Simple Harmonic Motion

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PERIODIC MOTION:

- When a body or a moving particle repeat its motion along a definite path after a regular interval of time (T) its motion is said to be Periodic Motion and interval of time is called time period (T). The path of motion may be circular, linear, elliptical or any other path. For example rotation of earth and an object moving in a vertical circular plane.

OSCILLATORY MOTION:

- To and fro type of motions is called oscillatory Motion. A particle has oscillatory motion when it moves about stable equilibrium position. It need not to be periodic and need not have fixed extreme positions.
- The oscillatory motion in which energy conserved are also periodic. For example: motion of pendulum of wall clock.
- The force and torque (directed towards equilibrium point) acting in oscillatory motion is called restoring force (F) and torque. Damped oscillations are those in which energy (E) consumed due to some resistive forces and total mechanical energy (T.M.E.) decreases and after some time oscillation will stops.
- **Oscillatory Equation:** Consider a particle free to move on x-axis is being acted upon by a force given by

  $F = -kx^n$

  Above equation is called oscillatory equation. Here k is a positive constant and x is the displacement from mean position.

  Now following cases are possible depending on the value of n.

  (i) If n is an even integer (0, 2, 4......... etc) force is always along negative x-axis whether x is positive or negative Hence, the motion of the particle is not oscillatory. If the particle is released from any position on the x-axis (except x = 0) a force in –ve direction

KEY POINTS

- Periodic Motion
- Oscillatory motion

Definitions

When a body or a moving particle repeats its motion along a definite path after regular intervals of time its motion is said to be Periodic Motion.

Definitions

To and Fro motion of a particle about its equilibrium position is known as oscillatory motion.

of x-axis acts on it and it moves rectilinearly along – ve x axis.

(ii) If n is an odd integer (1, 3, 5 .... etc), force is along – ve x-axis for $x > 0$ and along +ve x-axis for $x < 0$ and zero for $x = 0$. Thus the particle will oscillate about stable equilibrium position $x = 0$. The force in this case is called the restoring force.
If $n = 1$ i.e., $F = -kx$ the motion is said to be SHM (Simple Harmonic Motion)
If the restoring force/torque acting on the body in oscillatory motion is directly proportional to the displacement of body/particle w.r.t. mean position and is always directed towards equilibrium position then the motion is called Simple Harmonic motion. It is the simplest form of oscillatory motion.

Definitions

If the restoring force/torque acting on the body in oscillatory motion is directly proportional to the displacement of body/particle w.r.t. mean position and is always directed towards equilibrium position then the motion is called Simple Harmonic motion.

HARMONIC FUNCTIONS:

The function of constant amplitude and single frequency is define as harmonic function. (Among all the trigonometrical functions only 'sin' and 'cos' functions are taken as harmonic function in basic form)

$$\left.\begin{aligned} Y = A\sin\theta = A\sin\omega t \\ y = A\cos\theta = A\cos\omega t \end{aligned}\right\} \text{Harmonic function}$$

Note: The function will be non-harmonic if-

(i) Its amplitude is not constant.

(ii) It is basically formed by "tan", "cot", "sec", "cosec" functions.

SIMPLE HARMONIC MOTION

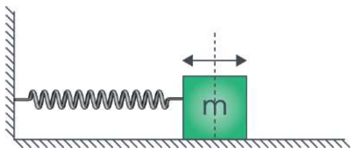
Simple harmonic motion is the simplest form of vibratory or oscillatory motion.

Types of SHM: SHM are of two types:

1. **Linear SHM:** To and fro motion about a fixed point along a straight line is known as linear SHM.

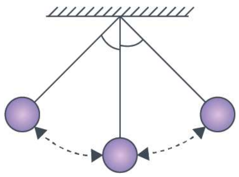
Rack your Brain

If a particle is executing simple harmonic motion, then acceleration of particle:
(1) Is uniform
(2) Varies linearly with time
(3) Is non uniform
(4) Both (2) & (3)



Ex. (i) Motion of block connected to a spring.
(ii) Motion of piston in automobiles.

2. **Angular SHM:** Angularly to and fro motion about a fixed position or axis is known as angular SHM.



Ex.: Motion of bob of simple pendulum.

Concept Reminder

There are basically two types of SHM-
(i) Linear SHM
(ii) Angular SHM

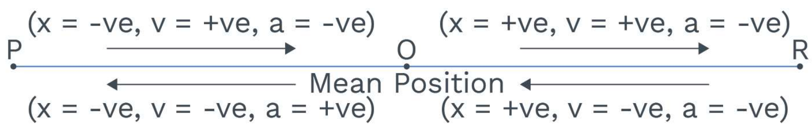
Necessary Condition to execute S.H.M.:

(a) Motion of particle should be oscillatory.
(b) Total mechanical energy of particle should be conserved (Kinetic energy + Potential energy = constant)
(c) Extreme positions should be well defined.

Linear SHM: In linear SHM restoring force or acceleration acting on the particle should always be proportional to the displacement of the particle and directed towards the equilibrium position.

$F \alpha -x \Rightarrow F = -kx$

Here: Negative sign. shows the opposite direction.



-A		+A
$x_{max} = -A$	$x_{min} = 0$	$x_{max} = +A$
$F_{max} = kA$	$F_{min} = 0$	$F_{max} = -kA$
$a_{max} = \frac{k}{m}A$	$a_{min} = 0$	$a_{max} = \frac{-k}{m}A$

where, F = Restoring force
x = displacement from mean position
k = restoring force constant

Some basic terms:

Mean Position: At mean position restoring force/ torque on the particle is zero and its potential energy is minimum, is known as its mean position (x = 0).

Restoring Force:

- The force acting on the object which wants to bring the object towards its mean position (M.P.), is known as restoring force (F).
- This force is always directed towards the mean position.
- Restoring force always acts in a direction opposite to that of displacement. Displacement is measured from the mean position.
- It is given by F = –kx and has dimension $[MLT^{-2}]$.

Amplitude:

- The maximum displacement of particle from mean position is defined as amplitude.

Time period (T):

- The time after which the particle keeps on repeating its motion is known as time period.
- It is given by $T = \frac{2\pi}{\omega}$, $T = \frac{1}{n}$ where ω is angular frequency and n is frequency.

Oscillation or Vibration

When a particle goes on one side from mean position and returns back and then it goes to other side and again returns back to mean position, then this process is known as one oscillation.

Note: Oscillation ⇒ low frequency, vibration ⇒ high frequency.

When a particle moves in sequence O → R → O → P → O, then it completes one oscillation.

Definitions

The force acting on the particle which tends to bring the particle towards its mean position, is known as restoring force.

Definitions

The time after which the particle keeps on repeating its motion is known as time period.

Concept Reminder

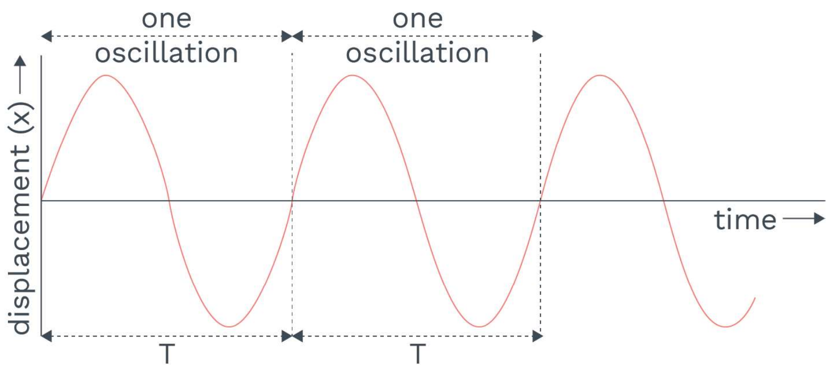
There is no significant difference between oscillations and vibrations. It seems that when the frequency is small, we call it oscillation (like the oscillation of a branch of a tree), while when the frequency is high, we call it vibration (like the vibration of a string of a musical instrument)

Where,

O = Mean position

R = Positive extreme

P = Negative extreme



Note:

(i) In one oscillation

Distance travelled = 4A and Displacement = 0

(ii) Length of line of SHM = 2A

Frequency (n or f)

(a) The number of oscillations per second is defined as frequency.

(b) It is given by $n = \frac{1}{T}$, $n = \frac{\omega}{2\pi}$

(c) **SI Unit:** Hertz (Hz)

1 Hertz = 1 cycle per second (cycle is a number not a dimensional quantity).

(d) **Dimension:** $[M^0L^0T^{-1}]$

Angular frequency (w)

- The rate of change of phase angle of a particle with respect to time is defined as its angular frequency.
- **SI UNIT:** radian/second.

Dimension: $[M^0L^0T^{-1}]$

Differential equation of linear SHM:

For linear SHM

$F \propto -x$

KEY POINTS

- Restoring force
- Frequency
- Time period
- Angular frequency

Definitions

The rate of change of phase angle of a particle with respect to time is defined as its angular frequency.

$\Rightarrow$ $F = -kx$

$\Rightarrow$ $ma = -kx$

$\Rightarrow$ $a = -\frac{k}{m}x$

$\Rightarrow$ $\frac{d^2x}{dt^2} = -\frac{k}{m}x$

$\Rightarrow$ $\boxed{\frac{d^2x}{dt^2} + \frac{k}{m}x = 0}$ (Differential equation of linear SHM.)

$\Rightarrow$ $\boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0}$ $\left\{\because \omega^2 = \frac{k}{m}\right\}$

(Standard differential equation)

After solving the above differential equation.

We get, $x = A\sin(\omega t + \phi)$ → (General equation of SHM)

Where,

x → Displacement from mean

A → Amplitude

ω → Angular frequency

$(\omega t + \phi)$ → Phase

ϕ → Initial phase

Phase: Phase is a parameter which tells about the direction of motion and displacement of oscillating particle.

$$x = A\sin\underbrace{(\omega t + \phi)}_{\text{Phase}}$$

Initial phase: Phase at t = 0 is known as initial phase or epoch.

SHM AS A PROJECTION OF UNIFORM CIRCULAR MOTION

Consider a particle Q, moving on a circle of radius A with constant angular velocity ω. The projection of Q on a diameter BC is P. It is clear from the figure that as Q moves around the circle the projection P excecutes a simple harmonic motion on the x-axis between B and C. The angle that the radius OQ makes with the +ve vertical in

Concept Reminder

Differential equation of linear SHM is

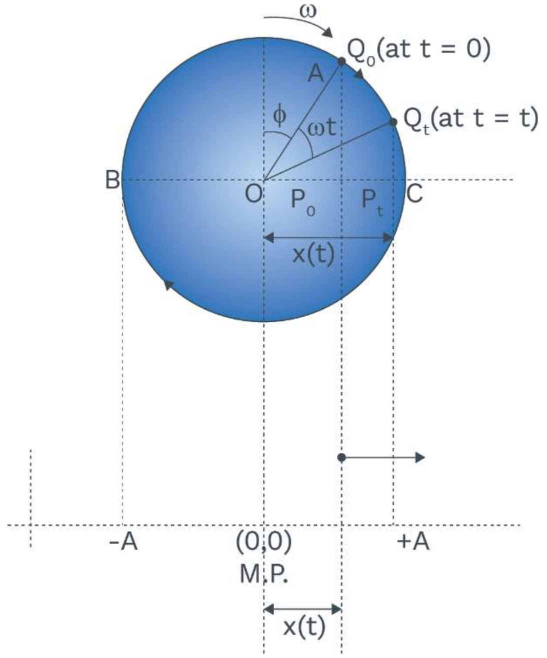
$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

Where $\omega = \sqrt{\frac{k}{m}}$.

clockwise direction in at t = 0 is equal to phase constant (ϕ).

Let the radius OQ_0 makes an angle ωt with the OQ_t at time t. Then

$$x(t) = A \sin(\omega t + \phi)$$



In the above discussion the foot of projection is x-axis so it is called horizontal phasor. Similarly the foot of perpendicular on y axis will also executes SHM of amplitude A and angular frequency ω[$y(t) = A \cos \omega t$]. This is called vertical phasor. The phaser of the two SHM differ by $\pi/2$.

Problem solving strategy in horizontal phasor:

(1) First assume circle of radius equal to amplitude of S.H.M.

(2) Assume a particle rotating in a circular path moving with constant ω same as that of S.H.M in clockwise direction.

(3) Angle made by the particle at t = 0 with the upper vertical is equal to phase constant.

(4) Horizontal component of velocity of particle gives you the velocity of particle performing SHM for example.

Concept Reminder

The method of reference circle can be used for obtaining instantaneous velocity and instantaneous acceleration of a particle undergoing SHM.

Rack your Brain

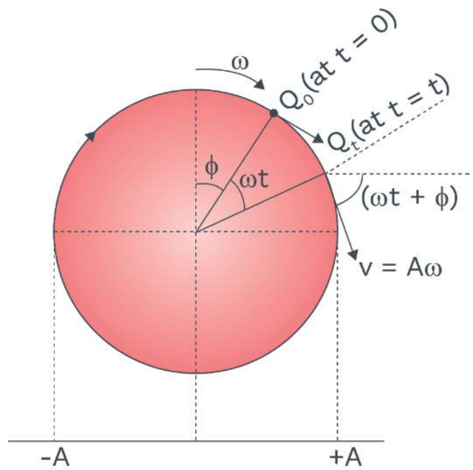
Select wrong statement about simple harmonic motion:

(1) The body is uniformly accelerated

(2) The velocity of the body changes smoothly at all instants

(3) The amplitude of oscillation is symmetric about the equilibrium position

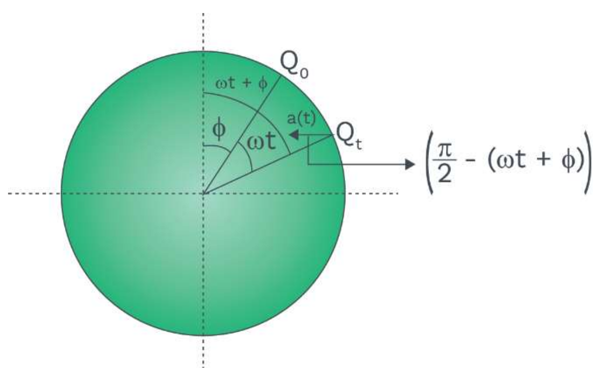
(4) The frequency of oscillation is independent of amplitude



From figure,

$$v(t) = A\omega\cos(\omega t + \phi)$$

(5) Component of acceleration of particle in horizontal direction is equal to the acceleration of particle performing S.H.M. The acceleration of a particle in uniform circular motion is only centripetal and has a magnitude $a = \omega^2 A$.



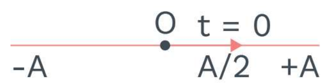
From figure,

$$a(t) = -\omega^2 A\sin(\omega t + \phi)$$

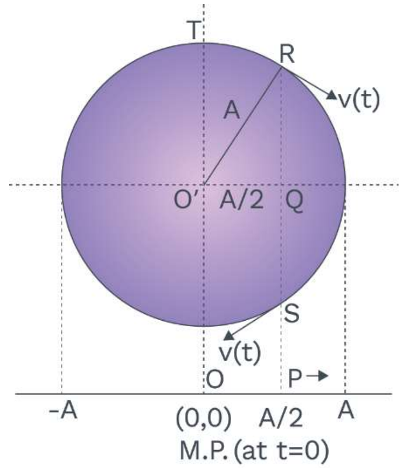
Ex. A particle starts from A/2 and moves towards positive extreme as shown below. Find the equation of the SHM. Given amplitude of SHM is A.

Concept Reminder

Very often the body undergoing periodic motion has an equilibrium position somewhere inside its path. When the body is at this position no net external force acts on it. Therefore, if it is left there at rest, it remains there forever. If the body is given a small displacement from the position, a force comes into play which tries to bring the body back to the equilibrium point, giving rise to oscillations or vibrations.



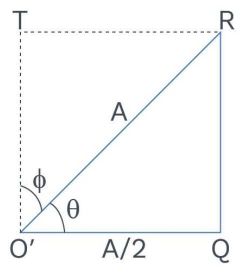
Sol. We will solve the above problem with the help of horizontal phasor.



Step-1: Draw a perpendicular line in upward direction from point P on the circle which cuts it at point R & S.

Step-2: Horizontal compoment of v(t) at R gives the direction P to A while at S gives P to O. So at t = 0 particle is at R.

Step-3: In DO' RQ

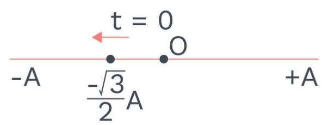


$$\cos\theta = \frac{A/2}{A} = \frac{1}{2} \Rightarrow \theta = 60° \text{ and } \phi = 30°$$

So equation of the SHM is

$$x = A \sin(\omega t + 30°)$$

Ex. A particle starts from point $x = \frac{-\sqrt{3}}{2}A$ and move towards negative extreme as shown.

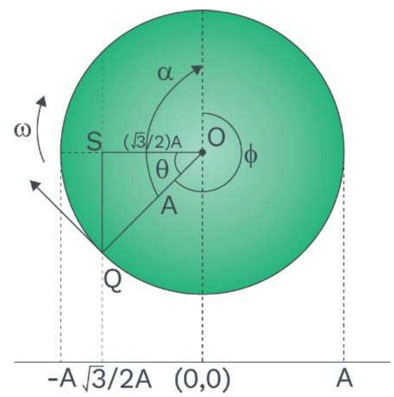


(a) Find the equation of the SHM.

(b) Find the time taken by the particle to go directly from its initial position to negative extreme (x = – A).

(c) Find the time taken by the particle to reach at mean position (x = 0).

Sol. (a) Diagram shows the solution of the problem with the help of phasor.



Horizontal component of velocity at Q gives the required direction of velocity at t = 0.

In ΔOSQ,

$$\cos\theta = \frac{\sqrt{3}/2A}{A} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad \theta = \frac{\pi}{6}$$

Now, $\phi = \frac{3\pi}{2} - \frac{\pi}{6} = \frac{8\pi}{6} = \frac{4\pi}{3}$

So, equation of SHM is

$$x = A\sin\left(\omega t + \frac{4\pi}{3}\right)$$

(b) Now to reach the particle at left extreme (x = – A) point it will travel angle θ along the circle. So, time taken.

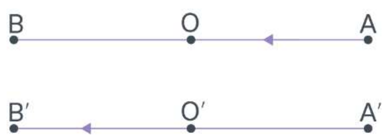
$$t = \frac{\theta}{\omega} = \frac{\pi}{6\omega} \quad \Rightarrow t = \frac{T}{12}\text{sec}$$

(c) To reach the particle at mean position (x = 0) it will travel an angle

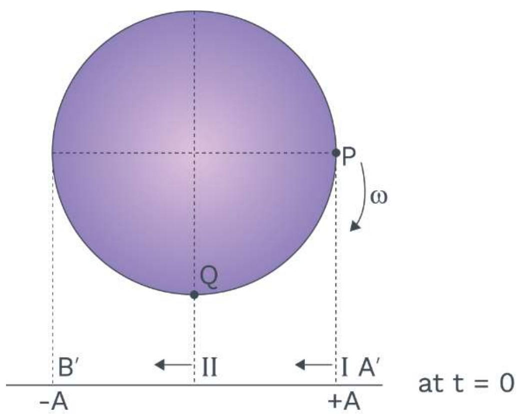
$$\alpha = \frac{\pi}{2} + = \frac{\pi}{6} = \frac{2\pi}{3}$$

So, time taken = $\frac{\alpha}{\omega} = \frac{T}{3}$ sec

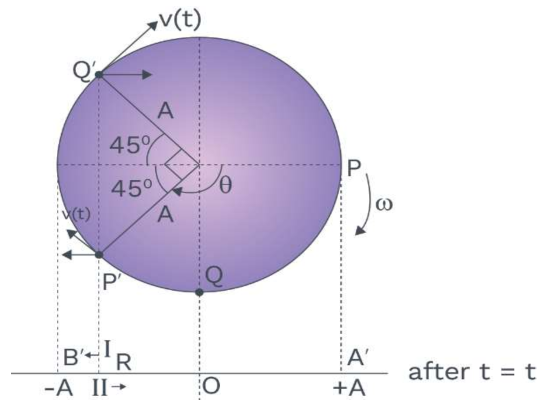
Ex. Two particles undergoes SHM along parallel lines with the same time period (T) and equal amplitudes (A). At a particular instant, one particle is at its extreme position (x = A) while the other is at its mean position (x = 0). They move in the same direction. They will cross each other after a further time.



Sol. In this problem is easy to solve with the help of phasor diagram.
First we draw the initial position of both the particle on the phasor as shown in figure.



From above figure phase difference between both the particles is π/2.
They will cross each other when their projection from the circle on the horizontal diameter meet at one point.
Let after time t both will reach at P'Q' point having phase difference π/2 as shown in figure.



Both will meet at $-\frac{A}{\sqrt{2}}$.

When they meet angular displacement of P is

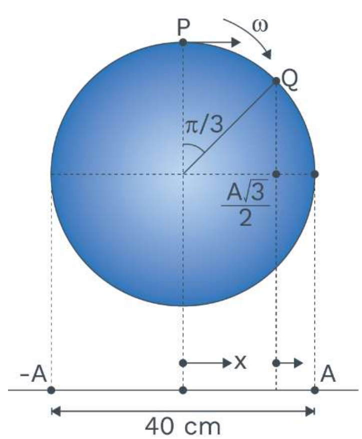
$$\theta = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$$

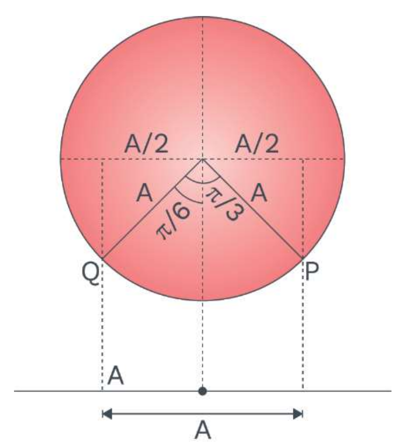
So, they will meet after time $t = \frac{3\pi}{4 \times \omega}$

$$t = \frac{3\pi}{4 \times 2\pi} \times T = \frac{3T}{8} \text{sec}$$

Ex. Two particles execute SHM of same amplitude (A) of 20 cm with same period along the same line about the same equilibrium position (x = 0). If phase difference is $\pi/3$ then find out the maximum distance between these two.

