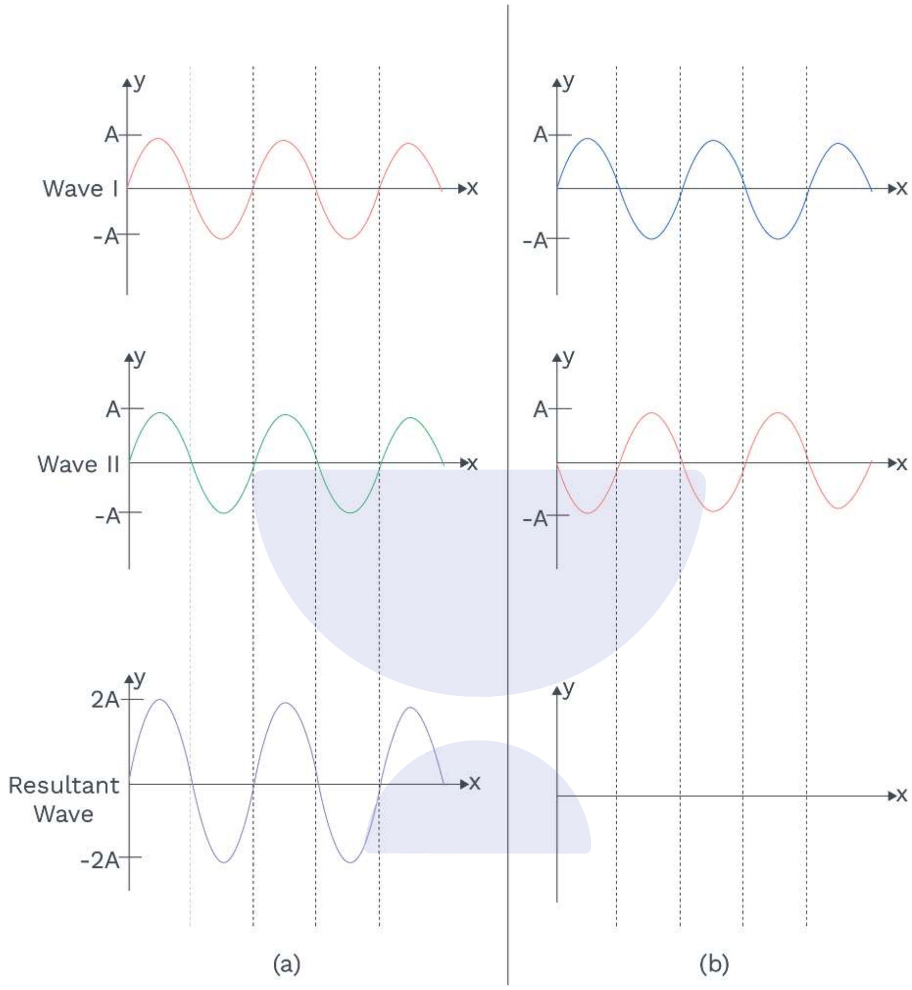
(1) Interference of Wave

(2) Stationary Waves

(3) Beats

(b) Interference of Waves:-

Suppose the two sinusoidal waves of the same wavelength and amplitude travel in the same direction along the same straight line (may be on a stretched string) then the superposition principle can be used to define resultant displacement of every medium particle. The resultant wave in medium depends on extent to which the waves are in the phase w.r.t. each other, that is, how much one wave form is shifted from other waveform. If the two waves are exactly in same phase, that is the shape of one wave exactly fits on to the other wave then they combine to double the displacement of the every medium particle as shown in the figure (a). This phenomenon we call as the constructive interference. If superposing waves are exactly out of the phase or in the opposite phase then they combine to cancel all displacements at every particle of medium particle and medium remains in the form of a straight line as shown in figure.

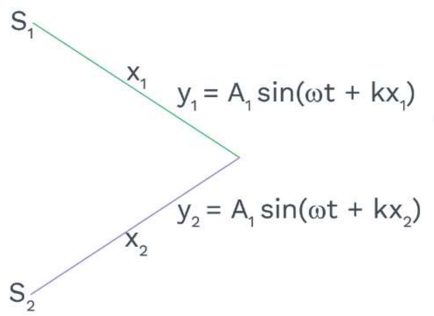


This phenomenon we call the destructive interference. Thus we can state that when the waves meet, they interfere constructively if they meet in the same phase and destructively if they meet in the opposite phase. In either case wave patterns don't shift relative to each other as they propagates. Such superposing waves which have the same form and the wavelength and have a fixed phase relation to each other, are called the coherent waves. Sources of the coherent waves are called the coherent source. Two independent sources can never be coherent in the nature because of practical limitations of the manufacturing process. Generally all the coherent sources are made either by splitting of the wave forms of a single source or different sources are fed by the single main energy source.

In simple words interference is phenomenon of superposition of the two coherent waves travelling in the same direction.

We've discussed that resultant displacement of a medium particle when the two coherent waves interfere at that point, as the sum or difference of the individual displacements by the two waves if they are in same phase (phase difference $= 0, 2\pi, ...$) or opposite phase (phase difference $= \pi, 3\pi, ...$) respectively. But two waves can also meet at a medium particle with the phase difference other then 0 or 2π, say if phase difference ϕ is such that $0 < \phi < 2\pi$, then how is the displacement of point of superposition given ? Now we discuss the interference of the waves in details analytically.

(c) Analytical Treatment of Interference of Waves



Interference implies super position of the waves. Whenever the two or more than two waves superimpose each other they give the sum of their individual displacement.

Let the two waves coming from the sources S_1 & S_2 be

$$y_1 = A_1 \sin(\omega t + kx_1)$$

$$y_2 = A_2 \sin(\omega t + kx_2) \quad \text{respectively}$$

Due to superposition

$$y_{net} = y_1 + y_2$$

KEY POINTS

- Superposition
- Interference
- Coherent sources

Concept Reminder

Assuming there is no absorption of energy by the boundary, the reflected wave has the same shape as the incident pulse but it suffers a phase change of π or 180°on reflection. This is because the boundary is rigid and the disturbance must have zero displacement at all times at the boundary.

Phase difference between

i.e.,

As

(where = path difference & = phase difference)

(as)

When the two displacements are in the phase, then the resultant amplitude will be sum of the two amplitude & I_{net} will be maximum, this is known of constructive interference.

For I_{net} to be maximum

where n = {0,1,2,3,4,5..........}

For constructive interference

When

$I_{net} = 4 I$

$A_{net} = A_1 + A_2$

When the superposing waves are in the opposite phase, the resultant amplitude is the difference of two amplitudes & I_{net} is minimum; this is known as destructive interference.

For I_{net} to be minimum,

where n = {0,1,2,3,4,5..........}

For destructive interference

If $I_1 = I_2$

$I_{net} = 0$

$A_{net} = A_1 - A_2$

Ratio of $\frac{I_{max}}{I_{min}} = \frac{(\sqrt{I_1} + \sqrt{I_2})^2}{(\sqrt{I_1} - \sqrt{I_2})^2}$

Generally, $I_{net} = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos\phi$

If $\quad I_1 = I_2 = I$

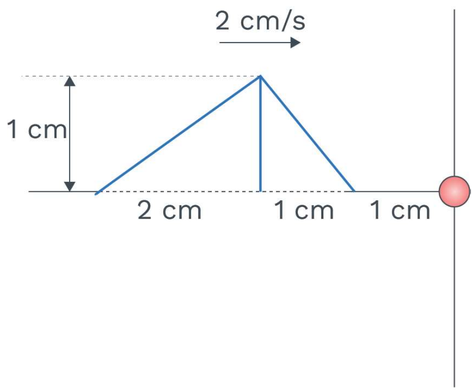
$I_{net} = 2I + 2I\cos\phi$

$I_{net} = 2I(1+\cos\phi) = 4I\cos^2\frac{\Delta\phi}{2}$

Ex. Wave from two source, each of the same frequency and travelling in the same direction, but with the intensity in the ratio 4 : 1 interfere. Find ratio of maximum to minimum intensity.