Sol. Consider us assume that one particle starts from mean position and another starts at a distance x having $\phi = \frac{\pi}{3}$. This condition is shown in figure.





Above figure shows the situation of maximum distance between them.
So maximum distance = A = 10 cm. (as 2A = 20 cm)

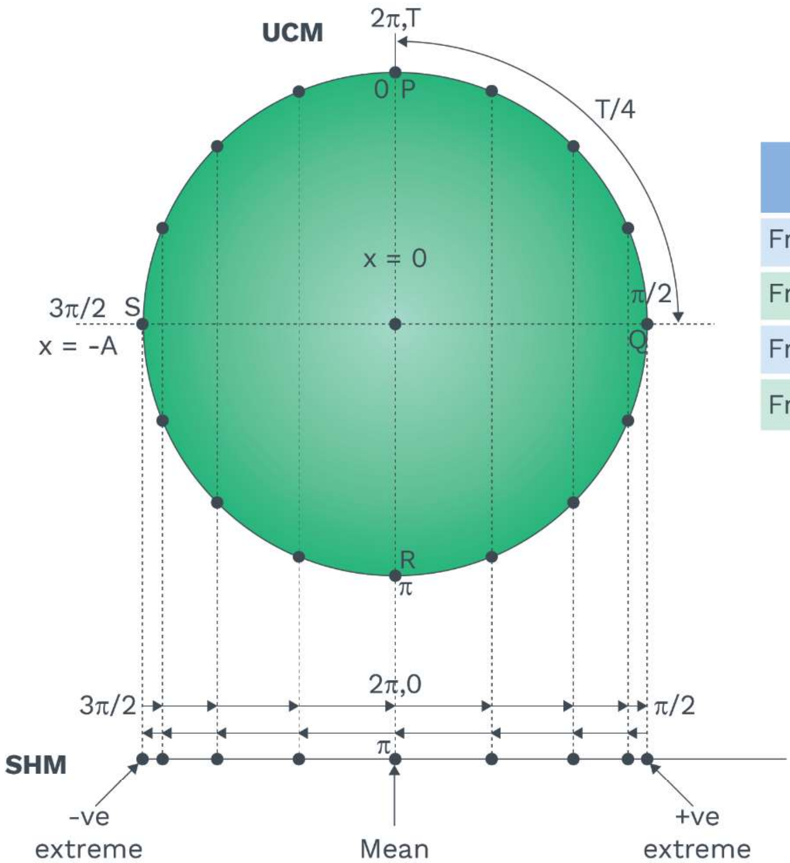
Ex. If two particles have time periods T and 5T/4. They start SHM at the same time from the mean position. After how many oscillations of the particle having smaller time period, they will be again in the same phase?

Sol. They will be again at mean position and moving in same direction when the particle having smaller time period makes n_1 oscillations and the other one makes n_2 oscillations.

$$\Rightarrow \quad n_1 T = \frac{5T}{4} \times n_2$$

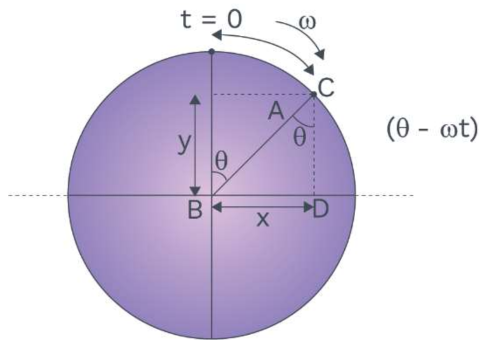
$$\frac{n_1}{n_2} = \frac{5}{4} \Rightarrow n_1 = 5, n_2 = 4$$

SHM as a projection of uniform circular motion:



UCM	SHM
From P to Q	Mean to +ve Extreme
From Q to R	+ve Extreme to Mean
From R to S	Mean to -ve Extreme
From S to P	-ve Extreme to Mean

In triangle BCD,

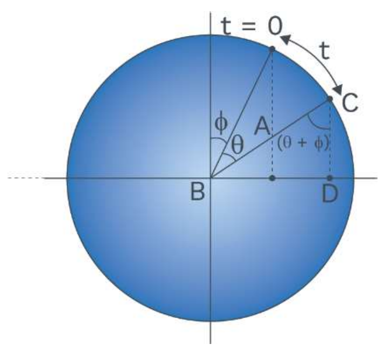


$$\sin\theta = \frac{x}{A}, \qquad \cos\theta = \frac{y}{A}$$

$\Rightarrow$ $x = A\sin\theta$ and $y = A\cos\theta$

$\Rightarrow$ $\boxed{x = A\sin\omega t}$, $\boxed{y = A\cos\omega t}$

In triangle BCD,



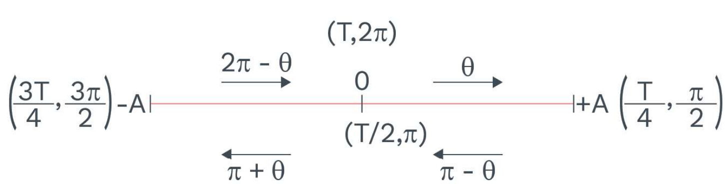
$$\sin(\theta + \phi) = \frac{x}{A}$$

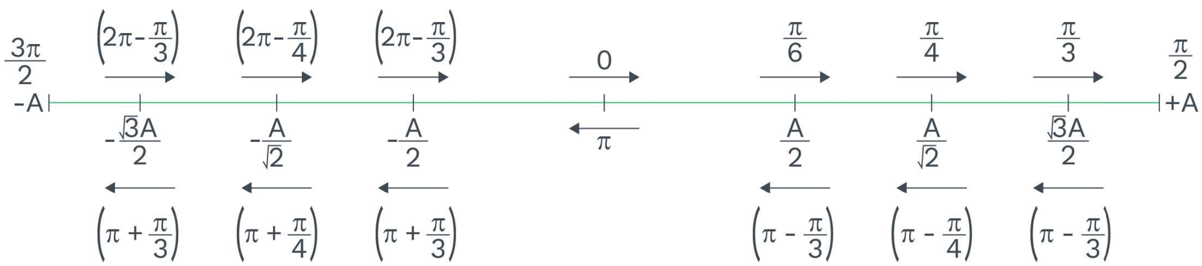
$\Rightarrow$ $x = A\sin(\theta + \phi)$

$\Rightarrow$ $\boxed{x = A\sin(\omega t + \phi)}$

This is general equation of SHM.

If equation of SHM of a particle is given by x = A sin(ωt + ϕ) then the initial phase is given by





Ex. A particle moves in SHM according to equation $16\frac{d^2x}{dt^2}+25x=0$. Find the nature of motion and its time period.

Sol. $16\frac{d^2x}{dt^2}+25x=0$

$$\Rightarrow \quad \frac{d^2x}{dt^2}+\left(\frac{25}{16}\right)x=0$$

Motion is linear SHM.

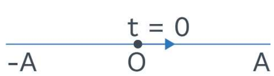
Comparing the above equation with $\frac{d^2x}{dt^2}+\omega^2x=0$.

We get,

$$\omega^2=\frac{25}{16} \quad \Rightarrow \quad \omega=\frac{5}{4}$$

$$\Rightarrow \quad \frac{2\pi}{T}=\frac{5}{4} \Rightarrow T=\frac{8\pi}{5}\,s$$

Ex. A particle starts from mean position (x = 0) and moves towards positive extreme as shown below. Find the equation of the SHM. Amplitude of SHM is A.



Sol. General equation of SHM can be written as

$x = A \sin(\omega t + \phi)$

At $t = 0$, $x = 0$ (mean position)

$\therefore$ $0 = A \sin \phi$

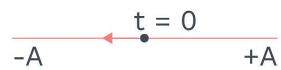
$\therefore$ $\phi = 0, \pi,$ $\quad \phi \in [0, 2\pi]$

Also; at $t = 0$, $v = +$ ve

$\therefore$ $A\omega \cos \phi = +$ ve or, $\phi = 0$

If the particle is at mean position (M.P.) at t = 0 sec and is moving towards +ve extreme, then the equation of SHM is given by $x = A \sin \omega t$.

Similarly for particle moving towards –ve extreme then



$\phi = \pi$

∴ Equation of SHM is $x = A \sin(\omega t + \pi)$

or, $x = -A \sin \omega t$

Ex. Write the equation of SHM for the situation shown below:

Sol. General equation of SHM can be written as

$$x = A\sin(\omega t + \phi)$$

At $t = 0, \; x = \frac{A}{2}$

$\Rightarrow \frac{A}{2} = A\sin\phi$

$\Rightarrow \phi = 30°, \; 150°$

Also at $t = 0$, $v = -$ ve

$A\omega\cos\phi = -\text{ve}$

$\Rightarrow \phi = 150°$

Hence, $x = A\sin(\omega t + 150°)$.

Ex. The equation of particle executing simple harmonic motion is $x = (5\,m)\sin\left[(\pi s^{-1})t + \frac{\pi}{3}\right]$. Write down the amplitude, time period and maximum speed. Also find the velocity at t = 1 s.

Sol. Comparing with equation $x = A \sin(\omega t + \delta)$, we see that the amplitude = 5 m, and time period $= \frac{2\pi}{\omega} = \frac{2\pi}{\pi s^{-1}} = 2\,s$.

The maximum speed $= A\omega = 5\,m \times \pi s^{-1} = 5\pi \; m/s$

The velocity at time $t = \frac{dx}{dt} = A\omega\cos(\omega t + \delta)$

At, t = 1 s,

$$v = (5\,m)(\pi s^{-1})\cos\left(\pi + \frac{\pi}{3}\right) = -\frac{5\pi}{2} \; m/s$$

Ex. A particle executing simple harmonic motion has angular frequency 6.28 s^{-1} and amplitude 10 cm. Find-

(a) The time period (T),

(b) The maximum speed (V_{max}),

(c) The maximum acceleration (a_{max}),

(d) The speed when the displacement is 6 cm from the mean position (x = 0),

(e) The speed (v) at t = 1/6 sec assuming that the motion starts from rest at t = 0.

Sol. (a) Time period $= \frac{2\pi}{\omega} = \frac{2\pi}{6.28} s = 1\ s$

(b) Maximum speed

$$= A\omega = (0.1\ m)(6.28\ s^{-1}) = 0.628\ m/s$$

(c) Maximum acceleration $= A\omega^2$

$$= (0.1\ m)\ (6.28\ s^{-1})^2 = 4\ m/s^2$$

(d) $v = \omega\sqrt{A^2 - x^2}$

$$= (6.28\,s^{-1})\sqrt{(10\,cm)^2 - (6\,cm)^2} = 50.2\ cm/s$$

(e) At t = 0, the velocity is zero i.e., the particle is at an extreme. The equation for displacement may be written as

$$x = A\cos\omega t$$

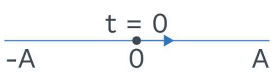
The velocity is $v = -A\omega \sin \omega t$.

At $t = \frac{1}{6} s$

$$v = -(0.1m)(6.28\,s^{-1})\sin\left(\frac{6.28}{6}\right)$$

$$= (-0.628\ m/s)\sin\frac{\pi}{3} = 54.4\ cm/s$$

Ex. A particle starts from mean position (x = 0) and moves towards positive extreme as given figure. Find out the equation of the SHM. Amplitude of SHM is A.



Sol. Standard equation of SHM can be written as $x = A\sin(\omega t + \phi)$

At t = 0 sec, x = 0 metre

$\therefore$ $0 = A\sin\phi$

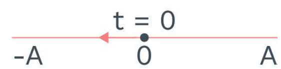
$\therefore$ $\phi = 0, \pi$ $\phi \in [0, 2\pi]$

Also, at t = 0 sec, v = positive

$\therefore$ $A\omega\cos\phi$ = positive

or, $\phi = 0$

Hence, if the particle (object) is at mean position (M.P.) at t = 0 sec and is moving towards positive extreme, then the equation of SHM is given by $x = A \sin \omega t$.
Similarly,



For, $\phi = \pi$

$\therefore$ Equation of SHM is $x = A \sin(\omega t + \pi)$

or, $x = -A \sin \omega t$

Note: If mean position is not at the origin, then we can replace x by $x - x_0$ and the equation becomes $x - x_0 = -A \sin \omega t$, where x_0 is the position co-ordinate of the mean position.

Ex. A particle is executing SHM of amplitude 'A' and time period 'T'. Find the time (T) taken by the particle to go from x = 0 to x = A/2.

Sol. Let equation of SHM be $x = A \sin \omega t$

when $x = 0$, $t = 0$

when $x = \frac{A}{2}$; $\frac{A}{2} = A \sin \omega t$

or $\sin \omega t = \frac{1}{2}$, $\omega t = \frac{\pi}{6}$

$\frac{2\pi}{T} t = \frac{\pi}{6}$, $t = \frac{T}{12}$

Hence, time taken is T/12, where T is time period of SHM.

Ex. Particle of mass 2 kilogram is moving on a straight line path and force acting is F = (8 – 2x) Newton. It is released at rest (v = 0) from x = 6 metre.

(a) Is the particle moving simple harmonically.

(b) Find the equilibrium position of the particle.

(c) Write the equation of motion of the particle.

(d) Find the time period of SHM.

Sol. $F = 8 - 2x$ (Newton)

or $F = -2(x - 4)$

For equilibrium position F = 0

$\Rightarrow$ x = 4 is equilibrium position.

Hence the motion of particle is SHM with force constant 2 and equilibrium position x = 4.

(a) Yes, motion is SHM

(b) Equilibrium position is x = 4

(c) At x = 6 m, particle is at rest i.e. it is one of the extreme position

Hence amplitude is A = 2 m and initially particle is at the extreme position.

∴ Equation of SHM can be written as

$$x - 4 = 2\cos\omega t$$

where, $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{2}{2}} = 1$

i.e., x = 4 + 2 cos t

(d) Time period,

$$T = \frac{2\pi}{\omega} = 2\pi \text{ sec}.$$

Ex. Find phase constant and equation of SHM of an oscillating particle if it starts motion from-

(a) Positive extreme position

(b) $\frac{A}{2}$

(c) $\frac{A}{\sqrt{2}}$

(d) $\frac{\sqrt{3}A}{2}$

Sol. (a) Positive extreme position

x = A at t = 0

x = A sin(ωt + ϕ)

⇒ A = A sin ϕ

⇒ $\phi = \frac{\pi}{2}$

So, $x = A\sin\left(\omega t + \frac{\pi}{2}\right)$

⇒ $x = A\cos\omega t$

(b) $x = \frac{A}{2}$ at $t = 0$

x = A sin(ωt + ϕ)

⇒ $\frac{A}{2} = A\sin\phi$

⇒ $\phi = \frac{\pi}{2}$

So, $x = A\sin\left(\omega t + \frac{\pi}{6}\right)$

(c) $x = \frac{A}{\sqrt{2}}$ at $t = 0$

$x = A\sin(\omega t + \phi)$

$\Rightarrow \frac{A}{\sqrt{2}} = A\sin(\omega t + \phi)$

$\Rightarrow \frac{A}{\sqrt{2}} = A\sin\phi \quad \Rightarrow \phi = \frac{\pi}{4}$

So, $x = A\sin\left(\omega t + \frac{\pi}{6}\right)$

(d) $x = \frac{\sqrt{3}A}{2}$ at $t = 0$

$x = A\sin(\omega t + \phi)$

$\Rightarrow \frac{\sqrt{3}A}{2}A = \sin\phi \quad \Rightarrow \phi = \frac{\pi}{3}$

So, $x = A\sin\left(\omega t + \frac{\pi}{3}\right)$

Ex. If $x = 10\sin\left(20\pi t + \frac{\pi}{4}\right)$ cm, then find:

(a) A

(b) ω

(c) T

(d) n

(e) Initial phase

(f) Value of x at t = 0, $t = \frac{1}{4}$ s

Sol. (a) A = 10 cm

(b) ω = 20π rad/sec

(c) $\frac{2\pi}{T} = 20\pi \quad \Rightarrow T = \frac{1}{10}$ sec

(d) $2\pi n = 20\pi \quad \Rightarrow n = 10$ Hz

(e) Initial phase $\phi = \frac{\pi}{4}$ rad

(e) At t = 0

$x = 10\sin\left(\frac{\pi}{4}\right) = \frac{10}{\sqrt{2}}$ cm

At $t = \frac{1}{4}$ s

$x = 10\sin\left(\frac{20\pi \times 1}{4} + \frac{\pi}{4}\right)$

$$= 10\sin\left(5\pi + \frac{\pi}{4}\right) = -10\sin\frac{\pi}{4} = -\frac{10}{\sqrt{2}}\text{cm}$$

Ex. What will be the equation of SHM if a particle starts from- (at t = 0)

(i) Mean towards positive extreme

(ii) Positive extreme

(iii) Mean towards negative extreme

(iv) Negative extreme

Sol. Equation of SHM is $x = A\sin(\omega t + \phi)$

(i) From mean towards positive extreme

$\because$ $\phi = 0$

$\therefore$ $x = A\sin\omega t$

(ii) From positive extreme

$\because$ $\phi = \frac{\pi}{2}$

$\therefore$ $x = A\sin\left(\omega t + \frac{\pi}{2}\right)$

$\Rightarrow$ $x = A\cos\omega t$

(iii) From mean towards negative extreme

$\because$ $\phi = \pi$

$\therefore$ $x = A\sin(\omega t + \pi)$

$\Rightarrow$ $x = -A\sin\omega t$

(iv) From negative extreme

$\because$ $\phi = \frac{3\pi}{2}$

$\therefore$ $x = A\sin\left(\omega t + \frac{3\pi}{2}\right)$

$\Rightarrow$ $x = -A\cos\omega t$

Ex. A particle executing SHM of time period 2s. Find time taken by particle to cover displacement $\frac{1}{2}$ of amplitude if it starts its motion from-

(i) Mean position **(ii)** Positive extreme

Sol. **(i)** Mean position

So, $x = A\cos\omega t$

$\Rightarrow$ $\frac{A}{2} = A\sin\omega t$

$\Rightarrow \quad \omega t = \dfrac{\pi}{6}$

$\Rightarrow \quad t = \dfrac{\pi}{6\omega} = \dfrac{\pi \times T}{6 \times 2\pi} = \dfrac{\pi \times 2}{6 \times 2\pi}$

$\Rightarrow \quad t = \dfrac{1}{6} s$

(ii) Positive extreme

So, $x = A \sin \omega t$

$\Rightarrow \quad \dfrac{A}{2} = A \cos \omega t$

$\Rightarrow \quad \omega t = \dfrac{\pi}{3}$

$\Rightarrow \quad t = \dfrac{\pi}{3\omega} = \dfrac{\pi \times T}{3 \times 2\pi} = \dfrac{\pi \times 2}{3 \times 2\pi}$

$\Rightarrow \quad t = \dfrac{1}{3} s$

Phase difference:

Equations of two SHMs are given as $x_1 = A \sin(\omega t + \phi_1)$ and $x_2 = A \sin(\omega t + \phi_2)$

Then the phase difference between these SHMs is $\Delta\phi = \phi_2 - \phi_1$.

Ex. Two SHMs are given by $x_1 = A \sin(\omega t)$ and $x_2 = A \cos(\omega t)$. Then find phase difference.

Sol. $x_1 = A \sin \omega t$

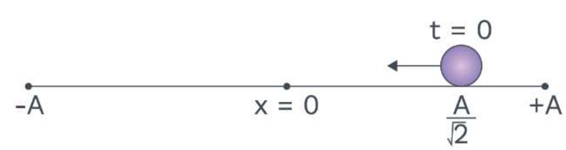
and $x_2 = A \cos \omega t$

$$= A \sin\left(\omega t + \frac{\pi}{2}\right)$$

So, phase difference

$$\Delta\phi = \left(\omega t + \frac{\pi}{2}\right) - \omega t \quad \Rightarrow \Delta\phi = \frac{\pi}{2}$$

Ex. Write equation of SHM for given particle:



Sol. $x = \dfrac{A}{\sqrt{2}}$

From positive extreme

$$\Rightarrow \quad x = A\cos(\omega t + \phi)$$

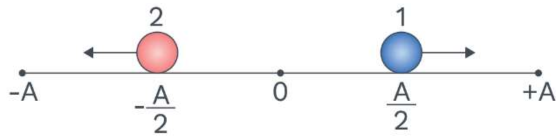
$$\Rightarrow \quad \phi = \frac{\pi}{4}$$

So, total phase $= \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$$\Rightarrow \quad x = A\cos\left(\omega t + \frac{\pi}{4}\right)$$

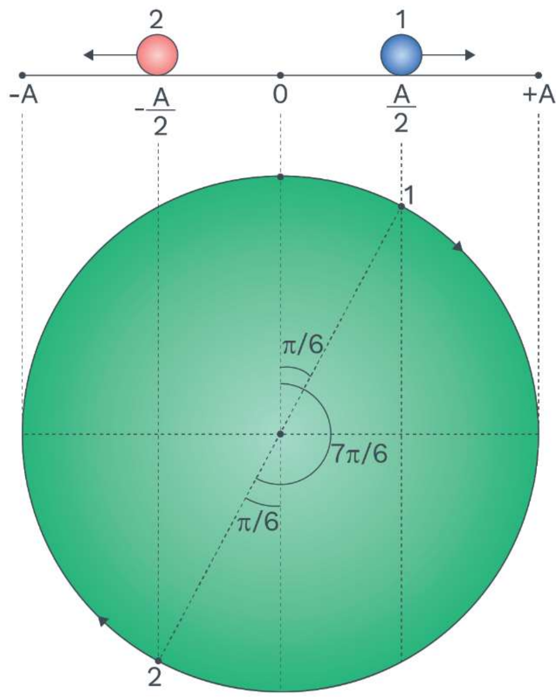
or $\quad x = A\sin\left(\omega t + \frac{3\pi}{4}\right)$

Ex. Write the equation of SHM for particle 1 and 2 as shown in figure.