Sol. $\frac{I_{max}}{I_{min}} = \left(\frac{\sqrt{I_1}+\sqrt{I_2}}{\sqrt{I_1}-\sqrt{I_2}}\right)^2 = \left(\frac{\sqrt{\frac{I_1}{I_2}}+1}{\sqrt{\frac{I_1}{I_2}}-1}\right)^2 = \left(\frac{2+1}{2-1}\right)^2 = \frac{9}{1}$

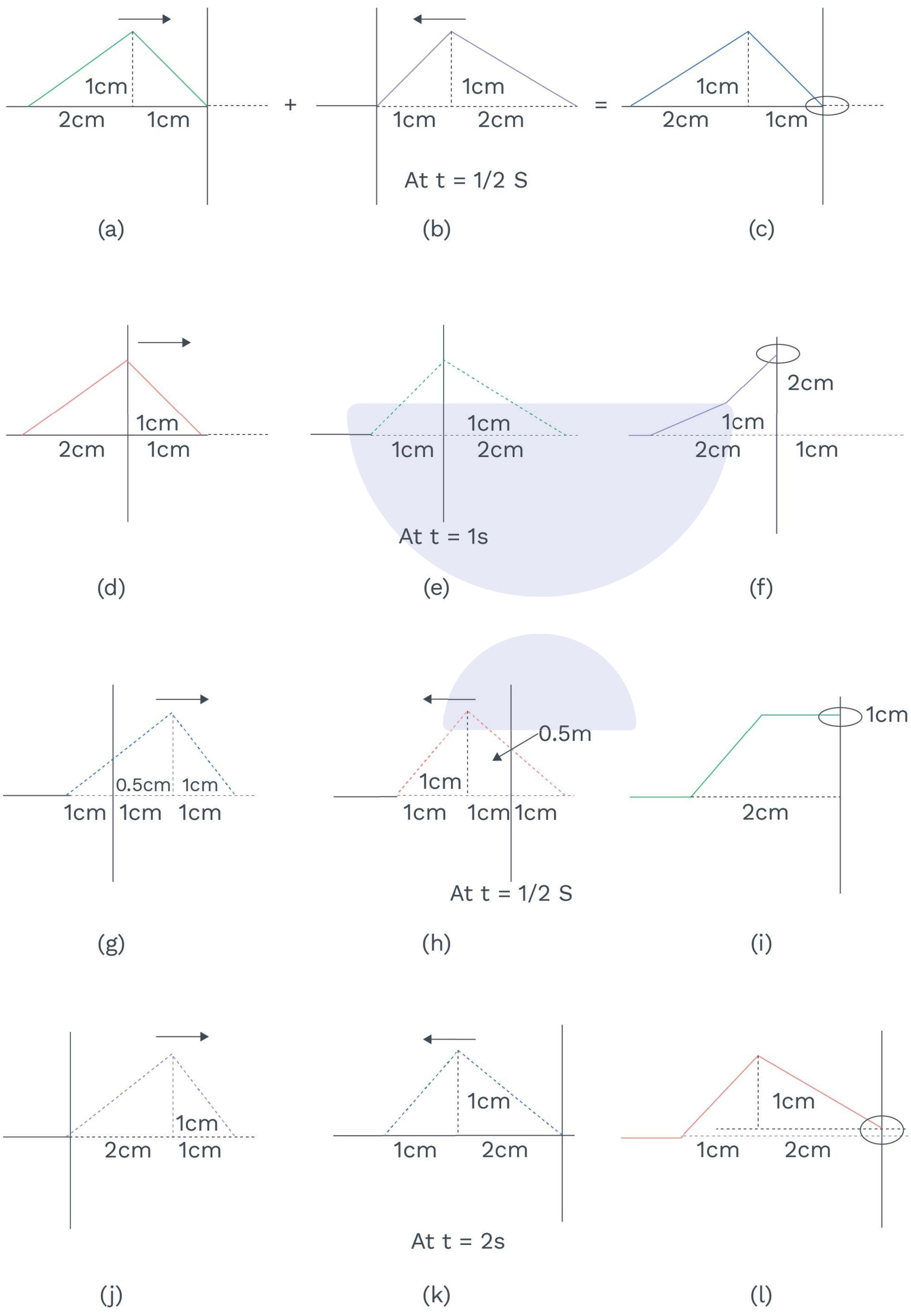
Ex. The triangular pulse moving at 2 cm/s on the rope approaches an end at which it's free to slide on a vertical pole.



(a) Draw the pulse at $\frac{1}{2}$s interval until it's completely reflected.

(b) What is the particle speed on trailing edge at the instant depicted ?

Sol. (a) Reflection of the pulse from a free boundary is really the superposition of two identical waves travelling in opposite direction. This can be shown as under.


1cm
2cm
1cm
+
1cm
1cm
2cm
=
1cm
2cm
1cm
At t = 1/2 S
(a)
(b)
(c)
1cm
2cm
1cm
1cm
1cm
2cm
2cm
1cm
2cm
1cm
At t = 1s
(d)
(e)
(f)
0.5cm
1cm
1cm
1cm
1cm
0.5m
1cm
1cm
1cm
1cm
1cm
2cm
At t = 1/2 S
(g)
(h)
(i)
1cm
2cm
1cm
1cm
1cm
2cm
1cm
1cm
2cm
At t = 2s
(j)
(k)
(l)

In every $\frac{1}{2}$ s, each pulse (one real moving towards the right and one imaginary moving towards the left travels a distance 1 cm, as the wave speed is 2 cm/s.)

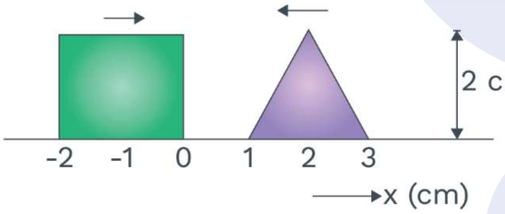
(b) Particle speed, $v_p = |-v \times (\text{slope})|$

Here, v = the wave speed = 2 cm/s and

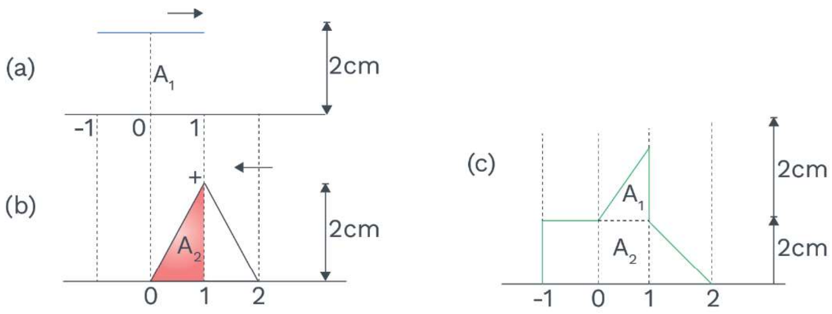
slope = $\frac{1}{2}$

$\therefore$ Particle speed = 1 cm/s

Ex. Diagram shows the rectangular pulse and the triangular pulse approaching towards each other. Pulse speed is 0.5 cm/s. draw the resultant pulse at t = 2 s



Sol. In 2 s each pulse travels a distance of 1 cm. The 2 pulses overlap between 0 and 1 cm as shown in the figure. So, A_1 and A_2 can be added as shown in the figure (c).



Resultant pluse
at t = 2s

Concept Reminder

Amplitude of standing wave is not constant but varies periodically with position.

Concept Reminder

The nodes divide the medium into segments (or loops). All the particles in a segment vibrate in same phase, but in opposite phase with the particles in the adjacent segment.

Standing Waves

The standing wave is formed when the two identical waves travelling in the opposite directions along the same line, interfere.

On the path of stationary wave, there are points where amplitude is zero, they are defined as NODES.

On other hand, there are points where amplitude is maximum, they are defined as ANTI-NODES.

The distance between two consecutive nodes or two consecutive anti-nodes is $\frac{\lambda}{2}$.

The distance between the node and the next anti-node is $\frac{\lambda}{4}$.

Consider the two waves of the same frequency, speed and amplitude, which are travelling in the opposite directions along the string. Two such waves can be represented by the equations

$y_1 = a\sin(kx - \omega t)$ and

$y_2 = a\sin(kx + \omega t)$

Hence the resultant may be written as

$y = y_1 + y_2 = a\sin(kx - \omega t) + a\sin(kx + \omega t)$

$y = 2a\sin kx\cos\omega t$

This is the equation of a standing wave.

Hence we can deduce :

1. As this equation satisfies the wave equation,

$$\frac{\partial^2 y}{dx^2} = \frac{1}{v^2}\frac{\partial^2 y}{dt^2}$$

it represents a wave. However, as it isn't of the form f (ax ± bt), the wave is not travelling and so it is called standing or stationary wave.

2. The amplitude of the wave

$A_s = 2A\cos kx$

isn't constant but changes periodically with position (and not with time as in beats).

KEY POINTS

- Standing wave
- Stationary wave
- Node
- Anti0node

Definitions

On the path of the stationary wave, there are points where the amplitude is zero, they are known as NODES. On the other hand, there are points where the amplitude is maximum, they are known as ANTINODES.

Concept Reminder

As in stationary waves nodes are permanently at rest, so no energy can be transmitted across them, i.e., energy of one region (segment) is confined in that region.

3. The points for which the amplitude is min. are called nodes and for these

$\cos kx = 0,$ i.e., $kx = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$

i.e., $x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, ... \left[as\, k = \frac{2\pi}{\lambda}\right]$

i.e., in a stationary wave, nodes are equally spaced.

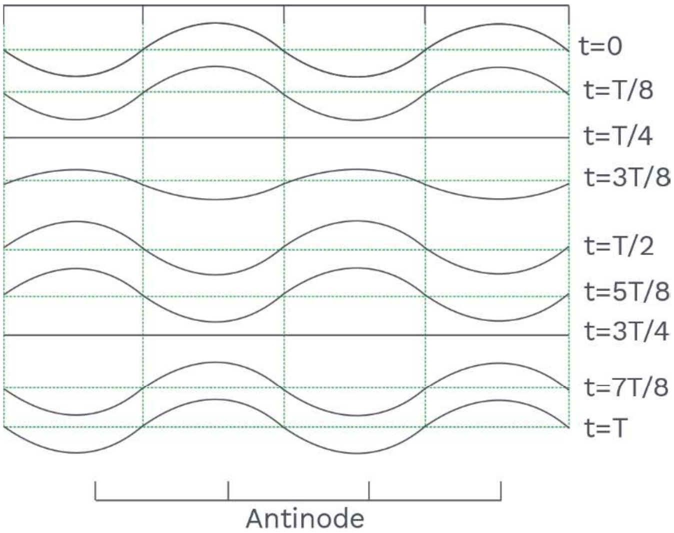
4. The points for which the amplitude is max. are called anti-nodes and for these,

$\cos kx = \pm 1$ i.e., $kx = 0, \pi, 2\pi, 3\pi, ...$

i.e., $x = 0, \frac{\lambda}{2}, \frac{2\lambda}{2}, \frac{3\lambda}{2}, ... \left[as\, k = \frac{2\pi}{\lambda}\right]$

like nodes, the anti-nodes are also equally spaced with the spacing $(\lambda/2)$ and A_{max} = ±2A. Furthermore, nodes and the anti-nodes are alternate with spacing $(\lambda/4)$.

5. The nodes divide the medium into the segments (or loops). All particles in a segment vibrate in the same phase, but in the opposite phase with the particles in adjacent segment. Twice in one period all particles pass through their mean position simultaneously with the maximum velocity $(A_s\omega)$, the direction of the motion being reversed after each half cycle.



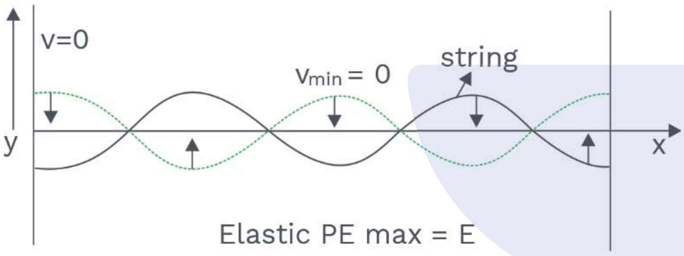
Rack your Brain

A second harmonic has to be generated in a string of length l stretched between two rigid supports. The points where the string has to be plucked and touched are respectively

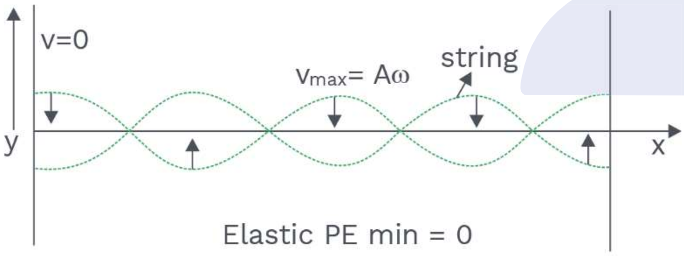
(1) $\frac{l}{4}, \frac{l}{2}$ (2) $\frac{l}{4}, \frac{3l}{4}$

(3) $\frac{l}{2}, \frac{l}{2}$ (4) $\frac{l}{2}, \frac{3l}{4}$

6. Standing wave can be transverse or longitude, e.g., in a string (under tension) if reflected wave exists, waves are transverse-stationary, while in athe organ pipes waves are longitudinal-stationary.
7. As in the stationary waves nodes are permanently at rest, so energy can not be transmitted across them, that means energy of one region (segment) is confined only in that region. However, this energy oscillates between the elastic potential energy and the kinetic energy of the particles of the medium (as shown in the figure A),and when all the particles (simultaneously) pass through their mean position, the kinetic energy will be maximum while the elastic PE will be minimum (figure B). The total energy confined in the segment (elastic PE + KE), always remains the same.



Elastic PE max = E
Kinetic energy min = 0
(A)



Elastic PE min = 0
Kinetic energy max = E
(B)

- **Different Equation for a Stationary Wave:-**

Consider the two equal amplitude waves travelling in opposite direction as

$$y_1 = A\sin(\omega t - kx) \quad \ldots (11)$$

and $$y_2 = A\sin(\omega t + kx) \quad \ldots (12)$$

The result of the superposition of these two waves is

$$y = 2A\cos kx \sin \omega t \quad \ldots (13)$$

Which is the equation of the stationary wave where 2A cos kx represents the amplitude of the medium particle situated at position 'x' and $\sin \omega t$ is the time sinusoidal factor. Equation (13) can be written in many ways depending on the initial phase differences in the component waves given by the equation (11) can (12). If the superposing waves are has the initial phase difference of π, then the component waves can be expressed as

$$y_1 = A\sin(\omega t - kx) \quad \text{... (14)}$$

$$y_2 = -A\sin(\omega t - kx) \quad \text{... (15)}$$

Superposition of above two waves will result

$$y = 2A\sin kx \cos\omega t \quad \text{.... 16)}$$

Equation (16) is also an equation of the stationary wave but here amplitude of the different medium particles in the region of the interference is given by

$$R = 2A \sin kx \quad \text{... (17)}$$

Similarly the possible equations of the stationary wave can be written as

$$y = A_0 \sin kx \cos(\omega t + \phi) \quad \text{... (18)}$$

$$y = A_0 \cos kx \sin(\omega t + \phi) \quad \text{... (19)}$$

$$y = A_0 \sin kx \sin(\omega t + \phi) \quad \text{... (20)}$$

$$y = A_0 \cos kx \cos(\omega t + \phi) \quad \text{... (21)}$$

Here A_0 is the amplitude of the anti-nodes. In a pure stationary wave it is given as

$$A_0 = 2A$$

Where the A is the amplitude of the component waves. If we care fully look at equation (18) to (21), we can see that in the equation (18) and (20), particle amplitude is given by

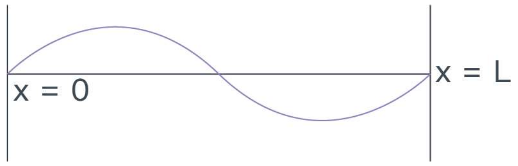
$$R = A_0 \sin kx \quad \text{...(22)}$$

Here at x = 0, there are nodes as R = 0 and in the equation (19) and (21) particle amplitude is given as

$$R = A_0 \cos kx \quad \text{... (23)}$$

Here at x = 0, there is anti-node as $R = A_0$. Thus we can state that in the given system of co- ordinates when origin of the system is at node we use either the equation (18) or (20) for analytical representation of the stationary wave and we use equation (19) or (21) for same when an anti-node is located at origin of the system.