Sol. Particle 1, is going from mean to positive extreme and is at displacement A/2 initially (at t = 0) so its equation is x = A sin (ωt + ϕ)

$$\Rightarrow \quad \frac{A}{2} = A\sin\phi \quad \Rightarrow \quad \phi = \frac{\pi}{6}$$



So, equation of SHM for particle 1 is

$$x = A\sin\left(\omega t + \frac{\pi}{6}\right)$$

For particle 2,

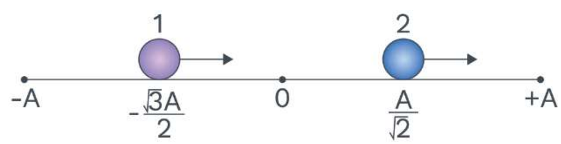
$$x = -A\sin(\omega t + \phi)$$

$$\Rightarrow \quad -\frac{A}{2} = -A\sin\phi \quad \Rightarrow \phi = \frac{7\pi}{6}$$

So, its phase is $\left(\pi + \frac{\pi}{6}\right) = \frac{7\pi}{6}$

$\therefore$ Equation for particle 2 is $x = A\sin\left(\omega t + \frac{7\pi}{6}\right)$.

Ex. Find the time taken by the particle to go from position 1 to position 2.



Sol. Particle first going to $-\frac{\sqrt{3}A}{2}$ to mean and then mean to $\frac{A}{\sqrt{2}}$.

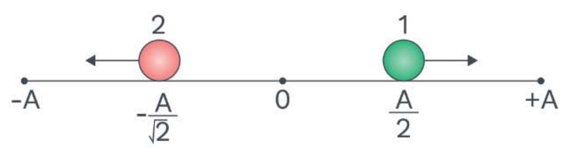
Total travelling

$$= \left(-\frac{\sqrt{3}A}{2} \text{ to mean}\right) + \left(\text{mean to } \frac{A}{\sqrt{2}}\right)$$

Travelled angle $= \frac{\pi}{3} + \frac{\pi}{4} = \frac{7\pi}{12} = \frac{7}{24}(2\pi)$

Total time $= \frac{7T}{24}$.

Ex. Find the phase difference between given particles.



Sol. Phase difference $\Delta\phi = \phi_2 - \phi_1$ ...(1)

Particle (2), moving from mean to –ve extreme

$\therefore \quad x = -A \sin\theta$

$\Rightarrow \quad -\frac{A}{\sqrt{2}} = -A\sin\theta \quad \Rightarrow \theta = \frac{\pi}{4}$

So, $\phi_2 = \pi + \theta = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$

Phase of particle (1) is $\phi_1 = \theta$

$\therefore \quad x = A \sin \theta$

$\Rightarrow \quad \frac{A}{2} = A\sin\theta \quad \Rightarrow \theta = \frac{\pi}{6}$

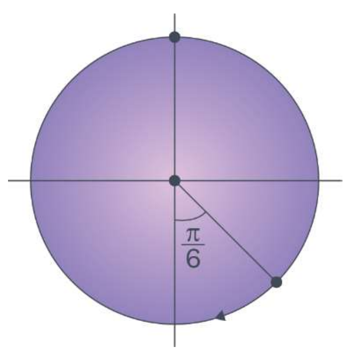
$\Rightarrow \quad \phi_1 = \frac{\pi}{6}$

Phase difference

$$= \phi_2 - \phi_1 = \frac{5\pi}{4} - \frac{\pi}{6}$$

$$= \frac{30\pi - 4\pi}{24} = \frac{26\pi}{24} = \frac{13\pi}{12}$$

Ex. Write the equation of given particle along x-axis.



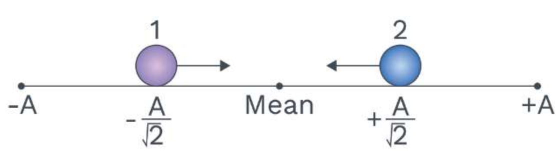
Sol. Equation of SHM

$$x = A \sin(\omega t + \phi)$$

Initial phase $\phi = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

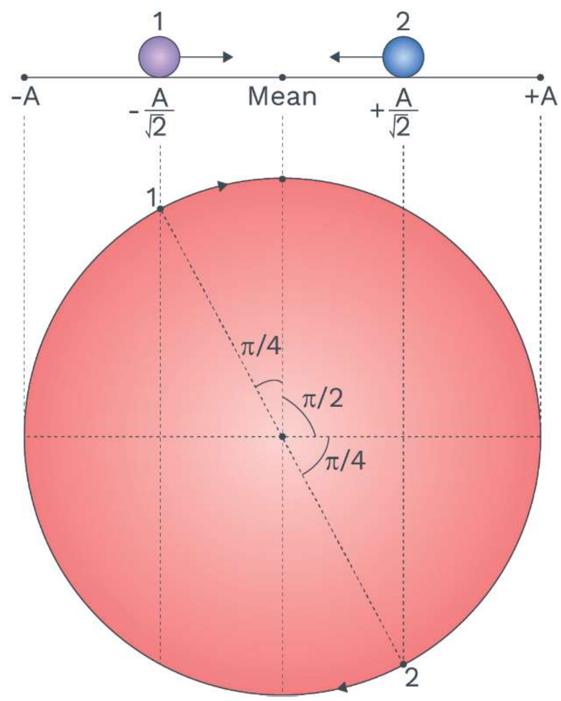
$$x = A\sin\left(\omega t + \frac{5\pi}{6}\right)$$

Ex. Find the time taken by the particle to reach from position 1 to 2.



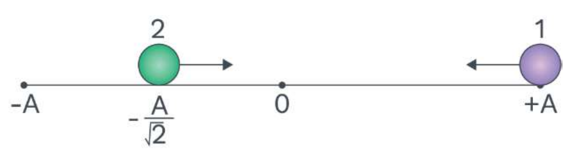
Sol. In given diagram we have to find the time taken by particle to move from position 1 to 2. So, angle travelled by particle on phasor is

$$= \frac{\pi}{4} + \frac{\pi}{2} + \frac{\pi}{4} = \pi = \frac{(2\pi)}{2}$$



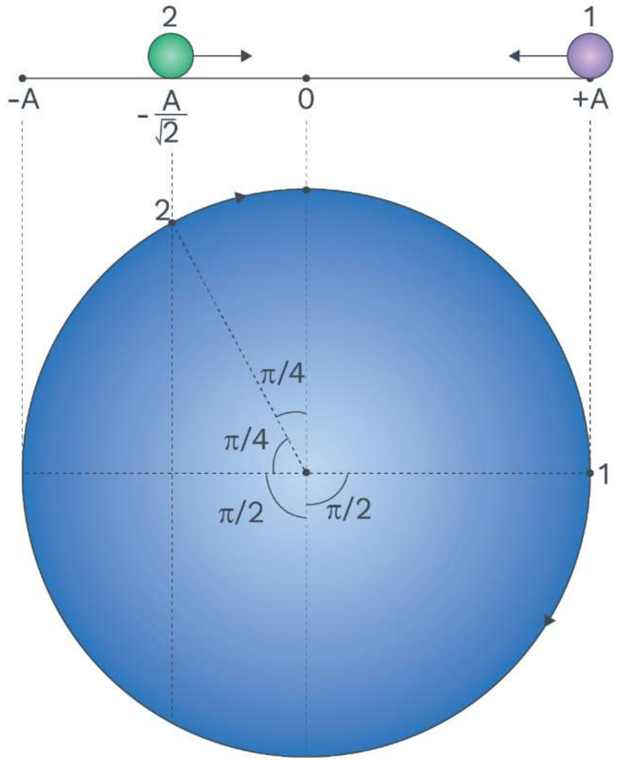
So, time taken by particle is $= \frac{T}{2}$.

Ex. Find the time taken by particle in moving from position 1 to 2.



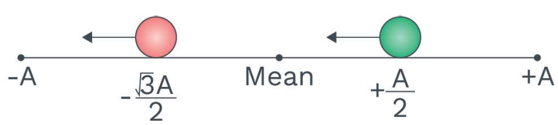
Sol. Total angle travelled

$$= \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{4} = \frac{5\pi}{4} = \frac{5}{8}(2\pi)$$



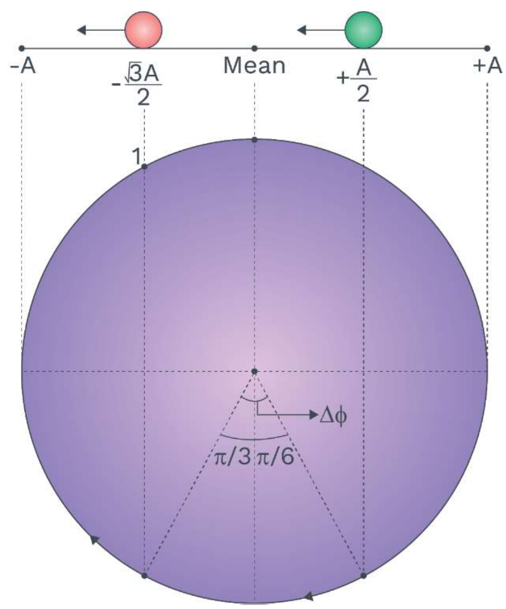
So, time taken by particle is = $\frac{5}{8}T$.

Ex. Find the phase difference between two particle which are shown in figure.

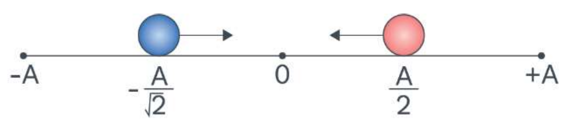


Sol. From phasor diagram phase difference between these two particle is

$$\Delta\phi = \frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}$$

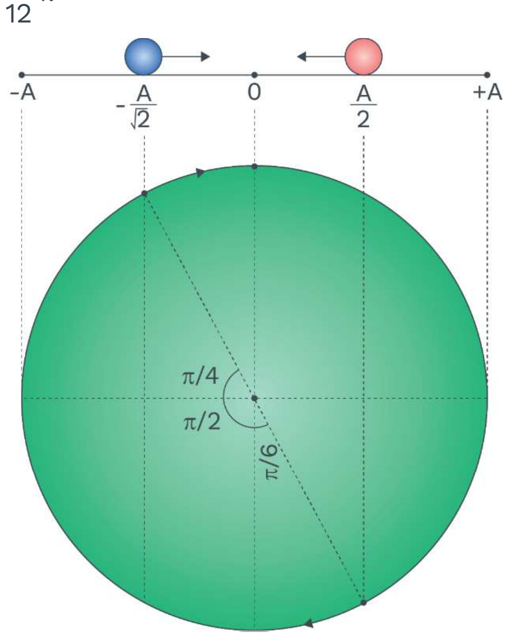


Ex. Repeat the above question with given diagram.



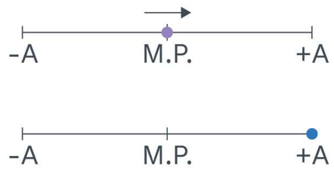
Sol. From phasor diagram phase difference between these two particle is

$$\Delta\phi = \frac{\pi}{6} + \frac{\pi}{2} + \frac{\pi}{4} = \frac{11}{12}\pi$$

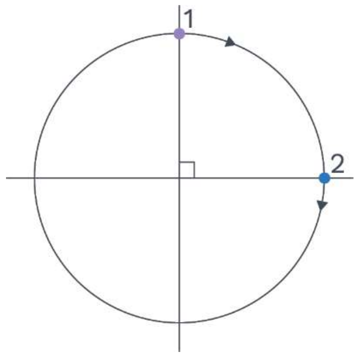


To calculate maximum separation between particle:

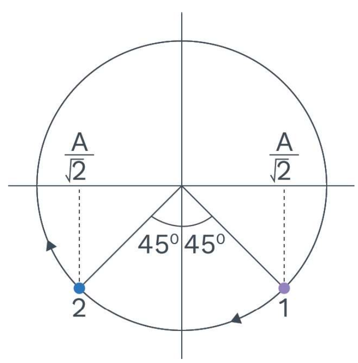
Ex. Two particle performing SHM in two near by lines according to the given diagram with same amplitude and time period. Find out maximum separation.



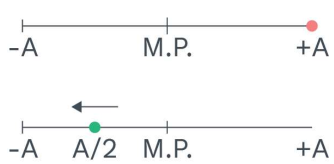
Sol. At the time of maximum separation, velocities of both particles are equal.



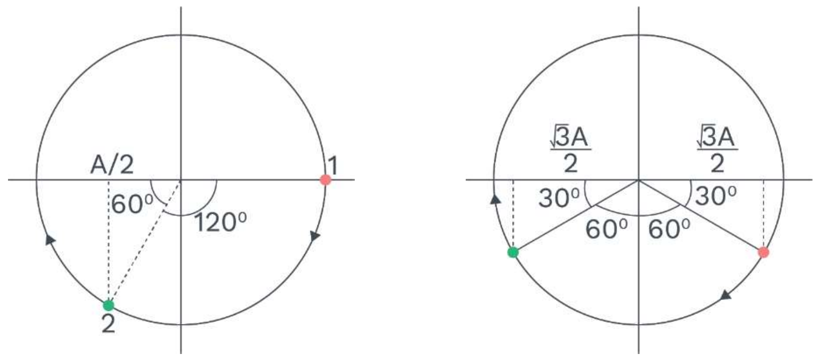
Maximum separation $= \frac{A}{\sqrt{2}} + \frac{A}{\sqrt{2}} = \sqrt{2}\,A$.



Ex. Two particle performing SHM in two near by lines according to the given diagram with same amplitude and time period. Find out maximum separation.



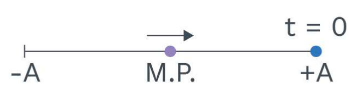
Sol.



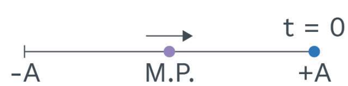
Maximum separation = $\frac{\sqrt{3}\,A}{2} + \frac{\sqrt{3}\,A}{2} = \sqrt{3}\,A$.

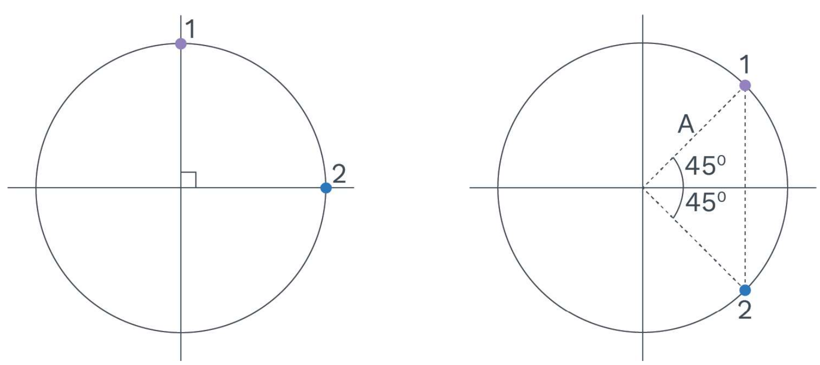
To find out time and position where particle collide:

Ex. Both particles have same ω and same A. Then find out position and time when they will collide.



Sol.





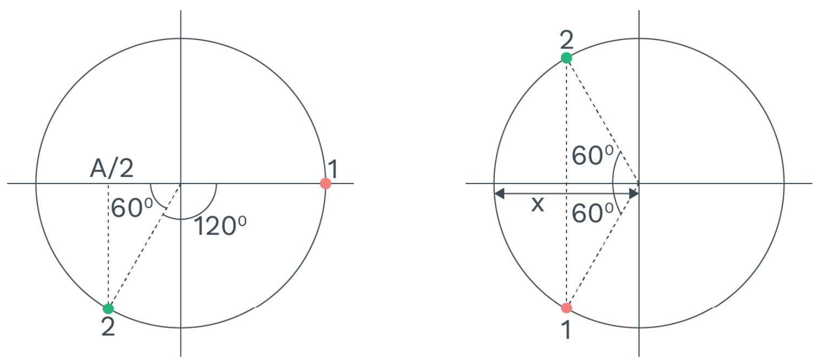
$$x = A\cos 45^\circ \quad \Rightarrow \quad x = \frac{A}{\sqrt{2}}$$

$$\Rightarrow \quad \theta = \omega t \quad \Rightarrow \quad \frac{\pi}{4} = \frac{2\pi}{T}\cdot t$$

$$\Rightarrow \quad t = \frac{T}{8}.$$

Ex. Both particle have same ω and same A. Then find out position and time when they will collide.

Sol.



$$x = -A\cos 60° \Rightarrow x = -\frac{A}{2} \Rightarrow \theta = \omega t$$

$$\Rightarrow \quad \frac{2\pi}{3} = \frac{2\pi}{T}\cdot t \quad \Rightarrow t = \frac{T}{3}$$

VELOCITY AND ACCELERATION OF PARTICLE IN SHM

Velocity:

It is the rate of change of particle displacement with respect to time at that instant. Let the displacement from mean position is given by x = A sin (ωt + ϕ).

Velocity,

$$v = \frac{dx}{dt} = A\omega\cos(\omega t + \phi)$$

$$v = A\omega\cos(\omega t + \phi)$$

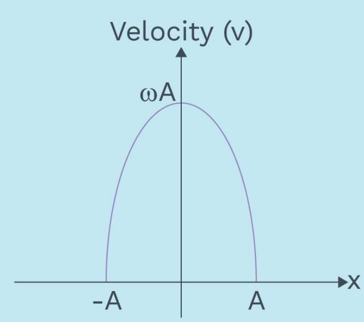
$$v = \omega\sqrt{A^2 - x^2}$$

At mean position (x = 0), velocity is maximum.

$$v_{max} = \omega A$$

Concept Reminder

$$v = \omega\sqrt{A^2 - x^2}$$



At extreme position (x = A), velocity is minimum.

v_{min} = zero.

Acceleration:

It is the rate of change of particle's velocity w.r.t. time at that instant.

x = -A	x = 0	x = +A
$a_{max} = \omega^2 A$	$v_{max} = A\omega$	$a_{max} = -\omega^2 A$

Acceleration, $a = \frac{dv}{dt} = \frac{d}{dt}[A\omega\cos(\omega t + \phi)]$

$$a = -\omega^2 A\sin(\omega t + \phi)$$

$$a = -\omega^2 x$$

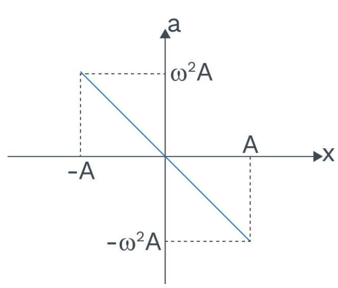
Note: Negative sign shows that acceleration is always directed towards the mean position. At mean position (x = 0), acceleration is minimum.

a_{min} = zero

At extreme position (x = A), acceleration is maximum.

$|a_{max}| = \omega^2 A$

Graph of displacement (x) v/s acceleration (a):



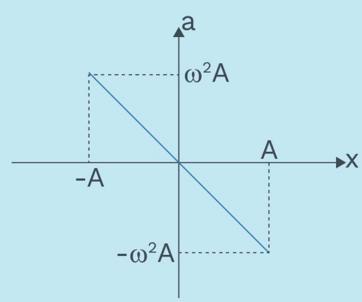
$a = -\omega^2 x$

Concept Reminder

Velocity of particle performing SHM is maximum at mean position and is zero at extreme positions.

Concept Reminder

$a = -\omega^2 x$



GRAPHICAL REPRESENTATION OF VELOCITY, DISPLACEMENT & ACCELERATION IN SHM

Displacement, $x = A\sin\omega t$

Velocity,

$$v = A\omega\cos\omega t = A\omega\sin\left(\omega t + \frac{\pi}{2}\right)$$

or $v = \omega\sqrt{A^2 - x^2}$

Acceleration,

$$a = -\omega^2 A\sin\omega t = \omega^2 A\sin(\omega t + \pi)$$

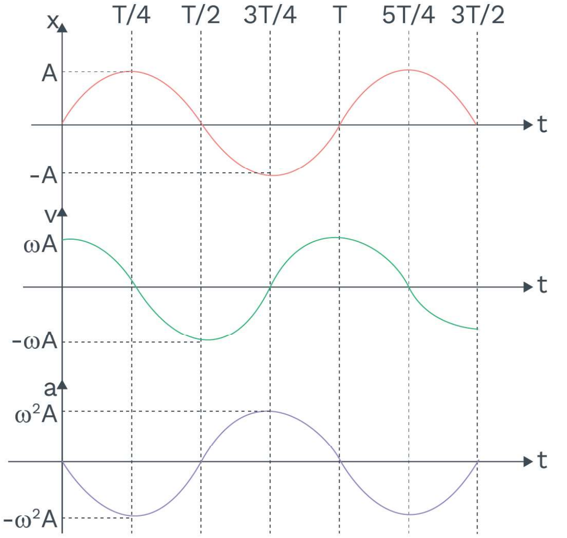
or $a = -\omega^2 x$

Note:

$$v = \omega\sqrt{A^2 - x^2}$$

$$a = -\omega^2 x$$

These relations are true for any equation of x.



- All the three quantities displacement, velocity and acceleration vary harmonically with time, having same period.
- The maximum velocity is ω times the amplitude ($v_{max} = \omega A$).
- The acceleration is ω^2 times the displacement amplitude ($a_{max} = \omega^2 A$).
- In SHM, the velocity is ahead of displacement by a phase angle of $\frac{\pi}{2}$.
- In SHM, the acceleration is ahead of velocity by a phase angle of $\frac{\pi}{2}$.

Rack your Brain

The circular motion of a particle with constant speed is:

(1) Simple harmonic but not periodic
(2) Periodic and simple harmonic
(3) Neither periodic nor simple harmonic
(4) Periodic but not simple harmonic

Concept Reminder

All the three quantities displacement, velocity and acceleration vary harmonically with time, having same period.