Ex. Find out equation of the standing waves for the following standing wave pattern.



Sol. General Equation of standing wave

$$y = A'\cos\omega t$$

where $A' = A\sin(kx + \theta)$

here $\lambda = L$

$\Rightarrow \quad k = \dfrac{2\pi}{L}$

$$A' = A\sin(kx+\theta) = A\sin\left(\frac{2\pi}{L}x + \theta\right)$$

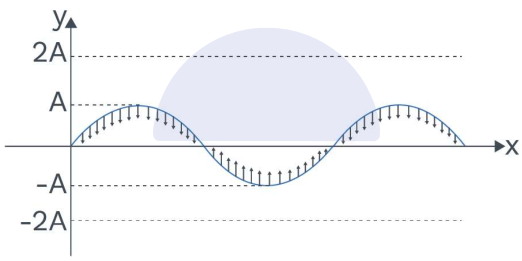
at $\quad x = 0$ node

$\Rightarrow \quad A' = 0$ at $x = 0$

$\Rightarrow \quad \theta = 0$

eq. of standing wave = $A\sin\left(\dfrac{2\pi}{L}x\right)\cos\omega t$

Ex. In the figure shows the standing waves pattern in a string at t = 0. Find out equation of the standing wave where amplitude of anti-node is 2A.



Sol. Let status we assume the equation of standing waves is $= A'\sin(\omega t + \phi)$

where A' $= 2A\sin(kx + \phi)$

$\because \quad$ x = 0 is node $\Rightarrow A' = 0$, at x = 0

$2A\sin\theta = 0 \Rightarrow \theta = 0$

at t = 0 The particle at is at y = A and going towards mean position.

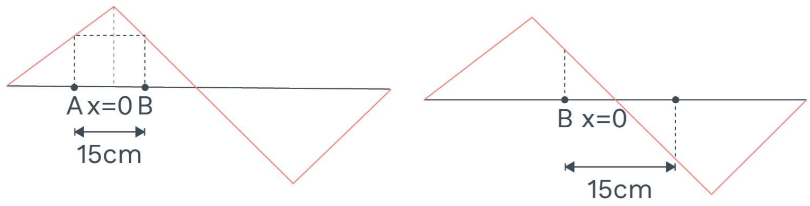
$\Rightarrow \quad \phi = \dfrac{\pi}{2} + \dfrac{\pi}{3} = \dfrac{5\pi}{6}$

so, eq. of standing waves is

$$y = 2A\sin kx \sin\left(\omega t + \frac{5\pi}{6}\right)$$

Ex. A string 120 cm in the length sustains standing wave with points of the string at which the displacement amplitude is 3.5 mm being separated by 15.0 cm.

Sol. In this problem two cases are possible :



Case - I is that 'A' and 'B' have the same displacement amplitude and **Case - II** is that 'C' and 'D' have the same amplitude viz 3.5 mm. If x = 0 is chosen at anti-node then

$$A = a \cos kx$$

if x = 0 is chosen at node, then

$$A = a \sin kx$$

But since nothing is given in question. Hence from the both the cases, result should be same. This is possible only when

$$a \cos kx = a \sin kx$$

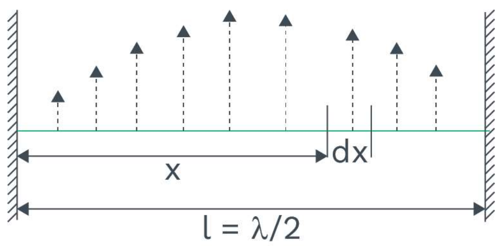
or $kx = \frac{\pi}{4}$

or $a = \frac{A}{\cos kx} = \frac{3.5}{\cos \pi / 4} = 4.95\,\text{mm}$

- **Energy of the standing wave in one loop:-**

When all particles of one loop are at the extreme position then the total energy in the loop is in form of the potential energy only, when particles reaches its mean position then the total potential energy converts into the kinetic energy of the particles. So we can say the total energy of the loop remains constant

Total kinetic energy at the mean position is equal to the total energy of the loop because the potential energy at the mean position is zero.



Small kinetic energy of the particle which is in element dx is

$$d(KE) = \frac{1}{2} dm v^2$$

$$dm = \mu\, dx$$

Velocity of particle at mean position

$$= 2A \sin kx\, \omega$$

then $\quad d(KE) = \frac{1}{2}\mu dx.\, 4A^2\omega^2 \sin^2 kx$

$\Rightarrow \quad d(KE) = 2A^2\omega^2\mu.\, \sin^2 kx\, dx$

$$\int d(K.E.) = 2A^2\omega^2\mu \int_0^{\lambda/2} \sin^2 kx dx$$

Total K.E. =

$$A^2\omega^2\mu \int_0^{\lambda/2} (1 - \cos 2kx)dx = A^2\omega^2\mu \left[x - \frac{\sin 2kx}{2k}\right]_0^{\lambda/2}$$

$$= \frac{1}{2}\lambda A^2\omega^2\mu$$

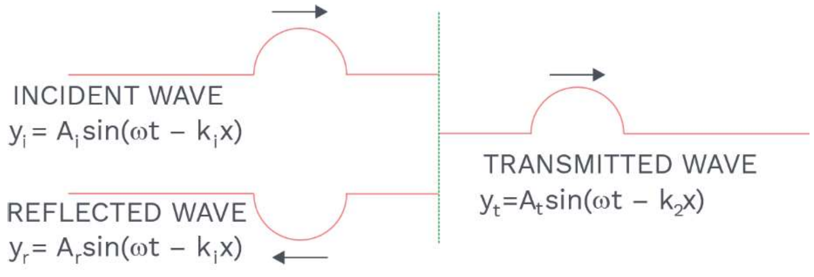
Concept Reminder

(i) $A_r = \frac{v_2 - v_1}{v_1 + v_2} A_i$

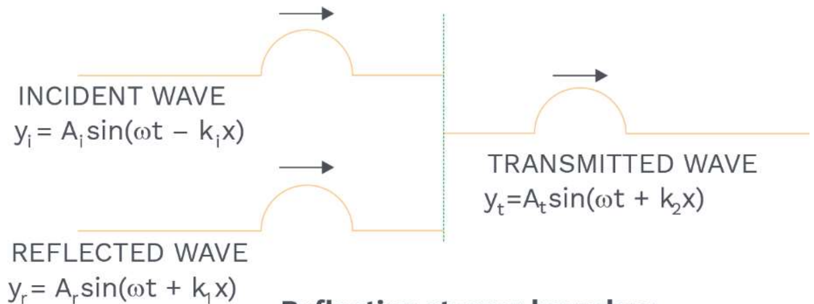
(ii) $A_t = \frac{2v_2}{v_1 + v_2} A_i$

Reflection and Transmission of the Waves

A travelling wave, at the rigid or denser boundary, is reflected with the phase reversal but the reflection at any open boundary (rarer medium) takes place without the any phase change. The transmitted wave is never inverted, but the propagation constant k is changed.



Reflection at denser boundary



Reflection at rarer boundary

Amplitude of Reflected And Transmitted Waves

If A_i is the amplitude of incident wave, A_r be the amplitude of the reflected wave and v_1 and v_2 are speeds of the wave in reflecting and transmitting mediums respectively then.