Ex. At a particular position the velocity of a particle in SHM with amplitude A is $\frac{\sqrt{3}}{2}$ that at its mean position. In this position, find its displacement?

Sol. Velocity, $v = \omega\sqrt{A^2 - x^2}$

$\Rightarrow \quad (A\omega)\frac{\sqrt{3}}{2} = \omega\sqrt{A^2 - x^2}$

$\Rightarrow \quad \frac{3}{4}A^2 = A^2 - x^2$

$\Rightarrow \quad x^2 = \frac{A^2}{4} \quad \Rightarrow \quad x = \pm\frac{A}{2}$

Ex. A particle is executing SHM. Its velocity is v_1 = 10 m/s at x_1 = 4 m position and v_2 = 8 m/s at x_2 = 5m position. Find-

(a) Amplitude and

(b) Angular frequency.

Sol. (a) $v_1 = \omega\sqrt{A^2 - x_1^2}$

$\Rightarrow \quad v_1^2 = \omega^2(A^2 - x_1^2)$ ...(i)

$v_2 = \omega\sqrt{A^2 - x_2^2}$

$\Rightarrow \quad v_2^2 = \omega^2(A^2 - x_2^2)$ ...(ii)

On dividing (i) and (ii),

$$\frac{v_1^2}{v_2^2} = \frac{A^2 - x_1^2}{A^2 - x_2^2}$$

$\Rightarrow \quad v_1^2 A^2 - v_1^2 x_2^2 = v_2^2 A^2 - v_2^2 x_1^2$

$\Rightarrow \quad v_2^2 x_1^2 - v_1^2 x_2^2 = A^2(v_2^2 - v_1^2)$

$\Rightarrow \quad A = \sqrt{\frac{v_2^2 x_1^2 - v_1^2 x_2^2}{v_2^2 - v_1^2}}$

$\Rightarrow \quad A = \sqrt{41}$ m

(b) $v_1^2 = \omega^2(A^2 - x_1^2)$

$\Rightarrow \quad v_1^2 = \omega^2 A^2 - \omega^2 x_1^2$ ...(iii)

$v_2^2 = \omega^2(A^2 - x_2^2)$

$\Rightarrow \quad v_2^2 = \omega^2 A^2 - \omega^2 x_2^2$ ...(iv)

On subtracting (4) from (3)

$$v_1^2 - v_2^2 = \omega^2(x_2^2 - x_1^2)$$

$$\Rightarrow \quad \omega = \sqrt{\frac{v_1^2 - v_2^2}{x_2^2 - x_1^2}} \Rightarrow \omega = 6 \text{ rad / sec}$$

Ex. A particle performing SHM is found at its equilibrium position at t = 1 sec and it is found to have a speed of 0.25 m/s at t = 2 sec. If the period of oscillation is 8 sec. Calculate the amplitude of oscillations.

Sol. $x = A \sin(\omega t + \phi)$

At t = 1 sec. particle at mean position

$$0 = A \sin\left(\frac{2\pi}{8} \times 1 + \phi\right) \Rightarrow \phi = -\frac{\pi}{4}$$

At t = 2 sec. velocity of particle is 0.25 m/s

$$0.25 = A\omega \cos\left(\frac{\pi}{4} \times 2 - \frac{\pi}{4}\right)$$

$$0.25 = \frac{A\omega}{\sqrt{2}} \Rightarrow A = \frac{\sqrt{2}}{\pi} m$$

Ex. The displacement of a particle executing SHM is given by $x = 10 \sin(6t + \pi/3)$. Here x is in metres and t is in seconds. Find its acceleration at x = 5m.

Sol. Acceleration in SHM

$a = \omega^2 x = (6)^2 \times 5 = 180 \text{ m/s}^2$

Phase difference between x, v and a:

$\because$ $x = A \sin \omega t$

$\therefore$ $v = A\omega \cos \omega t = A\omega \sin(\omega t + \pi/2)$

and $a = -A\omega^2 \sin \omega t = A\omega^2 \sin(\omega t + \pi)$

So, phase difference between

(i) Displacement and velocity

$$(\Delta\phi)_{x-v} = \frac{\pi}{2}$$

(ii) Velocity and acceleration

$$(\Delta\phi)_{v-a} = \frac{\pi}{2}$$

(iii) Displacement and acceleration

$$(\Delta\phi)_{x-a} = \pi$$

Concept Reminder

Any material medium can be pictured as a collection of a large number of coupled oscillators. The collective oscillations of the constituents of a medium manifest themselves as waves.

Relation between displacement and velocity:

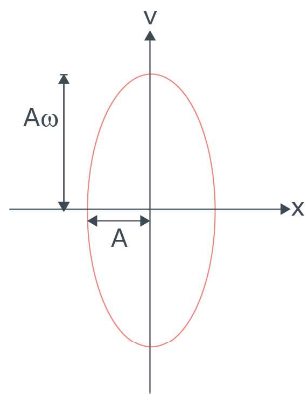
$\because \quad v = \omega\sqrt{A^2 - x^2} \quad \Rightarrow v^2 = \omega^2(A^2 - x^2)$

$\Rightarrow \quad \frac{v^2}{\omega^2} = A^2 - x^2$

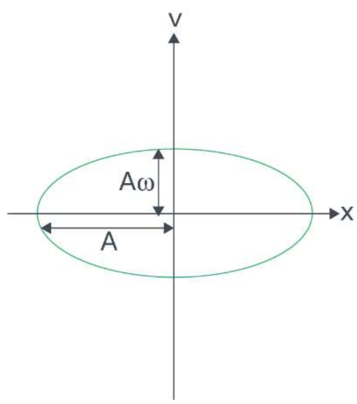
$\Rightarrow \quad \boxed{\frac{x^2}{A^2} + \frac{v^2}{(A\omega)^2} = 1}$ (equation of an ellipse)

So, the graph between displacement and velocity is an ellipse.

If $\omega > 1 (A\omega > A)$



If $\omega < 1 (A\omega < A)$

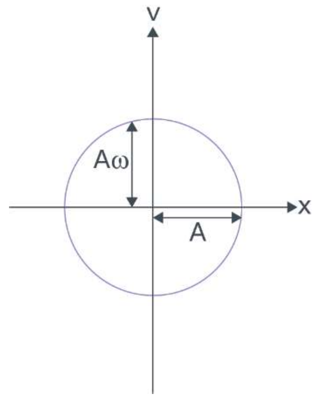


Rack your Brain

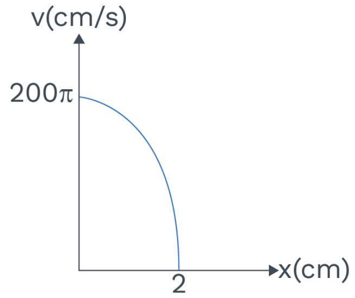
A particle moves under force $F = -5(x - 2)^3$. Motion of the particle is:

(1) Translatory
(2) Oscillatory
(3) S.H.M.
(4) All of these

If $\omega = 1 (A\omega = A)$



Ex. Find frequency of SHM for which graph between velocity and displacement is given:



Sol. A = 2 cm

$$v_{max} = A\omega \Rightarrow 200\pi = 2 \times \omega \Rightarrow \omega = 100\pi$$

$\Rightarrow \quad (2\ \pi n) = 100\ \pi \Rightarrow n = 50$ Hz

Ex. v-x relation for a SHM is given as $\frac{x^2}{4} + \frac{v^2}{400\pi^2} = 1$, where x is in cm and v is in cm/s. Find the time period and amplitude of this SHM.

Sol. $\frac{x^2}{4} + \frac{v^2}{400\pi^2} = 1$

Comparing this equation with $\frac{x^2}{A^2} + \frac{v^2}{(A\omega)^2} = 1$ gives

$A^2 = 4$ and $(A\omega)^2 = 400\pi^2$

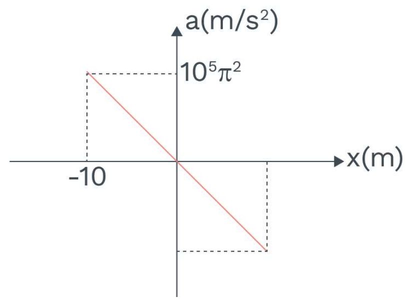
$\Rightarrow$ $A = 2$ and $A\omega = 20\pi$

$\therefore$ $2\omega = 20\pi$

$\Rightarrow$ $\frac{2\pi}{T} = 10\pi$

$\Rightarrow$ $T = \left(\frac{1}{5}\right)$ sec and $A = 2\,cm$

Ex. The acceleration displacement graph for a particle executing SHM is given by:



Find the frequency of SHM.

Sol. $\because$ $a = -\omega^2 x$ and $a_{max} = \omega^2 A$

$\Rightarrow$ $10^5\pi^2 = \omega^2(10)$

$\Rightarrow$ $\omega = 100\pi$

$\Rightarrow$ $n = 50$ Hz

Ex. Draw the graph of displacement, velocity and acceleration with time in SHM if equation of SHM is:

(a) $x = A \sin \omega t$

(b) $x = A \cos \omega t$

Sol. (a) $\because$ $x = A \sin \omega t$

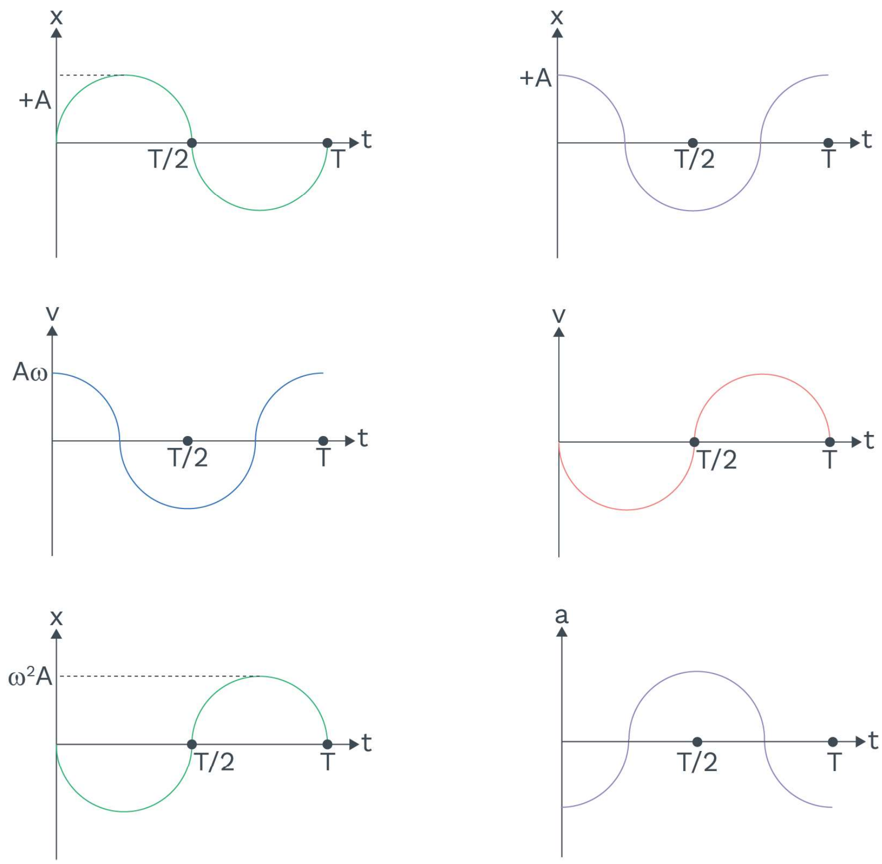
$\therefore$ $v = A\omega \cos \omega t$

and $a = -A\omega^2 \sin \omega t$

(b) $\because$ $x = A \cos \omega t$

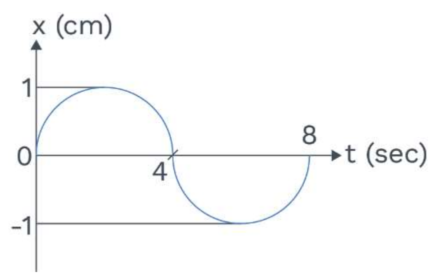
$\therefore$ $v = -A\omega \sin \omega t$

and $a = -A\omega^2 \cos \omega t$



(a) (b)

Ex. A particle is performing SHM. It's displacement-time graph is given below then find out acceleration of particle at t = 4/3 sec.



Sol. $x = A \sin \omega t \quad (\phi = 0)$

$$x = 1 \sin\left(\frac{2\pi}{T} \times t\right)$$

$$x = 1 \sin\left(\frac{2\pi}{8} \times t\right) \Rightarrow x = 1 \sin \frac{\pi}{4} t$$

$$v = \frac{\pi}{4}\cos\left(\frac{\pi}{4}t\right) \Rightarrow a = \frac{-\pi^2}{16}\sin\left(\frac{\pi}{3}\right)$$

$$a = \frac{-\sqrt{3}\,\pi^2}{32}$$

ENERGY IN SHM

1. Potential energy:

We known that $F = -\frac{dU}{dx}$

$\Rightarrow \quad \int dU = -\int F \cdot dx \quad \Rightarrow \quad U = \int kx\,dx$

$\Rightarrow \quad U = \frac{1}{2}kx^2 + c$

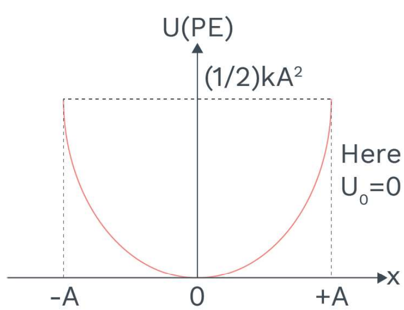
$\because \quad$ At $x = 0$, $U = U_0$, So, $c = U_0$

$\therefore \quad U = \frac{1}{2}kx^2 + U_0$

Where U_0 = Potential energy at equilibrium position.

If $U_0 = 0$ then $\boxed{U = \frac{1}{2}kx^2}$ (Potential energy in terms of displacement)

Graph between potential energy and displacement



(i) At x = 0 (mean position)

$PE_{min} = 0$

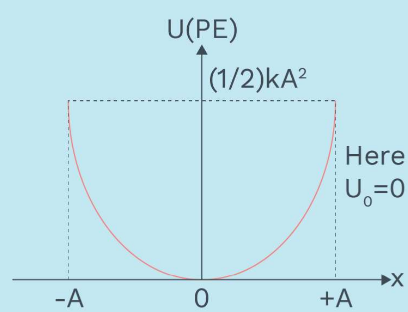
(ii) At x = ± A(extreme position)

$$\boxed{PE_{max} = \frac{1}{2}kA^2}$$

Concept Reminder

On an average a human heart is found to beat 75 times in a minute. Therefore, beat frequency of heart is 1.25 Hz.

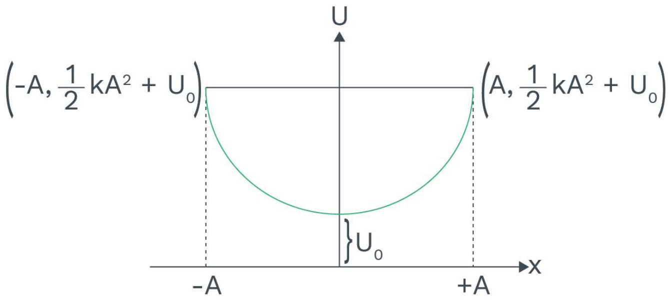
Concept Reminder



(i) At $x = 0 \Rightarrow PE_{min} = 0$

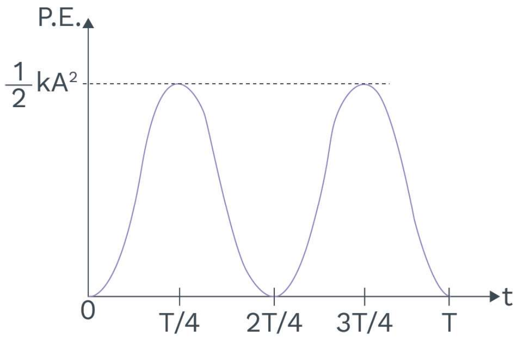
(ii) At $x = \pm A \Rightarrow PE_{max} = ½\,kA^2$

If potential energy is non zero at mean position then



Graph of PE with time:

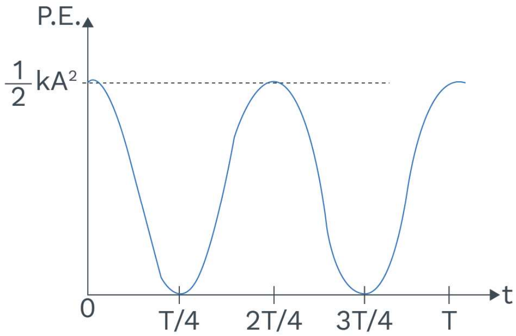
- If $x = A \sin(\omega t)$



then $PE = \frac{1}{2}k[A\sin(\omega t)]^2$

$\Rightarrow$ $\boxed{PE = \frac{1}{2}kA^2\sin^2(\omega t)}$ (Potential energy in terms of time)

- If $x = A\cos(\omega t)$



Rack your Brain

A particle is executing SHM along a straight line. Its velocities at distances x_1 and x_2 from the mean position are v_1 and v_2 respectively. Find its time period in terms of given quantities.

Concept Reminder

Any periodic function can be expressed as a superposition of sine and cosine functions of different time periods with suitable coefficients.

then $PE = \frac{1}{2}k[A\cos(\omega t)]^2$

$$\boxed{PE = \frac{1}{2}kA^2\cos^2(\omega t)}$$

Note: Frequency of P.E. is double the frequency of SHM.

2. Kinetic Energy:

We know that $KE = \frac{1}{2}mv^2$ and

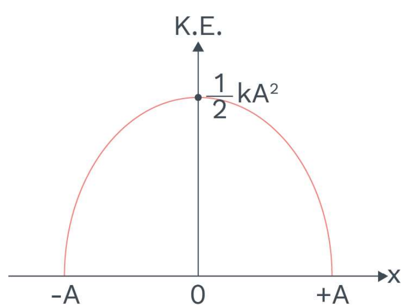
velocity in SHM $v = \omega\sqrt{A^2 - x^2}$

So, $KE = \frac{1}{2}m\left[\omega\sqrt{A^2 - x^2}\right]^2 = \frac{1}{2}m\omega^2(A^2 - x^2)$

$$\boxed{KE = \frac{1}{2}k(A^2 - x^2)} \qquad [\because k = m\omega^2]$$

(Kinetic energy in terms of displacement)

Graph between K.E. and displacement:



(i) At mean position (x = 0)

$KE_{max} = \frac{1}{2}kA^2$

(ii) At extreme position (x = ±A)

$KE_{min} = 0$

Graph of K.E. with time:

If v = Aω cos ωt (When x = A sin ωt)

then,

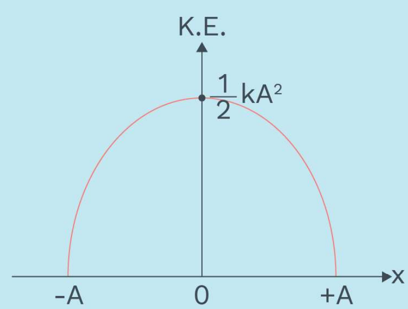
$$KE = \frac{1}{2}m[A\omega\cos\omega t]^2$$

Concept Reminder

$KE = \frac{1}{2}m\omega^2(A^2 - x^2)$

$= \frac{1}{2}K(A^2 - x^2)$

Concept Reminder

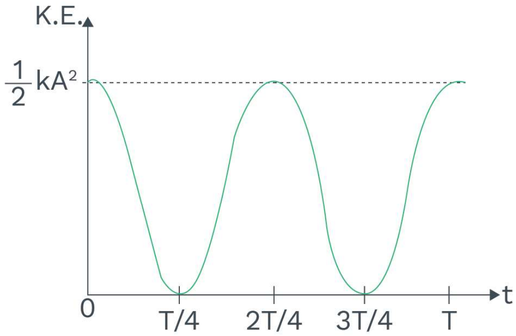


(i) At x = 0 ⇒ $KE_{max} = \frac{1}{2}kA^2$

(ii) At x = ± A ⇒ $KE_{min} = 0$

$$KE = \frac{1}{2}m\omega^2A^2\cos^2\omega t$$

$$\boxed{KE = \frac{1}{2}kA^2\cos^2\omega t}$$

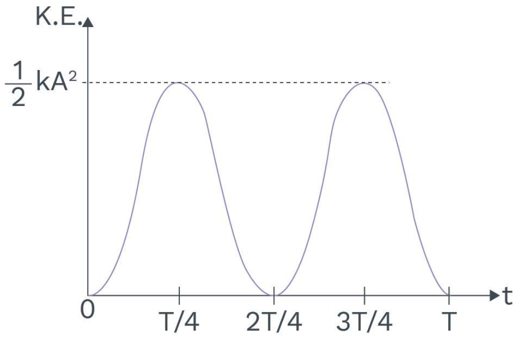


If v = –Aω sin ωt (when x = A cos ωt)
then,

$$KE = \frac{1}{2}m[-A\omega\sin\omega t]^2$$

$$KE = \frac{1}{2}m\omega^2A^2\sin^2\omega t$$

$$\boxed{KE = \frac{1}{2}kA^2\sin^2\omega t}$$



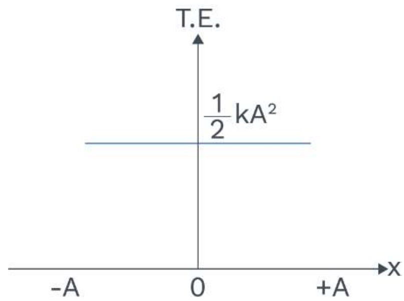
Note: Frequency of K.E. is double the frequency of SHM.

3. Total Energy: Total energy is SHM in given by
T.E. = K.E. + P.E.

Concept Reminder

Frequency of potential energy and kinetic energy is twice the frequency of SHM.

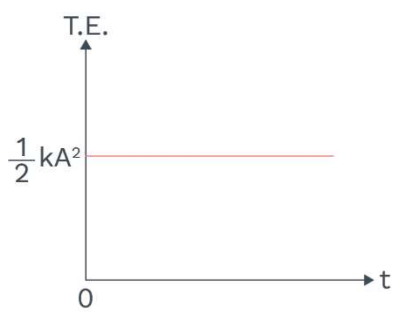
(i) In terms of displacement



$$TE = \frac{1}{2}kx^2 + \frac{1}{2}k(A^2 - x^2)$$

$$TE = \frac{1}{2}kA^2 = \text{constant}$$

(ii) In terms of time



$$TE = \frac{1}{2}kA^2\sin^2\omega t + \frac{1}{2}kA^2\cos^2\omega t$$

$$TE = \frac{1}{2}kA^2[\sin^2\omega t + \cos^2\omega t]$$

$$TE = \frac{1}{2}kA^2 = \text{constant.}$$

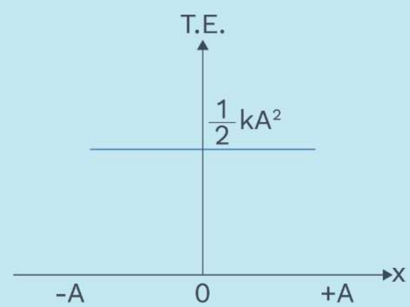
At mean position:

$$TE = \underbrace{\frac{1}{2}kA^2}_{KE_{max}} + \underbrace{U_0}_{PE_{min}}$$

At extreme position:

$$TE = \underbrace{\underbrace{\frac{1}{2}kA^2}_{PE_{extreme}} + \underbrace{U_0}_{PE_{mean}}}_{PE_{max}}$$

Concept Reminder



Total energy in SHM

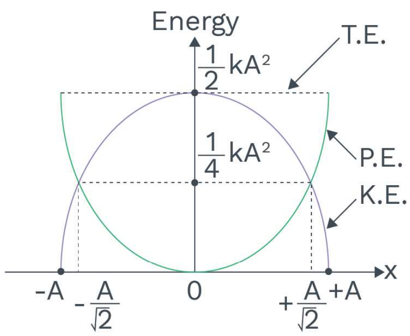
$$TE = \frac{1}{2}kA^2 = \text{constant}$$

Concept Reminder

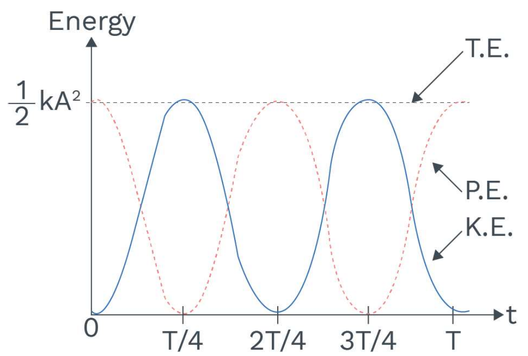
The conversion between radian and degree is not similar to that between metre and centimetre or mile. If the argument of a trigonometric function is stated without units, it is understood that the unit is radian. On the other hand, if degree is to be used as the unit of angle, then it must be shown explicitly.

Combined graphs of energy for SHM:

(i) With displacement



(ii) With time



(Here x = A sinωt)

Average value of K.E. and P.E.:

$\because \quad \langle \sin^2 \omega t \rangle = \langle \cos^2 \omega t \rangle = \frac{1}{2}$

So average value of K.E. and P.E. is

$$\boxed{\langle KE \rangle_t = \langle PE \rangle_t = \frac{1}{4}kA^2}$$

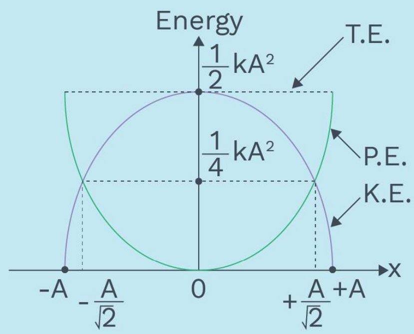
Ex. A particle doing SHM on a 4 cm long straight line. Find the position from mean, where its K.E. and P.E. is same.

Sol. Amplitude $= \frac{4}{2} = 2$ cm

According to question K.E. = P.E.

Concept Reminder

Variation of energy with displacement



$$\Rightarrow \quad \frac{1}{2}k(A^2 - x^2) = \frac{1}{2}kx^2$$

$$\Rightarrow \quad A^2 - x^2 = x^2 \quad \Rightarrow 2x^2 = A^2$$

$$\Rightarrow \quad x = \pm\frac{A}{\sqrt{2}} = \pm\frac{2}{\sqrt{2}} = \pm\sqrt{2}\text{ cm}$$

Ex. The P.E. of a particle in SHM is 2.5 J when its displacement is half of the amplitude. Determine the total energy of particle.

Sol. $TE = \frac{1}{2}kA^2$ and $PE = \frac{1}{2}kx^2$

$$\Rightarrow \quad 2.5 = \frac{1}{2}k\left(\frac{A^2}{4}\right)$$

$$\Rightarrow \quad \frac{1}{2}kA^2 = TE = 10\text{ J}$$

Ex. A particle starts oscillating simple harmonically from its equilibrium position with time period T. Determine ratio of K.E. and P.E. of the particle at time $t = \frac{T}{12}$.

Sol. At $t = \frac{T}{12}$

$$x = A\sin\left(\frac{2\pi}{T}\times\frac{T}{12}\right) = A\sin\frac{\pi}{6} = \frac{A}{2}$$

So, $KE = \frac{1}{2}k(A^2 - x^2) = \frac{3}{4}\times\frac{1}{2}kA^2$

and $PE = \frac{1}{2}kx^2 = \frac{1}{4}\times\frac{1}{2}kA^2$

$$\therefore \quad \frac{KE}{PE} = \frac{3}{1}$$

Ex. P.E. at mean position of a particle is 3J. If the mean K.E. is 4J, find its total energy.

Sol. $U_0 = 3\text{ J}$

Mean $KE = \frac{1}{4}kA^2 = 4\text{ J}$

$$\Rightarrow \quad \frac{1}{2}kA^2 = 8\text{ J}$$

So, total energy is

$$= \frac{1}{2}kA^2 + U_0 = 8\,J + 3\,J = 11\,J\,.$$

Ex. A particle of mass "m" is performing SHM. It's PE is given $U = U_0[1 - \cos(\alpha x)]$ then find time period of motion for small oscillations.

Sol. $U = U_0[1 - \cos(\alpha x)]$

$$F = \frac{-dU}{dx} \Rightarrow F = -U_0\alpha\sin(\alpha x)$$

For small oscillation

$\Rightarrow$ $\sin(\alpha x) \approx \alpha x$ [Since, $\sin\theta \approx \theta$]

$F = -(U_0\,\alpha^2)x$

$F = -kx$

Now, $T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{U_0\alpha^2}}$.

Ex. A particle of mass 1 kg is performing SHM along x-axis its PE is given by $U = (x^2 - 4x + 5)$ J. If it's maximum speed at mean position is $3\sqrt{2}$ m/s, then find:

(i) Mean position

(ii) Time period

(iii) Maximum amplitude

(iv) Maximum K.E.

(v) Maximum P.E.

(vi) U_{max}

Sol. (i) $F = -\frac{dU}{dx} = -\frac{d}{dx}(x^2 - 4x + 5)$

$F = -2x + 4$

For mean position

$\Rightarrow$ $F = -2x + 4 = 0 \Rightarrow x = 2m.$

(ii) $T = 2\pi\sqrt{\frac{1}{2}} = \sqrt{2}\,\pi\,\text{sec}$

(iii) $V_{max} = A\omega = 3\sqrt{2}$

$A = 3$ m

(iv) $KE_{max} = \frac{1}{2}kA^2 = \frac{1}{2} \times 2 \times 9 = 9$ Joule

(v) $U_{min} = (x^2 - 4x + 5) = (2^2 - 8 + 5) = 1$ Joule

(vi) $U_{max} = TE = 9 + 1 = 10$ J

Ex. A particle of mass 0.50 kg executes a simple harmonic motion under a force F = – (50 N/m)x. If it crosses the centre of oscillation with a speed of 10 m/s, find the amplitude of the motion.

Sol. The kinetic energy of the particle when it is at the centre of oscillation is

$$E = \frac{1}{2}mv^2 = \frac{1}{2}(0.50\,kg)(10\ m/s)^2 = 2.5\,J$$

The potential energy is zero here. At the maximum displacement x = A, the speed is zero and hence the kinetic energy is zero. The potential energy here is $\frac{1}{2}kA^2$. As there is no loss of energy,

$$\frac{1}{2}kA^2 = 2.5\,J$$

The force on the particle is given by

F = – (50 N/m) x

Thus, the spring constant is k = 50 N/m.

Equation (i) gives

$$\frac{1}{2}(50\ N/m)A^2 = 2.5\ J$$

or, $$A = \frac{1}{\sqrt{10}}\ m.$$

METHOD TO DETERMINE TIME PERIOD AND ANGULAR FREQUENCY IN SIMPLE HARMONIC MOTION

To understand the steps which are usually followed to find out the time period we will take one example.

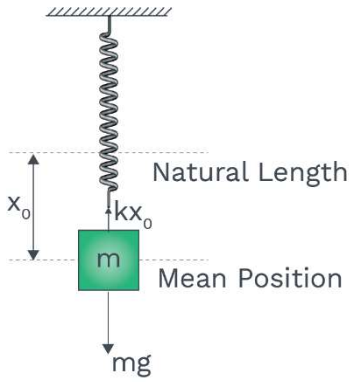
Case-I:

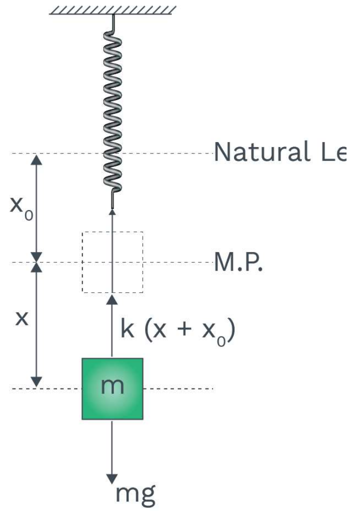
A mass m is attached to the free end of a massless spring of spring constant k with its other end fixed to a rigid support as shown in figure. Find out the time period (T) of the mass, if it is displaced slightly by an amount x downward.

The following steps are usually followed in this method:

Step-1: Find the stable equilibrium position which is usually known as the mean position. Net 'F' or torque on the particle at this position is zero. Potential energy is minimum.
In our example initial position is the mean position.
Step-2: Write down the mean position force relation. In above figure at mean position

$kx_0 = mg$ ...(1)





Step-3: Now displace the particle from its mean position by a small displacement x (in linear SHM)

Rack your Brain

The displacement of a particle along the x-axis is given by $x = a \sin^2\omega t$. The motion of the particle corresponds to:

(1) Simple harmonic motion of frequency $\frac{\omega}{\pi}$

(2) Simple harmonic motion of frequency $\frac{3\omega}{2\pi}$

(3) Non-simple harmonic motion

(4) Simple harmonic motion of frequency $\frac{\omega}{2\pi}$

Concept Reminder

The total mechanical energy of a harmonic oscillator is thus independent of time as expected for motion under any conservative force.

or angle θ (in case of an angular SHM) as shown in figure.

Step-4: Write down the net force on the particle in the displaced position. From the above figure.

$F_{net} = mg - k(x + x_0)$ ...(2)

Step-5: Now try to reduce this net force equation in the form of $F = -kx$ (in linear S.H.M.) or $\tau = -k\theta$ (in angular SHM) using mean position force relation in step 2 or binomial theorem.

From equation (2),

$F_{net} = mg - kx - kx_0$

Using equation (1) in above equation

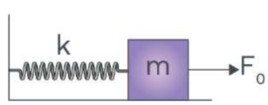
$F_{net} = -kx$ (Newton) ...(3)

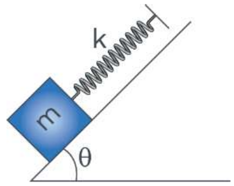
Equation (3) shows that the net force acting towards mean position and is proportional to x, but in this S.H.M. constant K_{SHM} is replaced by spring constant k. So,

$$T = 2\pi\sqrt{\frac{m}{K_{SHM}}} = 2\pi\sqrt{\frac{m}{k}}$$

Note: If we apply constant force on the string then time period T is always same

$$T = 2\pi\sqrt{\frac{m}{K_{SHM}}}$$





In above both cases $T = \left(2\pi\sqrt{\frac{m}{k}}\right)$.

Case-II:

The string, the spring and the pulley shown in figure are light. Find the time period of the mass m.

Rack your Brain

The phase difference between the instantaneous velocity and acceleration of a particle executing simple harmonic motion is:

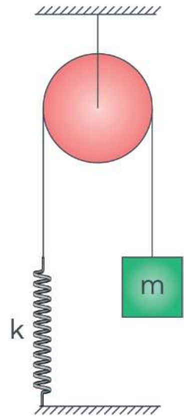
(1) Zero (2) 0.5 π

(3) π (4) 0.707 π

Concept Reminder

Time period of spring-block system is-

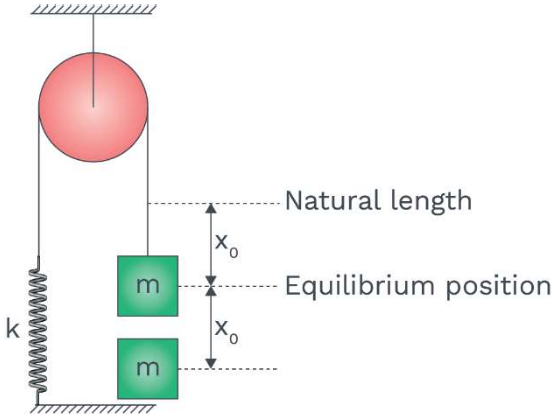
$$T = 2\pi\sqrt{\frac{m}{k}}$$



> **Concept Reminder**
>
> For both horizontal and vertical spring-block system, the value of time period remains same.

Let in equilibrium position of the block, extension in spring is x_0.

$\therefore \quad kx_0 = mg$ ...(1)



> **Concept Reminder**
>
> There are no physical examples of absolutely pure simple harmonic motion. In practice we come across systems that execute simple harmonic motion approximately under certain conditions.

Now if we displace the block by x in the downward direction, net force on the block towards mean position is

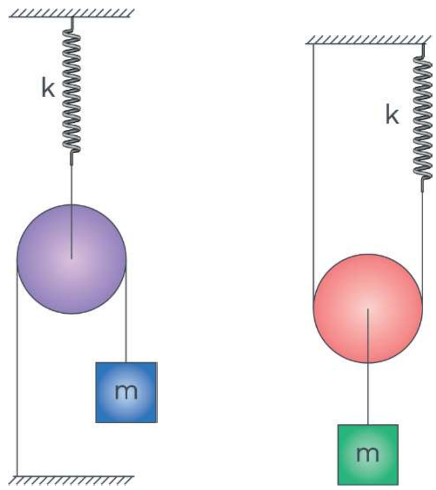
$$F = k(x + x_0) - mg = kx \text{ [using (1)]}$$

Hence the net force is acting towards mean position and is also proportional to x. So, the particle will perform S.H.M. and its time period would be

$$T = 2\pi\sqrt{\frac{m}{k}}$$

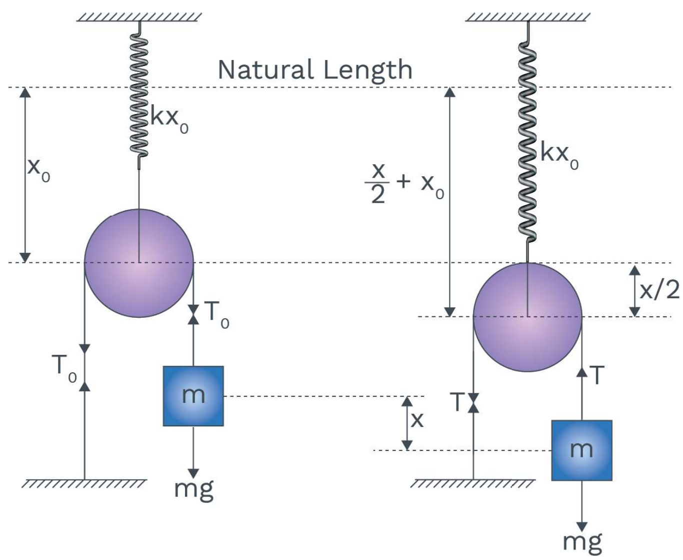
Case-III:

Diagram shows a system consisting of a massless pulley, a spring of force constant k and a block of mass m. If the block is slightly displaced vertically down (– y-direction) from its equilibrium position and then released, find the period of its vertical oscillation in cases (a) & (b).



Let us assume that in equilibrium condition spring is x_0 elongate from its natural length.

Step-1:



In equilibrium $T_0 = mg$ and $kx_0 = 2T_0$

$\Rightarrow \; kx_0 = 2mg$ ...(1)

If the mass m moves down (y-direction) a distance x from its equilibrium position then pulley will move down by $\frac{x}{2}$. So the extra force in spring will be $\frac{kx}{2}$. From figure

$$F_{net} = mg - T = mg - \frac{k}{2}\left(x_0 + \frac{x}{2}\right)$$

$$F_{net} = mg - \frac{kx_0}{2} - \frac{kx}{4}$$

From equation (1),

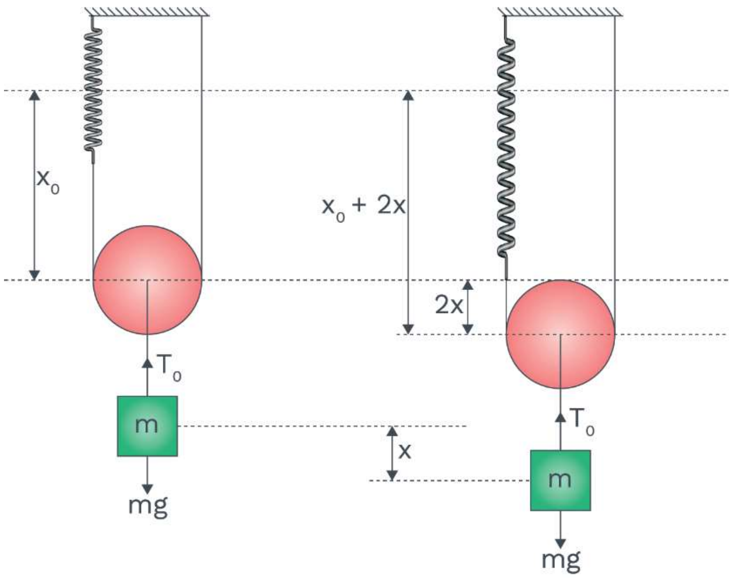
$$F_{net} = \frac{-kx}{4} \qquad ...(3)$$

Now compare eq. (3) with $F = -K_{SHM}\,x$

then $K_{SHM} = \frac{k}{4}$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{m}{K_{SHM}}} = 2\pi\sqrt{\frac{4m}{K}}$$

Step-2: In this situation if the mass m moves down (y-direction) distance x from its equilibrium position, then pulley will also move by x and so the spring will stretch by 2x.



At equilibrium $kx_0 = \frac{T_0}{2} = \frac{mg}{2}$

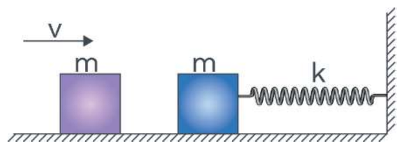
When block is displaced

$$F_{net} = mg - T = mg - 2k\,(x_0 + 2x)$$
$$= -\,4\,kx$$

Now $F = -\,K_{SHM}\,x$ then $K_{SHM} = 4\,K$

So, time period $T = 2\pi\sqrt{\dfrac{m}{4k}}$.

Ex. The left block in figure collides inelastically (0 < e < 1) with the right block of mass (m) and sticks to it. Find the 'A' of the resulting simple harmonic motion.

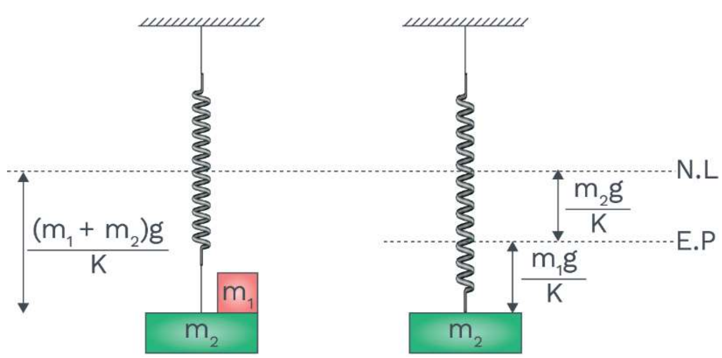


Sol. The collision is for a small 'T' only, we can apply the principal of conservation of momentum. The common velocity after the collision is $\dfrac{v}{2}$. The kinetic energy $= \dfrac{1}{2}(2m)\left(\dfrac{v}{2}\right)^2 = \dfrac{1}{4}mv^2$. This is also the total (energy) 'E' of vibration as the spring is unstretched at this moment. If the amplitude is 'A', the total energy can also be written as $\dfrac{1}{2}kA^2$. Thus,

$$\frac{1}{2}kA^2 = \frac{1}{4}mv^2,\ \text{giving}\ A = \sqrt{\frac{m}{2k}}\,v.$$

Case-IV:

The system is in equilibrium (F = 0) and at rest. Now mass m_1 is removed from m_2. Find the 'T' and 'A' of resultant motion. (Given: spring constant is K).



Initial extension in the spring

$$x = \frac{(m_1 + m_2)g}{k}$$

Now, if we remove mass (m_1), equilibrium position (E.P.) of m_2 will be $\frac{m_2 g}{K}$ below unstreached length (x) of spring.

At the I.P. (initial position), since velocity is zero i.e. it is the extreme position.