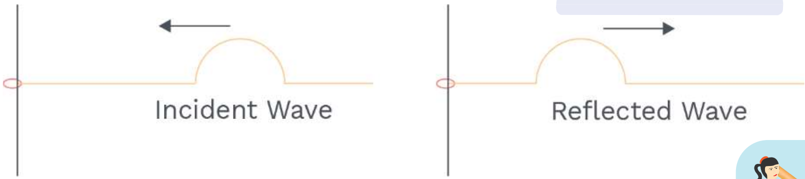
$$\boxed{A_r = \frac{v_2 - v_1}{v_1 + v_2} A_i} \; ; \; \boxed{A_t = \frac{2v_2}{v_1 + v_2} A_i}$$

'A_r' is positive if $v_2 > v_1$ i.e., the wave is reflected from a rarer medium.

Phase change during reflection of waves :

(a) Waves on reflection from a fixed end undergoes a phase change of 180°.

(b) A wave the reflected from a free end is reflected without a change in phase.



Vibration of String :

(a) Fixed at both ends :

Consider a string of length 'L' is kept fixed at the ends x = 0 and x = L. In such the system suppose we send a continuous sinusoidal wave of a fix frequency, say, toward right. When the wave reaches right end. It gets reflected and starts to travel back. The left-going wave then overlaps wave, which is still travelling to right. When the left-going wave reaches left end, it gets reflected again and the newly reflected wave starts to travel to the right. overlapping the left-going wave. This process will go on and, therefore, very

Concept Reminder

The most significant feature of stationary waves is that the boundary conditions constrain the possible wavelengths or frequencies of vibration of the system. The system cannot oscillate with any arbitrary frequency (contrast this with a harmonic travelling wave), but is characterized by a set of natural frequencies or normal modes of oscillation.

soon we will have many overlapping waves, which interfere with the one another. In such a system, at any point x and at any time 't', there are always two waves, one moving to left and another to right. We, therefore, have

$$y_1(x,t) = y_m \sin(kx - \omega t)$$

(wave travelling in positive direction of x-axis)

and $y_2(x,t) = y_m \sin(kx + \omega t)$

(wave travelling in negative direction of x-axis).

The principle of the superposition gives, for the combined wave

$$y'(x,t) = y_1(x,t) + y_2(x,t)$$

$$= y_m \sin(kx - wt) + y_m \sin(kx + \omega t)$$

$$= (2y_m \sin kx) \cos \omega t$$

It is seen that the points of the maximum or minimum amplitude stay at one position.

Nodes : The amplitude is zero for the values of kx that give sin kx = 0 i.e. for,

$kx = n\pi$, for n = 0, 1, 2, 3,.....

Substituting $k = 2\pi / \lambda$ in this equation, we get

$$x = n\frac{\lambda}{2}, \text{ for } n = 0, 1, 2, 3,.....$$

The positions of the zero amplitude are called the nodes. Note that a distance of $\frac{\lambda}{2}$ or half a wavelength separates the two consecutive nodes.

Anti-nodes :

The amplitude has maximum value of $2y_m$, which occurs for values of kx that give |sin kx| = 1.

Those values are as follows

$kx = (n + 1/2)\pi$ for n = 0, 1, 2, 3,....

Putting $k = 2\pi / \lambda$ in this equation, we get.

$$x = (n + 1/2)\frac{\lambda}{2} \text{ for } n = 0, 1, 2, 3,....$$

Concept Reminder

A distance of $\frac{\lambda}{2}$ or half a wavelength separates two consecutive nodes.

Rack your Brain

A uniform string resonates with a tuning fork, at a maximum tension of 32 N. If it is divided into two segments by placing a wedge at a distance one-fourth of length from one end, then to resonance with same frequency the maximum value of tension for string will be

(1) 2 N (2) 4 N

(3) 8 N (4) 16 N

Concept Reminder

Normal modes of a circular membrane rigidly clamped to the circumference as in a table are determined by the boundary condition that no point on the circumference of the membrane vibrates. Estimation of the frequencies of normal modes of this system is more complex. This problem involves wave propagation in two dimensions. However, the underlying physics is the same.

as the positions of maximum amplitude. These are called the anti-nodes. The anti-nodes are separated by $\frac{\lambda}{2}$ and are located half way between the pairs of nodes.

For a stretched string of length 'L', fixed at both ends, the two ends of ends is chosen as position 'x = 0', then other end is x = L. In order that this end is the node; the length 'L' must satisfy condition

$$L = n\,\frac{\lambda}{2}, \text{ for } n = 1, 2, 3,....$$

This condition shows that the standing waves on a string of length 'L' have restricted wavelength given by

$$L = n\frac{\lambda}{2}, \text{ for } n = 1, 2, 3,.....$$

This condition shows that the standing waves on a string of length 'L' have restricted wavelength given by

$$\lambda = \frac{2L}{n}, \text{ for } n = 1, 2, 3,.....$$

The frequencies related to these wavelengths follow from the Eq. as

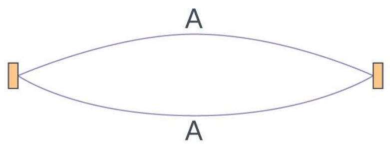
$$f = n\frac{v}{2L}, \text{ for } n = 1, 2, 3,.....$$

where 'v' is the speed of the travelling waves on the string. The set of the frequencies given by the equation are called the natural frequencies or the modes of oscillation of the system.

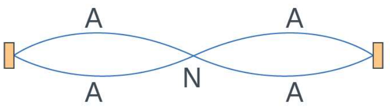
This equation tells us that natural frequencies of a string are the integral multiples of lowest frequency $f = \frac{v}{2L}$,

which corresponds to n = 1. The oscillation mode with that lowest frequency is known as the fundamental mode or first harmonic. The second harmonic or 1^{st} overtone is the oscillation mode with n = 2. The 3^{rd} harmonic and 2^{nd} overtone corresponds to n = 3 and so on. The frequencies linked with these modes are often labelled as ν_1, ν_2, ν_3 and so on. The collection of all possible modes is known as the harmonic series and 'n' is known as the harmonic number.

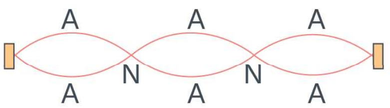
Some of the harmonic of the stretched string fixed at both ends are shown in figure.



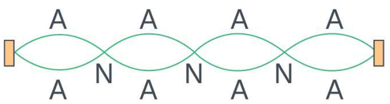
(a)
Fundamental
or first harmonic
$f_0 = v/2L$



(b)
Second harmonic
or first overtone =
$f_1 = 2f_0 = 2v/2L$



(c)
Third harmonic
or 2nd overtone =
$f_2 = 3f_0 = 3v/2L$



(d)
Fourth harmonic
or 3rd overtone =
$f_3 = 4f_0 = 4v/2L$

Ex. A middle 'C' string on a piano has the fundamental frequency 262 Hz, and the 'A' note has fundamental frequency of 440 Hz. (a) Calculate frequencies of next two harmonics of the 'C' string. (b) If the strings for the 'A' and 'C' notes are assumed to have same mass per unit length and same length, find the ratio of tensions in the two strings.

Sol. **(a)** Because f_1 = 262 Hz for the 'C' string, we can use Equation to find the frequencies f_2 and f_3 ;

$$f_2 = 2f_1 = 524\,\text{Hz}$$

$$f_3 = 3f_1 = 786\,\text{Hz}$$

Using Equation for two strings vibrating at their fundamental frequencies gives

$$f_{1A} = \frac{1}{2L}\sqrt{\frac{T_A}{\mu}} \Rightarrow f_{1C} = \frac{1}{2L}\sqrt{\frac{T_C}{\mu}}$$

$$\therefore \quad \frac{f_{1A}}{f_{1C}} = \sqrt{\frac{T_A}{T_C}} \Rightarrow \frac{T_A}{T_C} = \left(\frac{f_{1A}}{f_{1C}}\right)^2 = \left(\frac{440\,\text{Hz}}{262\,\text{Hz}}\right)^2 = 2.82$$

Ex. A wire having linear mass density of 5.0×10^{-3} kg/m is stretched between the two rigid supports with the tension of 450 N. The wire resonates at frequency of 420 Hz. The next higher frequency at which same wire resonates is 490 Hz. Determined the length of the wire.

Sol. Assume the wire vibrates at 420 Hz in its n^{th} harmonic and at 490 Hz in its $(n + 1)^{th}$ harmonic.

$$420\,s^{-1} = \frac{n}{2L}\sqrt{\frac{F}{\mu}} \qquad \text{.... (i)}$$

and $$490\,s^{-1} = \frac{(n+1)}{2L}\sqrt{\frac{F}{\mu}} \qquad \text{... (ii)}$$

This gives $\frac{490}{420} = \frac{n+1}{n}$

or $n = 6$.

Putting the value in (i),

$$420\,s^{-1} = \frac{6}{2L}\sqrt{\frac{450\,N}{5.0\times10^{-3}\,kg/m}} = \frac{900}{L}\,m/s$$

or $$L = \frac{900}{420}\,m = 2.1\,m$$

(b) Fixed at one end :

Standing waves can be produced on the string which is fixed at the one end & whose other end is free to move in the transverse direction. Such a free end can be nearly obtained by connecting the string to a very light thread.

If the vibrations are produced by a source of "correct" frequency, standing waves are produced. If the end x = 0 is fixed and x = L is free, the equation is again given by

$$y = 2A\sin kx\cos\omega t$$

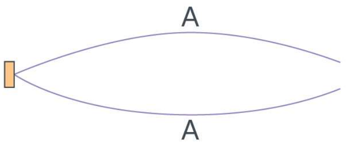
with boundary condition that "(x = L)" is anti-node. The boundary condition that "(x = 0)" is a node is automatically satisfied by above equation. For "(x = L)" to be anti-node,

$$\sin kL = \pm 1$$

or, $kL = \left(n+\frac{1}{2}\right)\pi$ or, $\frac{2\pi L}{\lambda} = \left(n+\frac{1}{2}\right)$

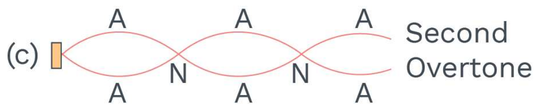
or, $\frac{2Lf}{v} = n+\frac{1}{2}$ or, $f = \left(n+\frac{1}{2}\right)\frac{v}{2L} = \frac{n+\frac{1}{2}}{2L}\sqrt{T/\mu}$

These are the normal frequencies of the vibration. The fundamental frequency is achieved when n = 0, i.e., $f_0 = \frac{v}{4L}$

(a) 

Fundamental

The overtone frequencies are

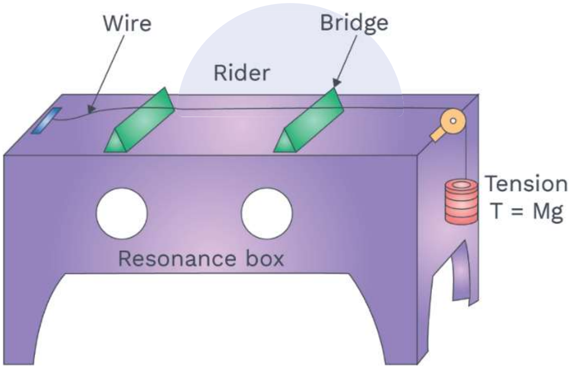


$$f_1 = \frac{3v}{4L} = 3f_0 \quad ; \quad f_2 = \frac{5v}{4L} = 5f_0$$

We see that all the harmonic of fundamental are not allowed frequencies for standing waves. Only the odd harmonics are overtones. Figure shows the shapes of string for some of the normal modes.

Sonometer :

Sonometer consists of the hollow rectangular box of the light wood. One end of experimental wire is fastened to the one end of the box. The wire passes over a frictionless pulley at the other end of box. The wire is stretched by a tension 'T'.



The box serves the purpose of the increasing loudness of the sound produced by vibrating wire. If the length wire between the two bridges is 'l', then the frequency of vibration is

$$f = \frac{1}{2l}\sqrt{\frac{T}{\mu}}$$

To test the tension of tuning fork and the string, a small paper rider is placed on string. When the vibrating tuning fork is placed on a box, and if the length between bridges is properly adjusted, then when the 2 frequencies are exactly equal, the string quickly picks up vibrations of the fork and rider is thrown off the wire.

Comment :

$$\mu = \frac{\text{mass of wire}}{\text{length of wire}} = \frac{\pi r^2 dl}{l} = \pi r^2 d$$

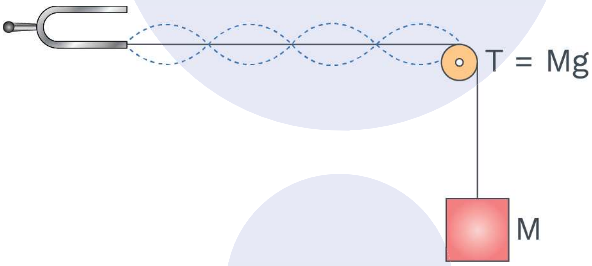
where 'r' is the radius of wire and 'd' is the density of the material of the wire. Thus the frequency of vibration of a given string under tension is

$$f \propto \frac{1}{\sqrt{r^2 d}}$$

Thus $f \propto \frac{1}{r}$ (for same material wires)

and $f \propto \frac{1}{\sqrt{d}}$ (for different wires of same radius).

Melde's Experiment :



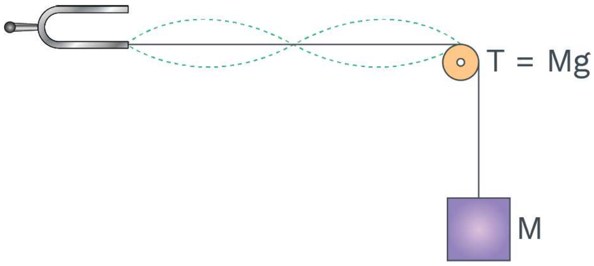
Case 1. In a vibrating string of the fixed length, the product of number of loops in a vibrating string and square root of tension is a constant or

$$n\sqrt{T} = \text{constant}$$

Case 2. When tuning fork is set vibrating as shown in the fig. then the prong vibrates at the right angles to the thread. As a result thread is set into the motion. The frequency of vibration of thread (string) is equal to the frequency of tuning fork. If the length and the tension are properly adjusted then, the standing waves are formed in the string. (This happens when the frequency of one of the normal modes of the string matched with the frequency of the tuning fork). Then, if n loops are formed in the thread, then the frequency of the tuning fork is given by

$$f = \frac{n}{2l}\sqrt{\frac{T}{\mu}}$$

Case 3. If the tuning fork is turned through the right angle, so that the prong vibrates along length of the thread, then string performs only a half oscillation for each complete vibrations of the prong. This is because the thread sags only when the prong moves towards the pulley i.e. only once in a vibration.



The thread performs the sustained oscillations when the natural frequency of given length of the thread under tension is half that of fork. Thus, if 'n' loops are formed in the thread, then the frequency of the tuning fork is $f = \frac{2n}{2l}\sqrt{\frac{T}{\mu}}$

EXAMPLES

Q1 **Consider the wave $y = (10\ mm)\ \sin[(5\pi\ cm^{-1})\ x - (60\pi\ s^{-1})\ t + \frac{\pi}{4}]$. Find**

(a) the amplitude
(b) the wave number
(c) the wavelength
(d) the frequency
(e) the time period
(f) the wave velocity
(g) phase constant of SHM of particle at x = 0.

Sol:

(a) Amplitude $A = 10$ mm

(b) wavenumber $k = 5\pi\ cm^{-1}$

(c) wavelength $\lambda = \frac{2\pi}{k} = \frac{2}{5}$ cm

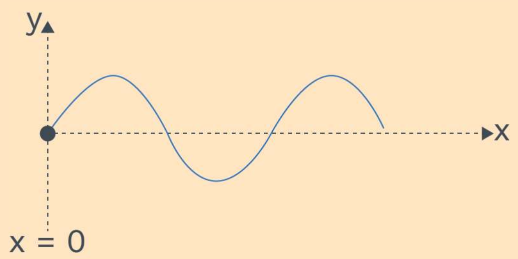
(d) frequency $f = \frac{\omega}{2\pi} = 30$ Hz

(e) time period $T = \frac{1}{f} = \frac{1}{30}$ s

(f) wave velocity $v = f\lambda = 12$ cm/s.