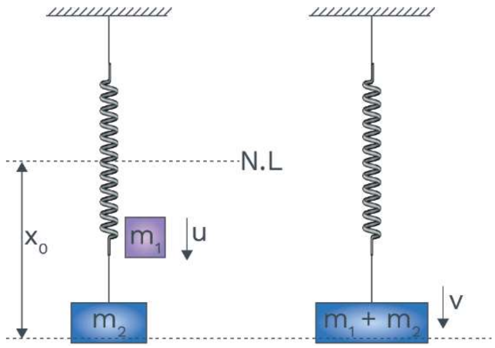
Hence amplitude = $\frac{m_1 g}{K}$

Time period = $2\pi\sqrt{\frac{m_2}{K}}$.

Ex. Block of mass m_2 is in equilibrium and at rest. The mass m_1 moving with velocity u m/s vertically downwards collides with m_2 & sticks to it. Find out the energy of oscillation.

Sol. At equilibrium position $m_2 g = kx_0$

$$\Rightarrow \quad x_0 = \frac{m_2 g}{K}$$



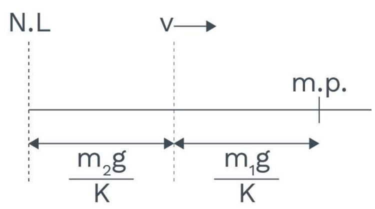
After collision m_2 sticks to m_1.

$\therefore$ By momentum conservation.

$$m_1 u = (m_1 + m_2)v$$

$$v = \frac{m_1 u}{m_1 + m_2}$$

Now both the blocks are executing S.H.M. which can be interpreted as follows:



Now, we know that

$$v^2 = \omega^2(A^2 - x^2) \qquad ...(1)$$

$$\omega^2 = \frac{k}{m_1 + m_2}$$

$$\Rightarrow \quad x = \frac{m_1 g}{k}$$

Put the values of v, ω^2 & x in eq. (1)

$$\left(\frac{m_1 u}{m_1 + m_2}\right)^2 = \left(\frac{k}{m_1 + m_2}\right)\left[A^2 - \left(\frac{m_1 g}{k}\right)^2\right]$$

$$\Rightarrow \quad kA^2 = \left[\left(\frac{m_1^2 u^2}{m_1 + m_2}\right) + \left(\frac{m_1^2 g^2}{k}\right)\right]$$

Energy of oscillation $= \frac{1}{2}kA^2$

$$= \frac{1}{2}\left[\left(\frac{m_1^2 u^2}{m_1 + m_2}\right) + \left(\frac{m_1^2 g^2}{k}\right)\right]$$

Ex. A body of mass 'm' falls from a height h on to the pan of a spring balance. The masses of the pan and spring are negligible. The spring constant of the spring is 'k'. Having stuck to the pan the body starts performing harmonic oscillations in the vertical direction. Find the 'A' and energy (E) of oscillation.

Sol. Suppose by falling down through a height 'h', the mass 'm' compresses the spring balance by a length 'x'.

$$x = \frac{mg}{k}, \quad \omega = \sqrt{\frac{k}{m}}$$

Velocity at Q, $v = \sqrt{2gh}$

$$\because \quad v = \omega\sqrt{A^2 - x^2}$$

$$\sqrt{2gh} = \sqrt{\frac{k}{m}}\sqrt{A^2 - \left(\frac{mg}{k}\right)^2}$$

$$\Rightarrow \quad A = \frac{mg}{k}\sqrt{1 + \frac{2kh}{mg}}$$

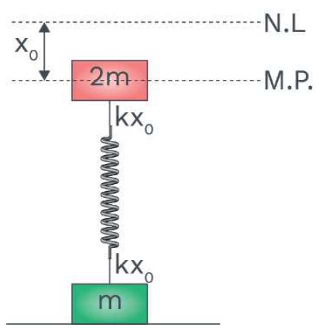
Energy of oscillation

$$= \frac{1}{2}kA^2 = \frac{1}{2}k\left(\frac{mg}{k}\right)^2\left(1+\frac{2kh}{mg}\right)$$

$$= mgh + \frac{(mg)^2}{2k}.$$

Ex. A body of mass '2m' is connected to another body of mass 'm' as shown in figure. The mass '2m' performs vertical S.H.M. Then find out the maximum amplitude of '2m' such that mass 'm' doesn't lift up from the ground.

Sol. Given situation 2m mass is in equilibrium condition. Consider spring is compressed x_0 distance from its natural length.



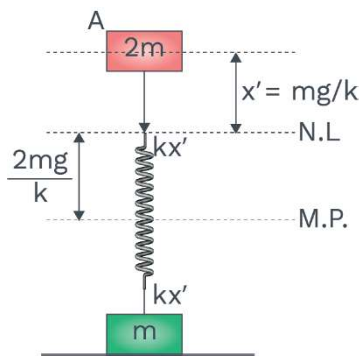
$\Rightarrow \quad kx_0 = 2mg$

$\Rightarrow \quad x_0 = \frac{2mg}{k}$

The lower block will be lift up, only in the case when the spring force on it will be greater than equal to mg and in upward direction

$\Rightarrow \quad kx' = mg \quad \Rightarrow \quad x' = \frac{mg}{k}$

Above situation arises when 2m block moves upward mg/k from natural length as shown in figure.



Block m doesn't lift up if the maximum amplitude of the 2m block is

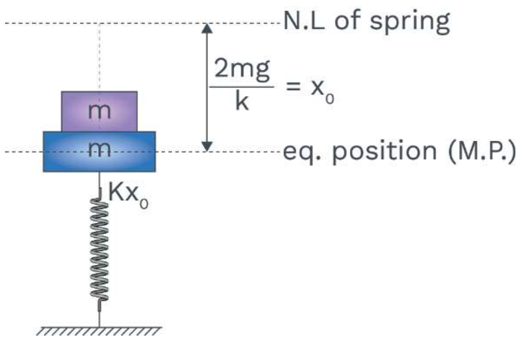
$$= \frac{2mg}{k} + \frac{mg}{k} = \frac{3mg}{k}.$$

Ex. A block of mass m is at rest on the another block of same mass as shown in figure. Lower block is attached to the spring then determine the maximum amplitude of motion so that both the block will remain in contact.

Sol. The blocks will remain in contact till the blocks do not go above the natural length of the spring, because after this condition the deceleration of lower block becomes more then upper block due to spring force. So they will get seprated.

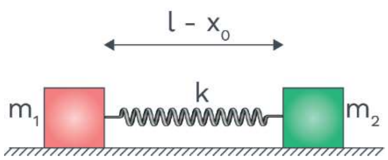
So maximum possible amplitude

$$x_0 = \frac{2mg}{k}.$$

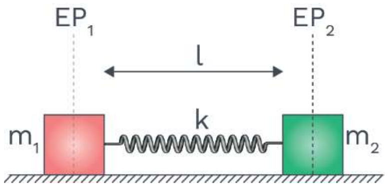


Case-V: Two block systems:

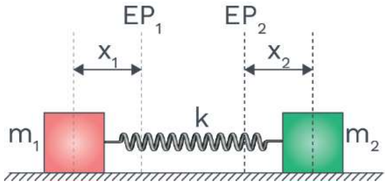
Ex. Two blocks of mass m_1 and m_2 are connected with a spring of natural length l and spring constant k. The system is lying on a smooth horizontal surface. Initially spring is compressed by x_0 as shown in figure.
Show that the two blocks will perform SHM about their equilibrium position. Also (a) find the time period, (b) find amplitude of each block and (c) length of spring as a function of time.



Sol. (a) Here both the blocks will be in equilibrium at the same time when spring is in its natural length. Let EP_1 and EP_2 be equilibrium positions of block A and B as shown in figure.



Let at any time during oscillations, blocks are at a distance of x_1 and x_2 from their equilibrium positions.



As no external force is acting on the spring block system

$\therefore \quad (m_1 + m_2)\Delta x_{cm} = m_1x_1 - m_2x_2 = 0$

or $\quad m_1x_1 = m_2x_2 \quad \ldots(i)$

For 1st particle, force equation can be written as

$$k(x_1 + x_2) = -m_1\frac{d^2x_1}{dt^2}$$

or $\quad k\left(x_1 + \frac{m_1}{m_2}x_1\right) = -m_1a_1 \qquad \left(\text{from equation (i)} \ \ x_2 = \frac{m_1x_1}{m_2}\right)$

or $\quad a_1 = -\frac{k(m_1 + m_2)}{m_1m_2}x_1$

$\therefore \quad \omega^2 = \frac{k(m_1 + m_2)}{m_1m_2}$

Hence,

$$T = 2\pi\sqrt{\frac{m_1m_2}{k(m_1 + m_2)}} = 2\pi\sqrt{\frac{\mu}{K}}$$

Where, $\mu = \frac{m_1m_2}{(m_1 + m_2)}$ which is known as reduced mass.

(b) Let the amplitude of blocks be A_1 and A_2.

$$m_1A_1 = m_2A_2$$

By energy conservation;

$$\frac{1}{2}k(A_1 + A_2)^2 = \frac{1}{2}kx_0^2$$

or $\quad A_1 + A_2 = x_0$

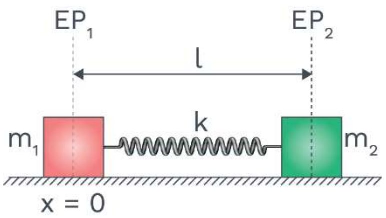
or $\quad A_1 + \frac{m_1}{m_2}A_1 = x_0 \qquad \because A_2 = \frac{m_1A_1}{m_2}$

or $\quad A_1 = \frac{m_2x_0}{m_1 + m_2}$

Similarly, $A_2 = \frac{m_1x_0}{m_1 + m_2}$

(c) Let equilibrium position of 1st particle be origin, i.e., x = 0. x co-ordinate of particles can be written as

$$x_1 = A_1 \cos \omega t \text{ and } x_2 = l - A_2 \cos \omega t.$$



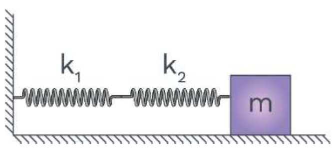
Hence, length of spring can be written as:

Length = $x_2 - x_1$

$= \ell - (A_1 + A_2)\cos\omega t$

COMBINATION OF SPRINGS:

1. Series Combination:



Total displacement $x = x_1 + x_2$

Tension in both springs = $k_1x_1 = k_2x_2$

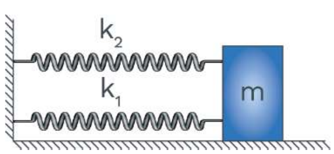
Equivalent constant in series combination K_{eq} is given by:

$$\boxed{\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}}$$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{m}{k_{eq}}}$$

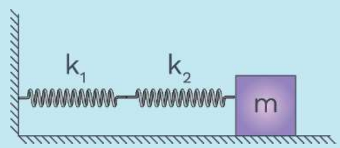
In series combination, tension is same in all the springs & extension will be different. (If k is same then deformation is also same)

2. Parallel combination:



Extension is same for both springs but force acting will be different. Force acting on the system = F

Concept Reminder

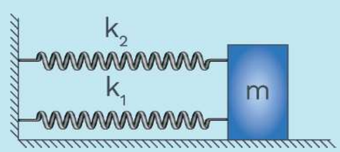


In series combination of spring

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}$$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{m}{k_{eq}}}$$

Concept Reminder



In parallel combination of spring

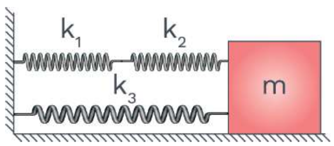
$$k_{eq} = k_1 + k_2$$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{m}{k_{eq}}}$$

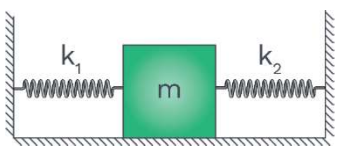
$$\therefore \quad F = -(k_1x + k_2x)$$
$$\Rightarrow \quad F = -(k_1 + k_2)\,x$$
$$\Rightarrow \quad F = -k_{eq}x$$
$$\therefore \quad \boxed{k_{eq} = k_1 + k_2}$$
$$\Rightarrow \quad T = 2\pi\sqrt{\frac{m}{k_{eq}}}.$$

Ex. Find the time period of the oscillation of mass m in figure a and b. What is the equivalent spring constant of the spring in each case?

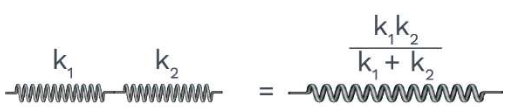
(a)



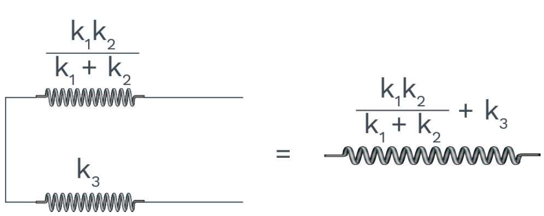
(b)



Sol. In figure (a)



Which gives



$$k_{eq} = \frac{k_1k_2}{k_1 + k_2} + k_3 = \frac{k_1k_2 + k_2k_3 + k_1k_3}{k_1 + k_2}$$

Now, $T = 2\pi\sqrt{\dfrac{m}{k_{eq}}} = 2\pi\sqrt{\dfrac{m(k_1 + k_2)}{k_1k_2 + k_2k_3 + k_1k_3}}$

Rack your Brain

A particle is executing a simple harmonic motion. Its maximum acceleration is α and maximum velocity is β. Then, its time period of vibration will be:

(1) $\dfrac{2\pi\beta}{\alpha}$ (2) $\dfrac{\beta^2}{\alpha^2}$

(3) $\dfrac{\alpha}{\beta}$ (4) $\dfrac{\beta^2}{\alpha}$

In figure (b):

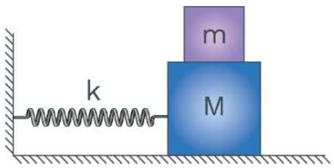
If the block is displaced slightly by an amount x then both the spring are displaced by x from their natural length so it is parallel combination of springs.

Which gives,

$$k_{eq} = k_1 + k_2$$

$$T = 2\pi\sqrt{\frac{m}{k_{eq}}} = 2\pi\sqrt{\frac{m}{k_1 + k_2}}$$

Ex. The coefficient of friction between the two blocks shown in figure is μ and the horizontal plane is smooth.



(a) If the system is slightly displaced and released, find the time period.

(b) Find the magnitude of the frictional force between the body when the displacement from the mean position is x.

(c) What can be the maximum amplitude if the upper block does not slip relative to the lower bodies?

Sol. (a) For small amplitude, the two bodies oscillate together. The angular frequency is

$\omega = \sqrt{\frac{k}{M+m}}$ and so, the time period

$$T = 2\pi\sqrt{\frac{M+m}{k}}$$

(b) The acceleration of the bodies at displacement x from the mean position is

Concept Reminder

In series combination, extension of springs will be reciprocal of its spring constant.

$$\therefore \quad k \propto \frac{1}{\ell}$$

$$\therefore \quad k_1\ell_1 = k_2\ell_2 = k_3\ell_3$$

If a spring is cut in n equal pieces then spring constant of one piece will be nk.

$$a = -\omega^2 x = \left(\frac{-kx}{M+m}\right)$$

The resultant force on the upper body is, therefore,

$$ma = \left(\frac{-mkx}{M+m}\right)$$

This force is provided by the friction of the lower block. Hence, the magnitude of the frictional force is $\left(\frac{mkx}{M+m}\right)$

(c) Maximum force of friction required for simple harmonic motion of the upper body is $\frac{mkA}{M+m}$ at the extreme positions. But the maximum frictional force can only be μ mg. Hence

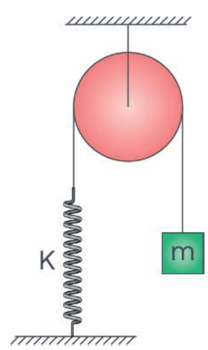
$$\frac{mkA}{M+m} = \mu mg$$

or $$A = \frac{\mu(M+m)g}{k}$$

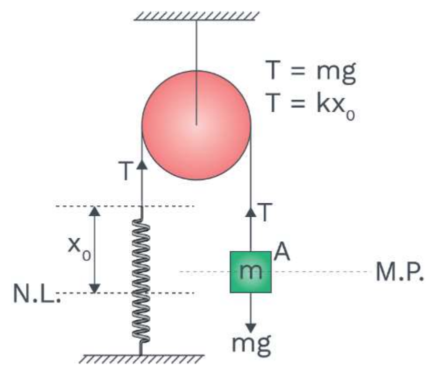
ENERGY METHOD:

Another method of finding time period of SHM is energy method. To understand this method we will consider the following example.

Ex. Figure shows a system consisting of pulley having radius R, a spring of force constant k and a block of mass m. Find the period of its vertical oscillation.



Sol. The following steps are usually followed in this method:



Step-1. Find the mean position. In following figure point A shows mean position.
Step-2. Write down the mean position force relation from figure.

$mg = kx_0$

Step-3. Assume that particle is performing SHM with amplitude A. Then displace the particle from its mean position.
Step-4. Find the total mechanical energy (E) in the displaced position since, mechanical energy in SHM remains constant $\frac{dE}{dt} = 0$.

$$E = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}k(x + x_0)^2 - mgx$$

$$E = \frac{1}{2}mv^2 + \frac{1}{2}I\frac{v^2}{R^2} + \frac{1}{2}k(x + x_0)^2 - mgx$$

$$\frac{dE}{dt} = \frac{2mv}{2}\frac{dv}{dt} + \frac{2Iv}{2R^2}\frac{dv}{dt} + \frac{2k(x + x_0)}{2}\frac{dx}{dt} - mg\frac{dx}{dt} \quad ...(1)$$

Put $\frac{dx}{dt} = v$ and $\frac{dv}{dt} = \frac{d^2x}{dt^2}$

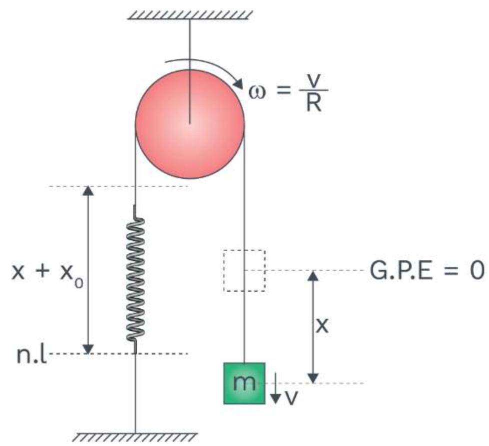
In equation (1) put,

$$\frac{dE}{dt} = 0$$

$$\Rightarrow \quad mv\frac{d^2x}{dt^2} + \frac{Iv}{R^2}\frac{d^2x}{dt^2} + kxv + kx_0v - mgv = 0$$

Which gives $\left(m + \frac{I}{R^2}\right)\frac{d^2x}{dt^2} + kx = 0$

$$\frac{d^2x}{dt^2} + \frac{k}{\left(m + \frac{I}{R^2}\right)}x = 0 \quad ...(2)$$

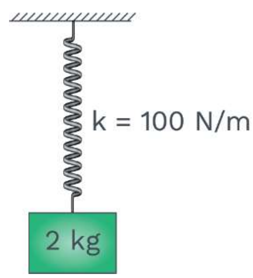


Compare eq. (2) with S.H.M eq. the

$$\omega^2 = \frac{k}{\left(m + \frac{I}{R^2}\right)}$$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{\left(m + \frac{I}{R^2}\right)}{k}}$$

Ex. Initially block is in equilibrium. Now we displace the block by a distance 0.4 m downward and then release. Find out:



(1) Elongation in spring when block is in equilibrium

(2) The period of oscillation

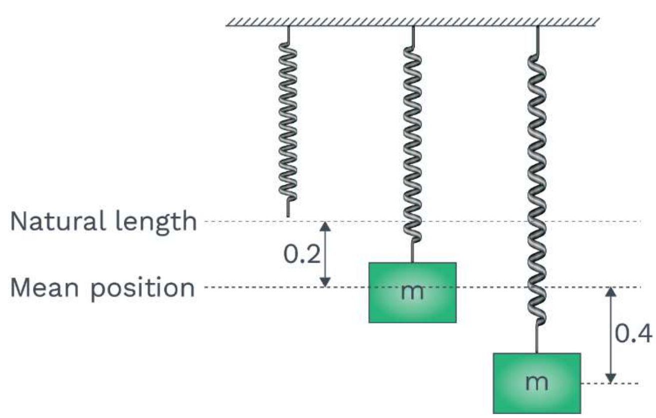
(3) Amplitude

(4) Maximum velocity of block

(5) Maximum elongation and compression in spring

Sol. **(1)** In equilibrium $mg = kx_0$

$$\Rightarrow \quad x_0 = \frac{m}{k} = \frac{2 \times 10}{100} = 0.2 \text{ m}$$



(2) Time period $T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{2}{100}} = \frac{\pi\sqrt{2}}{5}$ s s

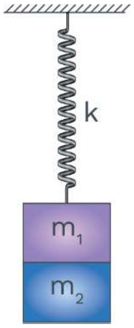
(3) Amplitude = 0.4 m

(4) $V_{max} = A\omega = 0.4 \times \frac{2\pi}{T} = 0.4 \times \frac{2\pi}{\sqrt{2}\pi} \times 5$

$A\omega = 2\sqrt{2}$ m/s

(5) Maximum elongation = 0.2 + 0.4 = 0.6 m
Maximum compression = 0.2 m

Ex. If 2 blocks of masses m_1 and m_2 are connected with spring of spring constant k and system is in equilibrium condition. Now a block of mass m_1 is removed from the system then it start to oscillate. Find time period and amplitude of motion.