Q2 **The string shown in the figure is driven at the frequency of 5.00 Hz. The amplitude of motion is 12.0 cm, and wave speed is 20.0 m/s. Furthermore, wave is such that y = 0 at x = 0 and t = 0. Determine**

(a) the angular frequency
(b) the wave number for this wave
(c) Give an expression for the wave function. Calculate
(d) the maximum transverse speed
(e) the maximum transverse acceleration of point on the string.



Sol: (a) $\omega = 2\pi f = 10\pi$ Red/sec.

(b) $\lambda \times 5 = 20 \quad \lambda = 4$ m, $\quad K = \frac{2\pi}{\lambda} = \frac{\pi}{2}$ rad/m.

(c) $y = 12 \times 10^{-2} \sin(\omega t - kx + \phi)$

$$= 12 \times 10^{-2} \sin\left(10\pi t - \frac{\pi}{2}x + \phi\right)$$

At $t = 0 \quad x = 0$ and $y = 0$

$0 = \sin\phi$

$\therefore \quad \phi = 0$

$$\frac{\partial y}{\partial v} = 12 \times 10^{-2}\left(-\frac{\pi}{2}\right)\cos\left(100\pi t - \frac{\pi}{2}x + \phi\right)$$

At $t = 0 \quad x = 0$

$$\frac{\partial y}{\partial x} = -12 \times 10^{-2}\left(\frac{\pi}{2}\right)\cos\phi$$

$\frac{\partial y}{\partial x}$ should be positive

$\therefore \quad \phi = \pi$

$$y = 12 \times 10^{-2} \sin\left(10\pi t - \frac{\pi}{2}x + \pi\right)$$

$$= 12 \times 10^{-2} \sin\left(\frac{\pi}{2}x - 10\pi t\right).$$

(d) $v_{max} = A\omega$

$= 12 \times 10^{-2} \times 10\pi$

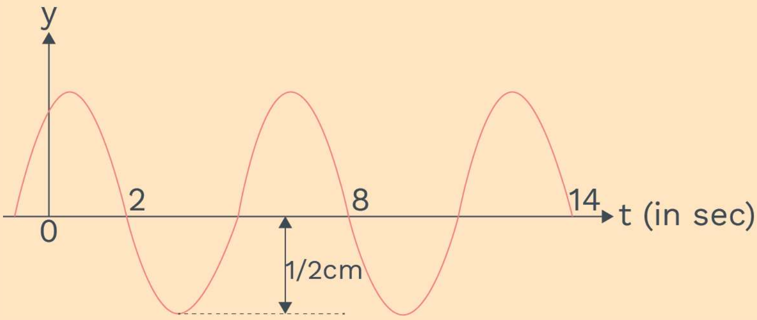
$= 1.2\pi = \frac{6}{5}\pi.$

(e) $A_{max} = A\omega^2$

$= 12 \times 10^{-2} \times 10\pi \times 10\pi$

$= 12\pi^2.$

Q3 **The sketch in the figure shows displacement time curve of a sinusoidal wave at x = 8 m. Taking velocity of wave v = 6 m/s along positive x-axis, write the equation of the wave.**



Sol:

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{8-2} = \frac{\pi}{3}$$

$$\therefore \quad K = \frac{\omega}{v} = \frac{\pi}{3\times 6} = \frac{\pi}{18}$$

$$\therefore \quad y = 0.5 \sin\left[\frac{\pi}{3}t - \frac{\pi}{18}x + \phi\right]$$

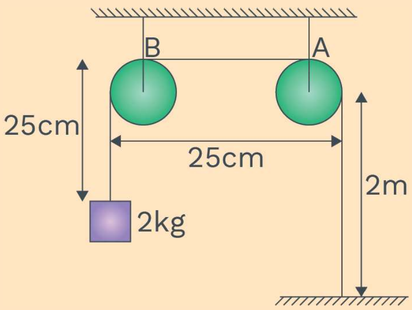
at t = 2, x = 8, y = 0

$$\therefore \quad \phi = \frac{7\pi}{9}$$

$$\therefore \quad y = 0.5 \sin\left[\frac{\pi}{3}t - \frac{\pi}{18}x + \frac{7\pi}{9}\right].$$

Q4

In the arrangement shown in figure, the string has mass of 5 g. How much time it will take for a transverse disturbance produced at floor to reach the pulley 'A' ? Take g = 10 m/s^2.



Sol: $v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{20 \times 2.5}{5 \times 10^{-3}}}$ = 100 m/sec

$t = \frac{D}{v} = \frac{2}{100} = \frac{1}{50}$ sec.

Q5 **A uniform rope of the length 20 m and mass 8 kg hangs vertically from the rigid support. A block of 2 kg is attached to free end of the rope. A transverse pulse of the wavelength 0.06 m is produced at the lower end of rope. Find the wavelength of the pulse when it reaches the top of the rope ?**

Sol: At the bottom end

$v = \frac{mg}{\mu} = f\,\lambda$ ...(i)

Now at the top

$v_1 = \sqrt{\frac{mg + \mu \ell g}{\mu}} = f\,\lambda'$ ...(ii)

μl = M = mass of string

From equation (i) & (ii)

$\lambda' = \sqrt{\frac{m + M}{m}}\lambda = \sqrt{\frac{2 + 8}{2}}\lambda = \sqrt{5}\,\lambda = \frac{3}{10\sqrt{5}}$ m.

Q6 **Two wires of the different densities but same area of the cross-section are soldered together at the one end and are stretched to a tension 'T'. The velocity of the transverse wave in the 1st wire is half of that in the 2nd wire. Determine the ratio of density of the first wire to that of the second wire.**

Sol: $v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{T}{A\rho}}$

$\frac{v_1}{v_2} = \sqrt{\frac{\rho_2}{\rho_1}} = \frac{1}{2}$

$\frac{\rho_2}{\rho_1} = \frac{1}{4} \Rightarrow \frac{\rho_1}{\rho_2} = 4.$

Q7 **A 6.00 m segment of the long string has a mass of 180 g. A high-speed photograph shows that the segment has four complete cycles of a wave. The string is vibrating sinusoidally with the frequency of 50.0 Hz and the peak-to-valley displacement of 15.0 cm. (The "peak-to-valley" displacement is vertical distance from the farthest positive displacement to farthest negative displacement.)**
(a) Write the function that describes the wave traveling in the positive 'x' direction.
(b) Determine average power being supplied to the string.

Sol: (a) f = 50 Hz

$\mu = \dfrac{180 \times 10^{-3}}{6} = 3 \times 10^{-2}$ Kg/m

$2A = 15 \times 10^{-2}$ m

$A = 7.5 \times 10^{-2}$ m

$4\lambda = 6$ $\lambda = 3/2$ m and $v = f\lambda = 75$ m/sec.

$\omega = 2\pi f = 100\pi$ also $K = \dfrac{\omega}{v} = \dfrac{4\pi}{3}$

If phase constant is ϕ then

$\therefore$ equation is $y = 7.5 \times 10^{-2} \sin\left[100\pi t - \dfrac{4\pi}{3}x + \phi\right]$.

Q8 **A transverse wave of amplitude 5 mm and the frequency 10 Hz is produced on the wire stretched to a tension of 100 N. If wave speed is 100 m/s, find the average power transmitted by the source to the wire ? ($\pi^2 = 10$)**

Sol: $P_{av} = 2\pi^2 f^2 A^2 \mu v$

$P_{av} = 2\pi^2 f^2 A^2 \mu \sqrt{T/\mu}$

Put value $P_{av} = 50$ mw

use $v = \sqrt{T/\mu}$

Q9 **The equation of a plane wave travelling along positive direction of x-axis is $y = a \sin \frac{2\pi}{\lambda}(\nu t - x)$ When this wave is reflected at the rigid surface and its amplitude becomes 80%, then find the equation of reflected wave**

Sol: Equation of reflected wave is

$$y = \left(\frac{80\,a}{100}\right)\sin\frac{2\pi}{\lambda}\left(\upsilon t + x + \frac{\lambda}{2}\right)$$

($\frac{\lambda}{2}$ path difference is added as the reflection is from hard rigid surface)

Q10 **Two waves, each having the frequency of 100 Hz and the wavelength of 2 cm, are travelling in same direction on a string. Calculate the phase difference between the waves**

(a) if the 2nd wave was produced 10 m sec later than the 1st one at the same place

(b) if two waves were produced at a distance of 1 cm behind 2nd one ?

(c) If each of the waves has the amplitude of 2.0 mm, what would be the amplitudes of resultant waves in part (a) and (b) ?

Sol: $v = f\lambda$

$v = 100 \times 2 = 200 \frac{cm}{sec}$

(a) $\frac{2\pi}{T} \times \Delta t = \frac{2\pi \times 0.01}{0.01} = 2\pi$

(b) $\phi = \frac{2\pi}{\lambda}\Delta t = \frac{2\pi}{2} = \pi$

(c) $a = a_1 + a_2 = 4$ mm

$a = a_1 - a_2 = 0$

Q11 **What are (i) the lowest frequency (ii) the 2nd lowest frequency and (iii) the 3rd lowest frequency for the standing waves on a wire that is 10.0 m long has mass of 100 g and is stretched under tension of 25 N which is fixed at the both ends ?**

Sol: $v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{25}{\frac{100 \times 10^{-3}}{10}}} = 50$

$\therefore \quad \nu_0 = \frac{v}{2L} = \frac{50}{2 \times 10} = 2.5 \text{ Hz}$

$\therefore \quad \nu_1 = 2\nu_0 = 5 \text{ Hz}$

$\nu_2 = 3\nu_0 = 7.5 \text{ Hz.}$

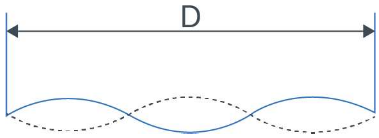
Q12 **A nylon guitar string has the linear density of 7.20 g/m and is under the tension of 150 N. The fixed supports are at distance D = 90.0 cm apart. The string is oscillating in standing wave pattern as shown. Calculate the**

(a) speed

(b) wavelength

(c) frequency of traveling waves whose superposition gives this standing wave.

Sol:

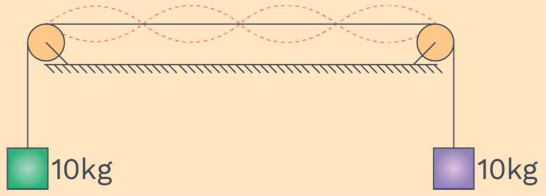


(a) $v = \sqrt{\frac{150}{7.2 \times 10^{-3}}} = \frac{250}{\sqrt{3}} \frac{m}{sec} \simeq 144 \text{ m/s}$

(b) $\frac{3\lambda}{2} = 90 \text{ cm} \quad \Rightarrow \lambda = 60 \text{ cm}$

(c) $v = f\lambda \quad \Rightarrow f = \frac{v}{\lambda} = \frac{250 \times 100}{\sqrt{3} \times 60} = \frac{1250}{3\sqrt{3}} \text{ Hz.}$

Q13 **The length of the wire shown in the figure between the pulleys is 1.5 m and its mass is 15 g. Determine the frequency of the vibration with which the wire vibrates in the four loops leaving the middle point of the wire between pulleys at rest. ($g = 10\ m/s^2$).**



Sol: $\frac{4\lambda}{2} = 1.5$

$\lambda = 0.75$ m

$v = \sqrt{\frac{10 \times 10 \times 1.5}{15 \times 10^{-3}}} = 100$ m/s

$f = \frac{v}{\lambda} = \frac{100}{0.75} = \frac{400}{3}$ Hz.

Q14 **A string oscillates according to the equation**

$$y' = (0.50\ \text{cm}) \sin \left[\left(\frac{\pi}{3}\text{cm}^{-1}\right)x\right] \cos [(40\pi\ s^{-1})t].$$

What are the

(a) Amplitude

(b) Speed of two waves (identical except for the direction of travel) whose superposition gives this oscillation?

(c) What is distance between nodes?

(d) What is transverse speed of a particle of string at the position x = 1.5 cm when $t = \frac{9}{8}$ s?

Sol: (a) $A_s\omega = 0.5 \sin \frac{\pi}{3} xt$

$= 0.5 \sin \frac{\pi}{3} (1.5 = 0.5$ cm

But the amplitude of component wave is A

(b) $2A = 0.5$

$A = 0.25$ cm

$K = \frac{\pi}{3}$ and $\omega = 40\pi \Rightarrow f = 20$ Hz

$\therefore \quad v = \frac{\omega}{K} = 120$ cm/sec

(c) $d = \frac{\lambda}{2} = \frac{v}{2f} = 3$ cm

(d) At x = 1.5 cm

At $t = \frac{9}{8}$ sec

$$y = 0.5 \sin\left(\frac{\pi}{3} \times 1.5\right) \cos\left(40\pi \times \frac{9}{8}\right)$$

$$= 0.5 \sin\frac{\pi}{2} \cos 45\pi$$

$$= 0.5(-1) = -0.5$$

So, particle is at negative extreme position that is why speed is zero.

Q15 **The vibration of the string of length 60 cm is represented by equation, y = 3 cos (πx/20) cos (72 πt) where 'x' and 'y' are in cm and 't' in sec.**
(i) Write down component waves whose superposition gives above wave.
(ii) Give the position of the nodes and anti-nodes located along the string.
(iii) What is the velocity of particle of the string at position x = 5 cm and t = 0.25 sec.

Sol:

(i) $y_1 = 1.5 \cos\left(\frac{\pi x}{20} + 72\pi t\right)$

$y_2 = 1.5 \cos\left(\frac{\pi x}{20} - 72\pi t\right)$

(ii) Nodes $\Rightarrow$

$$\frac{\pi x}{20} = (2n+1)\frac{\pi}{2}$$

$x = (2n + 1)\,10 \Rightarrow x = 10, 30, 50$ ....

(iii) Antinodes $\Rightarrow \frac{\pi x}{20} = n\pi \Rightarrow x = 20n \Rightarrow x = 0, 20, 40, 60$ ....

MIND MAP

Wave on a String

Power Transmission in Travelling Wave on String

$P = -T\left(\frac{\delta y}{\delta t}\right)\left(\frac{\delta y}{\delta x}\right)$ valid for all waves on string

$\langle P \rangle = \frac{1}{2}\sqrt{T\mu}\,A^2\omega^2$

Velocity of Wave on String

$v_W = \sqrt{\frac{T}{\mu}}$

μ = mass per unit length

General Equation of Standing Wave

$y = 2A\sin(kx + \phi_1)\sin(\omega t + \phi_2)$

Amplitude of component wave = A

Coefficeient of x = k

Wavelength of component waves = $\frac{2\pi}{k}$

Loop length = $\frac{\lambda}{2} = \frac{\pi}{k}$

Velocity and Acceleration of Particle

$v_P = \frac{\delta y}{\delta t} = \omega\sqrt{A^2 - y^2}$

$y = A\sin(\omega t - kx)$ → Displacement

$a_P = \frac{\delta v_P}{\delta t} = -\omega^2 y$

Waves

- **Mechanical Waves**
  Medium is required
  Waves on Stirng.
  Sound Wave
- **Non-Mechanical Waves**
  Medium is not required
  Light Wave

Mechanical waves are of two types:

- **Transverse Wave**
  Constituents of the medium oscillates perpendicular to the direction of wave propagation.
  Waves on String
- **Longitudinal Wave**
  Constituents of the medium oscillates along the direction of wave propagation.
  Sound Wave

Relation between Particle Velocity & Wave Velocity

$v_p = -v_W\left(\frac{\delta y}{\delta x}\right)$ → Slope of the string at point x.

Wave function $y = f(x,t)$

Velocity of wave $v_W = -\frac{\text{Coefficient of } t}{\text{Coefficient of } x}$

Sinusoidal Wave Equation

$y = A\sin(\omega t - kx + \phi)$

$\lambda = \frac{2\pi}{k}$ $\quad T = \frac{2\pi}{\omega}$

Angular wave number, $k = \frac{2\pi}{\lambda}$