Sol. Initially $(m_1 + m_2)\,g = -\,k(x + y)$ ...(i)

After removing m_1

$m_2g = -\,ky$

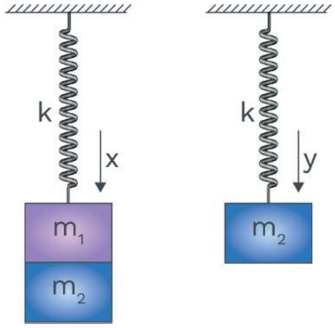
So in equation (i)

$m_1g + m_2g = -\,kx - ky$

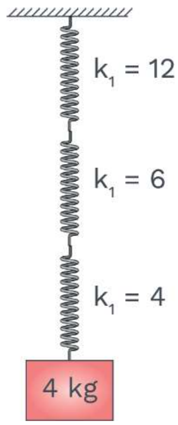
$\Rightarrow \quad m_1g + m_2g = -\,kx + ky$

$\Rightarrow \quad m_1g = -\,kx$

$\Rightarrow \quad x = \frac{m_1g}{k}$



Ex. Find the time period of given arrangement.



Sol. In series combination

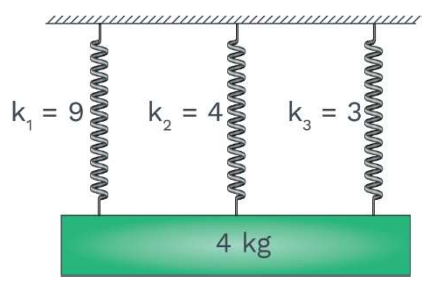
$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}$$

$$\frac{1}{k_{eq}} = \frac{1}{12} + \frac{1}{6} + \frac{1}{4} = \frac{1+2+3}{12} = \frac{6}{12} = \frac{1}{2}$$

$$\Rightarrow \quad k_{eq} = 2$$

$$T = 2\pi\sqrt{\frac{m}{k_{eq}}} = 2\pi\sqrt{\frac{4}{2}} = 2\sqrt{2}\pi \text{ s.}$$

Ex. Find time period of given arrangement.



Sol. In parallel combination

$$k_{eq} = k_1 + k_2 + k_3 = 9 + 4 + 3 = 16$$

$$T = 2\pi\sqrt{\frac{m}{k_{eq}}} = 2\pi\sqrt{\frac{4}{16}} = \pi \text{ s}$$

Ex. If a block of mass m is connected with spring of spring constant (k) and time period T. Now this spring is divided into 4 equal parts and all parts are arranged in parallel order and same mass is connected with the system. Find out time period of system.

Sol. $T = 2\pi\sqrt{\frac{m}{k}}$

$T' = 2\pi\sqrt{\frac{m}{k'_{eq}}}$ where $k' = 4k \Rightarrow k'_{eq} = 16\,k$

$\therefore \quad T' = 2\pi\sqrt{\frac{m}{16k}} = \frac{1}{4}\left(2\pi\sqrt{\frac{m}{k}}\right) \Rightarrow T' = \frac{T}{4}.$

Ex. A mass M attached to a horizontal spring, executes S.H.M. with amplitude A_1. When the mass M passes through its mean position then a smaller mass m is placed over it and both of them move together with amplitude A_2. Find the ratio of $\left(\frac{A_1}{A_2}\right)$.

Sol. Applying momentum conservation

$Mv_1 = (M + m)v_2$

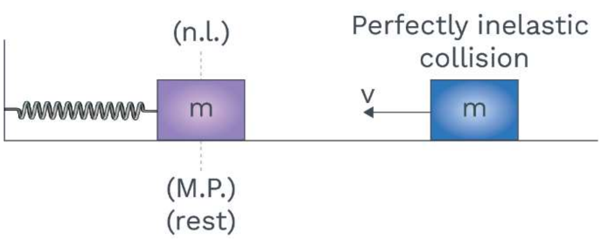
$M(A_1\omega_1) = (M + m)(A_2\omega_2)$

$$\frac{A_1}{A_2} = \left(\frac{M+m}{M}\right)\left(\frac{\omega_2}{\omega_1}\right)$$

$\because \quad \omega \propto \frac{1}{\sqrt{m}}$

$$\therefore \quad \frac{A_1}{A_2} = \left(\frac{M+m}{M}\right)\sqrt{\frac{M}{M+m}} = \sqrt{\frac{M+m}{M}} = \left(\frac{M+m}{M}\right)^{1/2}$$

Ex. Find speed of block immediately after collision and amplitude of motion.



Sol. Applying conservation of linear momentum

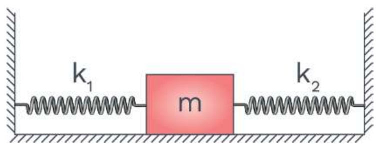
$\Rightarrow \quad mv = (m + M)v'$

$\Rightarrow \quad v' = \frac{v}{2}$

Now, $v_{max} = A\omega = \frac{v}{2} \Rightarrow A = \frac{v}{2\omega} = \frac{v}{2}\sqrt{\frac{2m}{k}}$

$= v\sqrt{\frac{m}{2k}}$.

Ex. Frequency of oscillation of a mass when it is connected with spring of spring constant k_1 is 3Hz. The same mass is connected with another spring of spring constant k_2 then the frequency is 6Hz. Find the resultant frequency when this mass is connected with both spring as shown in diagram.



Sol. This is a parallel combination

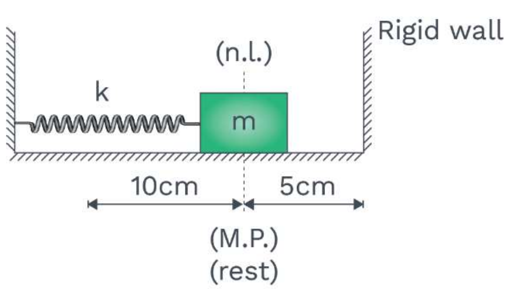
$\because$ displacement same.

So, for parallel combination

$$\nu^2 = \nu_1^2 + \nu_2^2 = (3)^2 + (6)^2$$

$\Rightarrow \quad \nu = \sqrt{9+36} = \sqrt{45} = 3\sqrt{5}$ Hz.

Ex. If block is moved to compress the spring by 10 cm and released then find out time period of its motion.

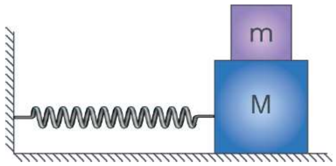


Sol. Time period of motion $= \frac{T}{4} + \frac{T}{4} + \frac{T}{12} + \frac{T}{12} = \frac{2T}{3}$

$$= \frac{2T}{3} = \frac{2}{3}\left(2\pi\sqrt{\frac{m}{k}}\right) = \frac{4\pi}{3}\sqrt{\frac{m}{k}}.$$

Ex. A block of mass m is on a horizontal slab of mass M which is moving horizontally and executing S.H.M. The coefficient of static friction between block and slab is μ. If block is not separated from slab then determine angular frequency of oscillation.

Sol. If block is not separated from slab then restoring force due to S.H.M. should be less than frictional force between slab and block.



$F_{restoring} \le F_{friction} \Rightarrow m\, a_{max.} \le \mu\, mg$

$\Rightarrow \quad a_{max.} \le \mu g$

$\Rightarrow \quad \omega^2 A \le \mu g \Rightarrow \omega \le \sqrt{\dfrac{\mu g}{A}}$.

Ex. Infinite springs with force constants k, 2k, 4k, 8k, ... respectively are connected in series. Calculate the effective force constant of the spring.

Sol. $\dfrac{1}{k_{eff}} = \dfrac{1}{k} + \dfrac{1}{2k} + \dfrac{1}{4k} + \dfrac{1}{8k} + \ldots\ldots\ldots\ldots\infty$ (For infinite G.P. $S_\infty = \dfrac{a}{1-r}$ where a = First term, r = common ratio)

$$\frac{1}{k_{eff}} = \frac{1}{k}\left[1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\ldots\ldots\right]$$

$$= \frac{1}{k}\left[\frac{1}{1-\frac{1}{2}}\right] = \frac{2}{k} \text{ so } k_{eff} = k/2.$$

Ex. Frequency of oscillation of a body is 6 Hz when force F_1 is applied and 8 Hz when F_2 is applied. If both forces F_1 and F_2 are applied together then find out the frequency of oscillation.

Sol. According to question $F_1 = -K_1x$ and $F_2 = -K_2x$

So $\quad n_1 = \dfrac{1}{2\pi}\sqrt{\dfrac{K_1}{m}} = 6$ Hz;

$n_2 = \dfrac{1}{2\pi}\sqrt{\dfrac{K_2}{m}} = 8$ Hz.

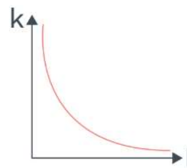
Now $F = F_1 + F_2 = -(K_1 + K_2)x$.

Therefore $n = \frac{1}{2\pi}\sqrt{\frac{K_1 + K_2}{m}}$

$$\Rightarrow \quad n = \frac{1}{2\pi}\sqrt{\frac{4\pi^2 n_1^2 m + 4\pi^2 n_2^2 m}{m}}$$

$$= \sqrt{n_1^2 + n_2^2} = \sqrt{8^2 + 6^2} = 10 \text{ Hz.}$$

Dependency of spring constant on length of spring :



Ex. If spring of constant K is cut into 2 parts of l_1 and l_2 such that length of one part is n times of other (l_1 = n l_2). Then, what is the constant of each part in terms of n and K?

Sol. $l = l_1 + l_2$ and $K \propto \frac{1}{\ell}$

$$\frac{K_2}{k} = \frac{\ell}{\ell_2}$$

$$\frac{K_2}{k} = \frac{\ell_1 + \ell_2}{\ell_2}$$

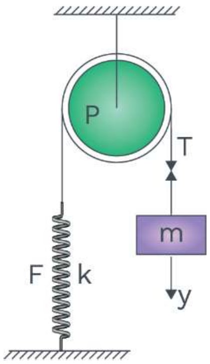
$$= \frac{\ell_1}{\ell_2} + 1$$

$$= \left(\frac{n\ell_2}{\ell_2} + 1\right)$$

K_2 = (n + 1) K.

Ex. Figure shows a system consisting of a massless pulley, a spring of force constant k = 4000 N/m and a block of mass m = 1 kg. If the block is slightly displaced vertically down from its equilibrium position and released find the frequency of its vertical oscillation in given cases.

Sol. **Case (A):** As the pully is fixed and string is inextensible, if mass m is displaced by y the spring will stretch by y, and as there is no mass between string and spring (as pully is massless) F = T = ky i.e., restoring force is linear and so motion of mass m will be linear simple harmonic with frequency



$$n_A = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{4000}{1}} \approx 10 \text{ Hz}$$

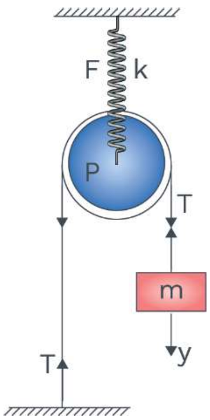
Case (B):

The pulley is movable and string inextensible, so if mass m moves down a distance y, the pulley will move down by (y/2). So the force in the spring F = k(y/2). Now as pully is massless F = 2T

$\Rightarrow$ T = F/2 = (k/4)y.

So, the restoring force on the mass

$$mT = \frac{1}{4} ky = k' = \frac{1}{4} K$$

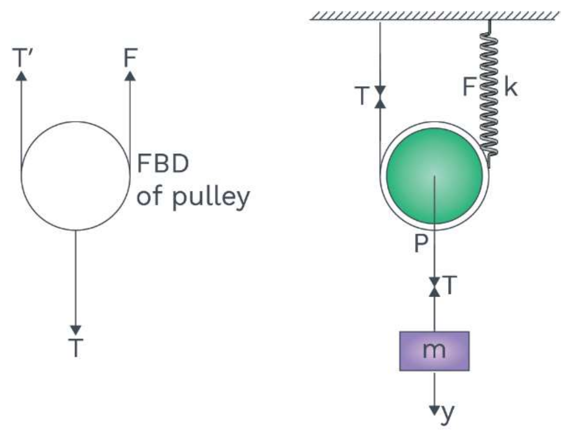


So, $n_B = \frac{1}{2\pi}\sqrt{\frac{k'}{m}} = \frac{1}{2\pi}\sqrt{\frac{k}{4\pi}} = \frac{n_A}{2} = 5 \text{ Hz}$

Case (C):

In this situation if the mass m moves by y the pully will also move by y and so the spring will stretch by 2y (as string is inextensible) and so T′ = F = 2ky. Now as pulley is massless so T = F + T′ = 4ky, i.e., the restoring force on the mass m

T = 4ky = k′y $\Rightarrow$ k′ = 4k



So, $n_c = \frac{1}{2\pi} \frac{1}{2\pi}\sqrt{\frac{k'}{m}} = \frac{1}{2\pi}\sqrt{\frac{4k}{m}} = 2n_A = 20$ Hz.

Angular SHM

To and fro motion about a fixed axis or position angularly is known as angular SHM.

In angular SHM the restoring torque (or angular acceleration) acting on the particle should always be proportional to the angular displacement of the particle and directed towards the equilibrium position.

$\therefore \quad \tau \propto -\theta \qquad$ or $\qquad \alpha \propto -\theta$

$\Rightarrow \quad \tau = -c\theta$

(where c = restoring torque constant, α = angular acceleration and I = Moment of Inertia)

$$\Rightarrow \quad \alpha = -\frac{c}{I}\theta$$

$$\Rightarrow \quad \boxed{\frac{d^2\theta}{dt^2} + \frac{c}{I}\theta = 0}$$

$$\Rightarrow \quad \boxed{\frac{d^2\theta}{dt^2} + \omega^2\theta = 0} \qquad \left[\because \omega = \frac{c}{I}\right]$$

Solution of above differential equation is

$$\boxed{\theta = \theta_0 \sin(\omega t + \phi)}$$

Which is an equation of angular displacement.

$\therefore$ Angular velocity

Concept Reminder

The damping force depends on the nature of the surrounding medium. If the block is immersed in a liquid, the magnitude of damping will be much greater and the dissipation of energy much faster.

Concept Reminder

In angular SHM

- $\tau = -c\theta$
- $\alpha = -\omega^2\theta$

$$\frac{d\theta}{dt} = \theta_0\omega\cos(\omega t + \phi)$$

and angular acceleration

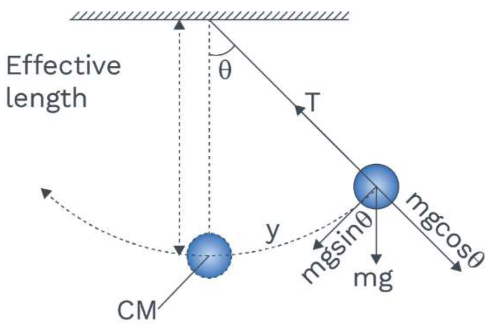
$$\alpha = \frac{d^2\theta}{dt^2} = -\theta_0\omega^2\sin(\omega t + \phi)$$

(iii) Comparison between linear and angular S.H.M.

Linear S.H.M.	Angular S.H.M.
$F \propto -x$ $F = -kx$ Where k is the restoring force constant	$\tau \propto -\theta$ $\tau = -C\theta$ Where C is the restoring torque constant
$a = -\frac{k}{m}x$	$\alpha = -\frac{C}{I}\theta$
$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$ It is known as differential equation of linear S.H.M.	$\frac{d^2\theta}{dt^2} + \frac{C}{I}\theta = 0$ It is known as differential equation of angular S.H.M.
$x = A\sin\omega t$ $a = \omega^2 x$ where ω is the angular frequency $\omega^2 = \frac{k}{m}$ $\omega = \sqrt{\frac{k}{m}} = \frac{2\pi}{T} = 2\pi n$ where T is time period and n is frequency	$\theta = \theta_0 \sin\omega t$ $\alpha = -\omega^2\theta$ $\omega^2 = \frac{C}{I}$ $\omega = \sqrt{\frac{C}{I}} = \frac{2\pi}{T} = 2\pi n$
$T = 2\pi\sqrt{\frac{m}{k}}$ $n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ This concept is valid for all types of linear S.H.M.	$T = 2\pi\sqrt{\frac{I}{C}}$ $n = \frac{1}{2\pi}\sqrt{\frac{C}{I}}$ This concept is valid for all types of angular S.H.M.

SIMPLE PENDULUM:-

If a heavy point mass 'm' is suspended by a inextensible, weightless and perfectly inextensible string from a rigid support, then this arrangement is called a simple pendulum.



Expression for time period

Method-I (Force Method)

For small angular displacement, $\sin\theta \approx \theta$, so that

$$F = -mg\sin\theta$$
$$= -mg\theta$$
$$= -\left(\frac{mg}{\ell}\right)y$$
$$= -ky \Rightarrow k = \frac{mg}{\ell}$$

(because $y = l\theta$), Thus, the time period of the simple pendulum is

$$T = 2\pi\sqrt{\frac{m}{k}} \text{ or } T = 2\pi\sqrt{\frac{\ell}{g}}$$

Method-II (Torque Method)

Bob of pendulum moves along the arc of circle in vertical plane. Here motion involved is angular and oscillatory where restoring torque is provided by gravitational force.

$\tau = -(mg)(l\sin\theta)$ (Negative sign shows opposite direction of τ and angular displacement)

$\tau = -mgl\theta$ (If angular displacement is small, then $\sin\theta \equiv \theta$)

$$I\alpha = -mgl\theta$$

$\alpha = -\dfrac{mg\ell}{I}\theta$ (where I = moment of inertia of bob about point of suspension so $I = ml^2$)

$$\text{Now } \alpha = \frac{mg\ell}{m\ell^2}\theta \Rightarrow \alpha = -\frac{g}{\ell}\theta \Rightarrow \frac{d^2\theta}{dt^2} + \frac{g}{\ell}\theta = 0$$

Concept Reminder

When a system (such as a simple pendulum or a block attached to a spring) is displaced from its equilibrium position and released, it oscillates with its natural frequency ω, and the oscillations are called free oscillations.

Concept Reminder

Time period of simple pendulum is

$$T = 2\pi\sqrt{\frac{\ell}{g}}$$

It is differential equation of angular SHM of simple pendulum.

Comparing with standard differential equation of angular SHM $\left(\frac{d^2\theta}{dt^2} + \omega^2\theta = 0\right)$

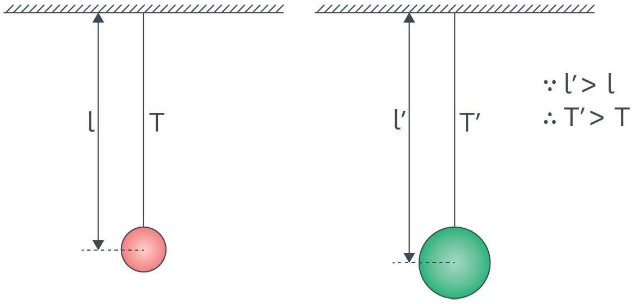
Therefore $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\ell}{g}}$

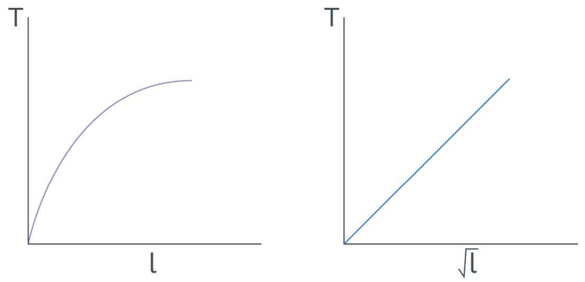
Note: Simple pendulum is the example of SHM but only when its angular displacement is very small.

Important points:

1. The time period of simple pendulum is independent from mass of the bob but it depends on size of bob (position of centre of mass). So in simple pendulum when a solid iron bob is replaced by light aluminium bob of same radius then time period remains unchanged.

2. Time period of simple pendulum is directly proportional to square root of length.





Concept Reminder

The time period of simple pendulum is independent from mass of the bob but it depends on size of bob (position of centre of mass).

Concept Reminder

When a person sitting on an oscillating swing comes in standing position then centre of mass raises upwards and length decreases, so time period decreases and frequency increases means swing oscillates faster.

When a person sitting on an oscillating swing comes in standing position then centre of mass raises upwards and length decreases, so time period decreases and frequency increases means swing oscillates faster.

3. When a hollow spherical bob of simple pendulum is completely filled with water and a small hole is made in bottom of it, then as water drain out, at first its time period increases, after that it decrease and when sphere becomes empty then finally it becomes as before (T).

4. If simple pendulum is shifted to poles, equator or hilly areas, then its time period may be different $\left(T \propto \frac{1}{\sqrt{g}}\right)$.

5. If a clock based on oscillation of simple pendulum is shifted from earth to moon then it becomes slow because its time period increases and becomes $\sqrt{6}$ times compare to earth. $\frac{g_M}{g_E} = \frac{1}{6} \Rightarrow T_M = \sqrt{6}T_E$

6. Periodic time of simple pendulum in reference system

$$T = 2\pi\sqrt{\frac{\ell}{g_{eff}}}$$

where, $g_{eff.}$ =effective gravity acceleration in reference system or total downward acceleration.

(a) If reference system is lift

(i) If velocity of lift v = constant acceleration a = 0 and $g_{eff.} = g$

$$\therefore \quad T = 2\pi\sqrt{\frac{\ell}{g}}$$

(ii) If lift is moving upwards with acceleration a

$$g_{eff.} = g + a$$

Rack your Brain

The period of oscillation of a mass M suspended from a spring of negligible mass is T. If along with it another mass M is also suspended, the period of oscillation will now be:

(1) T (2) $\frac{T}{\sqrt{2}}$

(3) 2T (4) $\sqrt{2}$ T

Concept Reminder

In case of free fall, satellite motion and at centre of earth $g_{eff} = 0$. Therefore simple pendulum will not oscillate.

$$T = 2\pi\sqrt{\frac{\ell}{g+a}} \Rightarrow \text{T decreases}$$

(iii) If lift is moving downwards with acceleration a

$g_{eff.} = g - a$

$$\therefore \quad T = 2\pi\sqrt{\frac{\ell}{g-a}} \Rightarrow \text{T increases}$$

(iv) If lift falls downwards freely

$g_{eff.} = g - g = 0 \quad \Rightarrow \quad T = \infty$

simple pendulum will not oscillate

If simple pendulum is shifted to the centre of earth, freely falling lift, in artificial satellite then it will not oscillate and its time period is infinite. ($\because g_{eff.} = 0$)

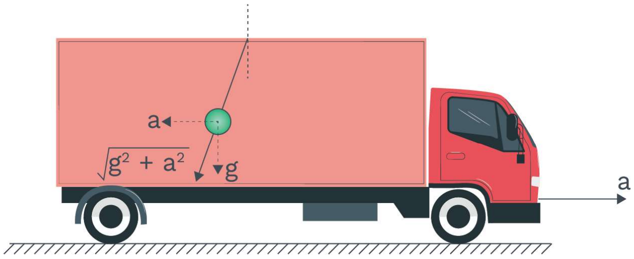
(b) A simple pendulum is mounted on a moving truck

(i) If truck is moving with constant velocity, no pseudo force acts on the pendulum and time period remains same $T = 2\pi\sqrt{\frac{\ell}{g}}$.

(ii) If truck accelerates forward with acceleration a then a pseudo force acts in opposite direction.

So effective acceleration,

$$g_{eff} = \sqrt{g^2 + a^2} \text{ and } T' = 2\pi\sqrt{\frac{\ell}{g_{eff}}}$$



Concept Reminder

A most important fact of forced periodic oscillations is that the system oscillates not with its natural frequency ω, but at the frequency ω_d of the external agency; the free oscillations die out due to damping.

Time period,

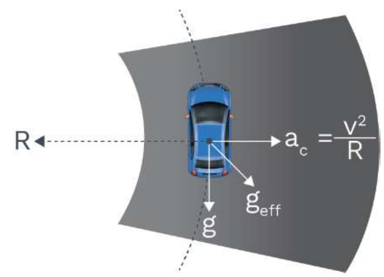
$$T' = 2\pi\sqrt{\frac{\ell}{\sqrt{g^2 + a^2}}} \Rightarrow T' \text{ decreases}$$

Ex. A simple pendulum of length 'L' and mass 'M' is suspended in a car. The car is travelling on a circular track of radius 'R' with a uniform speed 'v'. If the pendulum makes oscillation in a radial direction about its equilibrium position, then calculate its time period.

Sol. Centripetal acceleration $a_c = \frac{v^2}{R}$ and Acceleration due to gravity = g.

So, $$g_{eff} = \sqrt{g^2 + \left(\frac{v^2}{R}\right)^2}$$

$$\Rightarrow \quad \text{Time period } T = 2\pi\sqrt{\frac{L}{g_{eff}}} = 2\pi\sqrt{\frac{L}{\sqrt{g^2 + \frac{v^4}{R^2}}}}$$

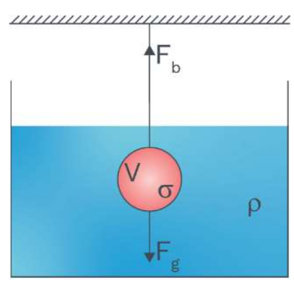


7. If a simple pendulum of density σ is made to oscillate in a liquid of density ρ then its time period will increase as compare to that of air and is given by $F_{net} = F_g - F_b$.

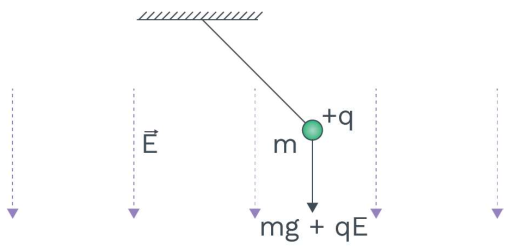
$$\frac{mg_{net}}{m} = \frac{mg}{m} - \frac{V\rho g}{m},$$

$$g_{net} = g - \frac{V\rho g}{V\sigma} = g\left(1 - \frac{\rho}{\sigma}\right)$$

$$T = 2\pi\sqrt{\frac{\ell}{\left[1 - \frac{\rho}{\sigma}\right]g}}$$

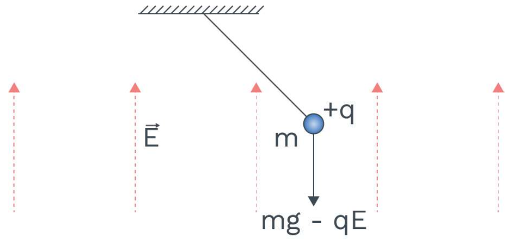


8. (a) If the bob of simple pendulum has positive charge q and pendulum is placed in uniform electric field which is in downward direction then time period decreases



$$T = 2\pi\sqrt{\frac{\ell}{g + \frac{qE}{m}}}$$

(b) If the bob of simple pendulum has positive charge q and is made is oscillate in uniform electric field acting in upward direction then time period increases



$$T = 2\pi\sqrt{\frac{\ell}{g - \frac{qE}{m}}}$$

9. $T = 2\pi\sqrt{\frac{\ell}{g}}$ is valid when length of simple pendulum (l) is negligible as compare to radius of earth (l << R) but if l is comparable to radius of earth then time period

$$T = 2\pi\sqrt{\frac{\ell}{g\left[1+\frac{\ell}{R}\right]}} = 2\pi\sqrt{\frac{1}{\left[\frac{1}{\ell}+\frac{1}{R}\right]g}}$$

The time period of oscillation of simple pendulum of infinite length

$$T = 2\pi\sqrt{\frac{R}{g}} \simeq 84.6 \text{ minute} \approx 1\frac{1}{2} \text{ hour}$$

It is maximum time period.

10. Second's pendulum:

If the time period of a simple pendulum is 2 second then it is called second's pendulum. Second's pendulum take one second to go from one extreme position to other extreme position.

For second's pendulum, time period

$$T = 2 \Rightarrow 2\pi\sqrt{\frac{\ell}{g}} = 2$$

Definitions

If the time period of a simple pendulum is 2 second then it is called second's pendulum.

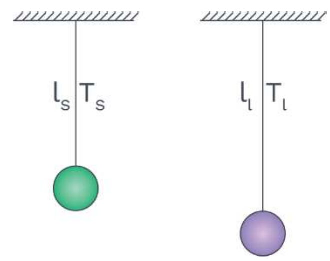
At the surface of earth $g = 9.8\ m/s^2 \approx \pi^2\ m/s^2$,
So, length of second pendulum at the surface of earth $l \approx 1$ metre.

11. When a long and short pendulum start oscillation simultaneously then both will be in same phase in minimum time when short pendulum complete one more oscillation compare to long pendulum.
After starting, if long pendulum completes N oscillation to come in same phase in minimum time then short will complete (N + 1) oscillation.

$$t = NT_{\ell} = (N+1)T_s$$

$$N\left(2\pi\sqrt{\frac{\ell_{\ell}}{g}}\right) = (N+1)\left(2\pi\sqrt{\frac{\ell_s}{g}}\right)$$

$$\boxed{N\sqrt{\ell_{\ell}} = (N+1)\sqrt{\ell_s}}$$



12. If bob of simple pendulum is suspended by a metallic wire. If 'α' is the coefficient of linear expansion and change in temperature is $d\theta$, the effective length of wire becomes $l' = l\,(l + \alpha d\theta)$

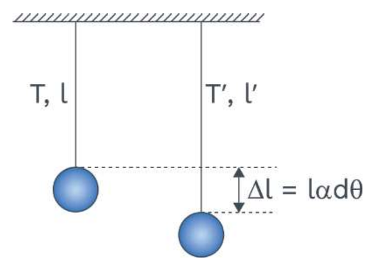
$$T' = 2\pi\sqrt{\frac{\ell'}{g}} \text{ and } T = 2\pi\sqrt{\frac{\ell}{g}}$$

Hence, $\frac{T'}{T} = \sqrt{\frac{\ell'}{\ell}} = (1+\alpha d\theta)^{1/2} = 1 + \frac{1}{2}\alpha d\theta$

$\therefore$ Percentage increase in time period

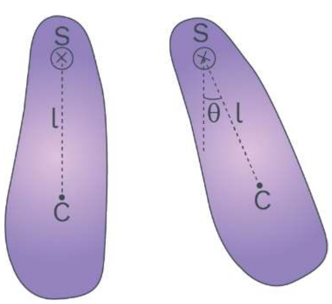
$$= \left[\frac{T' - T}{T}\right] \times 100 = \left[\frac{T'}{T} - 1\right] \times 100$$

$$= \left[1 + \frac{\alpha d\theta}{2} - 1\right] \times 100 = 50\ \alpha d\theta.$$



COMPOUND PENDULUM / PHYSICAL PENDULUM

When a rigid body is suspended from an axis and made to oscillate about that then it is called compound pendulum.



Definitions

When a rigid body is suspended from an axis and made to oscillate about that then it is called compound pendulum.

C = Position of center of mass

S = Point of suspension

l = Distance between point of suspension and center of mass (it remains constant during motion)

For small angular displacement "θ" from mean position.

The restoring torque is given by

$$\tau = mg\ell \sin\theta$$

$$\tau = -mg\ell\theta \quad \because \text{ For small } \theta, \sin\theta = \theta$$

or, $I\alpha = -mgl\theta$

where, I = Moment of inertia about point of suspension.

or $\alpha = -\dfrac{mg\ell}{I}\theta$

or $\omega^2 = \dfrac{mg\ell}{I}$

Time period,

$$T = 2\pi\sqrt{\frac{I}{mg\ell}}, \quad I = I_{CM} + m\ell^2$$

Where I_{CM} = moment of inertia relative to the axis which passes from the center of mass & parallel to the axis of oscillation.

$$T = 2\pi\sqrt{\frac{I_{CM} + m\ell^2}{mg\ell}}$$

Where $I_{CM} = mk^2$

k = gyration radius (about axis passing from centre of mass)

$$T = 2\pi\sqrt{\frac{mk^2 + m\ell^2}{mg\ell}}$$

$$T = 2\pi\sqrt{\frac{k^2 + \ell^2}{\ell g}} = 2\pi\sqrt{\frac{L_{eq}}{g}}$$

$L_{eq} = \dfrac{k^2}{\ell} + \ell$ = equivalent length of simple pendulum.

T is minimum when l = k.

$$T_{min} = 2\pi\sqrt{\frac{2k}{g}}$$

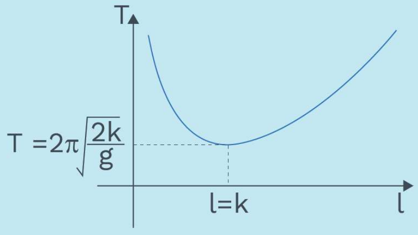
Concept Reminder

Time period of physical pendulum is

$$T = 2\pi\sqrt{\frac{I}{mg\ell}}$$

Concept Reminder

Variation of T with ℓ in case of physical pendulum is



Graph of T vs l:

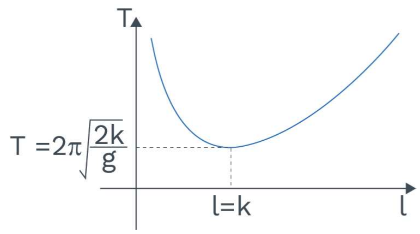
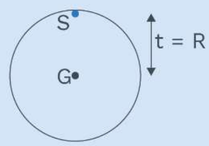
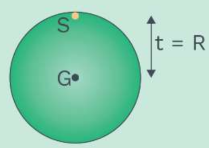
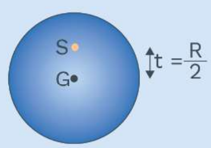
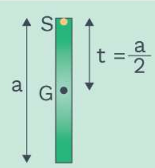
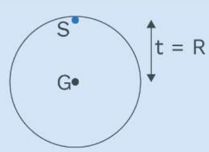
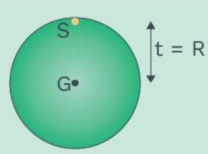


Table showing time periods for some rigid body pendulums:

S.No.	Shape	Figure showing position of axis	l	k^2	$L = \frac{k^2}{l} + l$	$T = 2\pi\sqrt{\frac{L}{g}}$	For second pendulum T = 2 s, L = 1 m
1.	Circular Ring		R	R^2	2R	$T = 2\pi\sqrt{\frac{2R}{g}}$	2R = 1m or R = 0.5m
2.	Circular Disc		R	$\frac{R^2}{2}$	$\frac{3R}{2}$	$T = 2\pi\sqrt{\frac{3R}{2g}}$	$\frac{3}{2}$ R = 1 m or R = $\frac{2}{3}$ m
3.	Circular Disc		$\frac{R}{2}$	$\frac{R^2}{2}$	$\frac{3R}{2}$	$T = 2\pi\sqrt{\frac{3R}{2g}}$	$\frac{3}{2}$ R = 1 m or R = $\frac{2}{3}$ m
4.	Light Rod		$\frac{a}{2}$	$\frac{a^2}{12}$	$\frac{2a}{3}$	$T = 2\pi\sqrt{\frac{2a}{3g}}$	$\frac{2}{3}$ a = 1 m or a = $\frac{3}{2}$ m
5.	Spherical shell		R	$\frac{2}{3}R^2$	$\frac{5}{3}R$	$T = 2\pi\sqrt{\frac{5R}{3g}}$	$\frac{5}{3}$ R = 1 m or R = $\frac{3}{5}$ m
6.	Solid Sphere		R	$\frac{2}{5}R^2$	$\frac{7}{5}R$	$T = 2\pi\sqrt{\frac{7R}{3g}}$	$\frac{7}{5}$ R = 1 m or R = $\frac{5}{7}$ m

Ex. A uniform rod of length 1.00 metre is suspended through an end and is set into oscillation with small amplitude under gravity. Find the time period of oscillation. ($g = 10\ m/s^2$)

Sol. For small amplitude the angular motion is nearly simple harmonic and the time period is given by

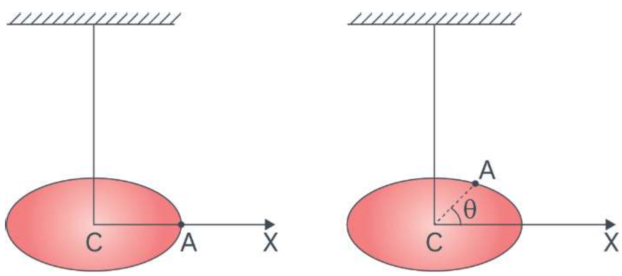
$$T = 2\pi\sqrt{\frac{I}{mg(\ell/2)}} = 2\pi\sqrt{\frac{(m\ell^2/3)}{mg(\ell/2)}}$$

$$= 2\pi\sqrt{\frac{2\ell}{3g}} = 2\pi\sqrt{\frac{2\times 1.00\,m}{3\times 10\ m/s^2}} = \frac{2\pi}{\sqrt{15}}\ s$$

TORSIONAL PENDULUM

In torsional pendulum, an extended object is suspended at the centre by a light torsion wire. A torsion wire is essentially inextensible, but is free to twist about its axis. When the lower end of the wire is rotated by a slight amount, the wire applies a restoring torque causing the body to oscillate rotationally when released.

The restoring torque produced is given by



$\tau = -C\theta$ (where C = torsional constant)

or $I\alpha = -C\theta$ (where I = moment of inertia about the vertical axis)

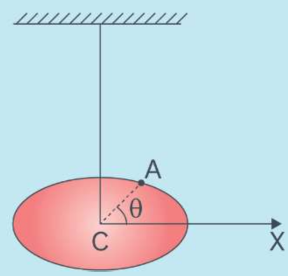
or $\alpha = -\frac{C}{I}\theta$

$\therefore$ Time period, $T = 2\pi\sqrt{\frac{I}{C}}$

Concept Reminder

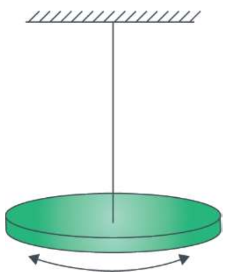
Time period of torsional pendulum is

$$T = 2\pi\sqrt{\frac{I}{C}}$$



Ex. A uniform disc of mass 200 g and radius 5.0 cm is fixed at its centre to a metal wire, the other end of which is fixed to a ceiling. The hanging disc is rotated about the wire through an angle and is released. If the disc makes torsional oscillations with time period 0.20 s, find the torsional constant of the wire.

Sol. The situation is shown in figure. The moment of inertia of the disc about the wire is



KEY POINTS

- Simple pendulum
- Physical pendulum
- Torsional pendulum

$$I = \frac{mr^2}{2} = \frac{(0.200\ \text{kg})(5.0 \times 10^{-2}\text{m})^2}{2}$$

$$= 2.5 \times 10^{-4}\ \text{kg-m}^2$$

The time period is given by

$$T = 2\pi\sqrt{\frac{I}{C}} \quad \text{or} \quad C = \frac{4\pi^2 I}{T^2}$$

$$= \frac{4\pi^2(2.5 \times 10^{-4}\ \text{kg} - \text{m}^2)}{(0.20\ \text{s})^2} = 0.25\frac{\text{kg} - \text{m}^2}{\text{s}^2}$$

Ex. The length of simple pendulum is increased by 2% then find the percentage change in time period.

Sol. $\left(\frac{\Delta T}{T} \times 100\right) = \frac{1}{2}\left(\frac{\Delta \ell}{\ell} \times 100\right)$

$$\Rightarrow \quad \left(\frac{\Delta T}{T} \times 100\right) = \frac{1}{2} \times 2\% = 1\%$$

Ex. If the length of simple pendulum is increased by 21% then find the percentage change in time period.

Sol. $l_1 = l$

$$\ell_2 = \ell + \frac{21}{100}\ell = \frac{121}{100}\ell = 1.21\ell$$

$$\frac{T_1}{T_2} = \sqrt{\frac{\ell_1}{\ell_2}} \Rightarrow \frac{T_1}{T_2} = \sqrt{\frac{\ell}{1.21\ell}} = \sqrt{\frac{100}{121}} = \frac{10}{11}$$

$$\Rightarrow \quad T_2 = \frac{11}{10} T_1$$

So, percentage change $= \frac{T_2 - T_1}{T_1} \times 100$

$$= \left(\frac{11 - 10}{11}\right) \times 100 = 10\%$$

Ex. Two pendulums of length 1 m and 1.21 m starts their motion simultaneously from same position. Then find out after how much oscillation of smaller pendulum they will be in same phase again?

Sol. After 1 oscillation they are in same phase again. So, time taken is equal

$$t = NT_\ell = (N + 1)T_s$$

$$\Rightarrow \quad N\left(2\pi\sqrt{\frac{\ell_\ell}{g}}\right) = (N + 1)\left(2\pi\sqrt{\frac{\ell_s}{g}}\right)$$

$$\Rightarrow \quad N\sqrt{\ell_\ell} = (N + 1)\sqrt{\ell_s}$$

So, $\quad N\sqrt{1.21} = (N + 1)\sqrt{1}$

$$\Rightarrow \quad \frac{11}{10}N = N + 1 \Rightarrow N = 10$$

So, number of oscillation of smaller pendulum = N + 1 = 11.

Ex. Find the length of second's pedumum-

(i) On earth (ii) On moon

Sol. (i) $T = 2\pi\sqrt{\frac{\ell}{g}}$

$$\Rightarrow \quad 2^2 = 4\pi^2\left(\frac{\ell}{g}\right)$$

$$\Rightarrow \quad l = 1 \text{ m}$$

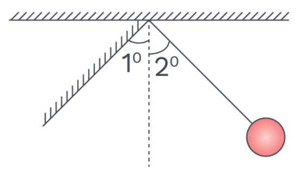
(ii) $T = 2\pi\sqrt{\frac{\ell}{g} \times 6}$

$$\Rightarrow \quad 2^2 = 4\pi^2\left(\frac{6\ell}{g}\right)$$

$$\Rightarrow \quad \ell = \frac{1}{6}\text{m}$$

Ex. A simple pendulum of length '1' metre is allowed to oscillate with amplitude 2°. It collides elastically with a wall inclined at 1° to the vertical. Then find time period of simple pendulum. ($g = \pi^2$)

Sol. $T = 2\pi\sqrt{\frac{\ell}{g}} = 2\pi\sqrt{\frac{1}{\pi^2}} = 2s$



$\because \quad \theta = \theta_0 \sin \omega t$

$\Rightarrow \quad 1^\circ = 2^\circ \sin\left(\frac{2\pi}{2} \times t\right) \Rightarrow \frac{1}{2} = \sin(\pi t)$

$\Rightarrow \quad \frac{\pi}{6} = \pi t \Rightarrow t = \frac{1}{6}$

$t_2 = 2t = 2 \times \frac{1}{6} = \frac{1}{3} \sec$

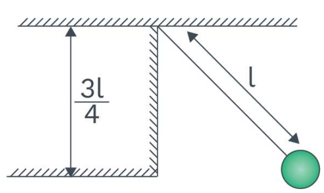
So, time period $= t_1 + t_2 = 1 + \frac{1}{3} = \frac{4}{3}$ s.

Ex. A simple pendulum is suspended in a car. It's starts moving on horizontal road according to equation $x = \frac{\sqrt{3}}{2} gt^2$. Find out time period of oscillation of pendulum.

Sol. $x = \frac{\sqrt{3}}{2} gt^2 \Rightarrow v = \sqrt{3}\, gt \Rightarrow a = \sqrt{3}\, g$

So, $T = 2\pi\sqrt{\frac{\ell}{(g^2 + a^2)^{1/2}}} = 2\pi\sqrt{\frac{\ell}{2g}}$

Ex. Find the time period for one oscillation of given simple pendulum.



Sol. $T_1 = 2\pi\sqrt{\frac{\ell}{g}} = T$ and $T_2 = 2\pi\sqrt{\frac{\ell}{4g}} = \frac{T}{2}$

So, time period is

$= \frac{T_1}{2} + \frac{T_2}{2} = \frac{T}{2} + \frac{T}{4} = \frac{3T}{4}$.

SUPERPOSITION OF SHM:

A simple harmonic motion is produced when a force (called restoring force) proportional to the displacement acts on an object. If an object is acted upon by two such forces the resultant motion of the object is a combination of two simple harmonic motions.

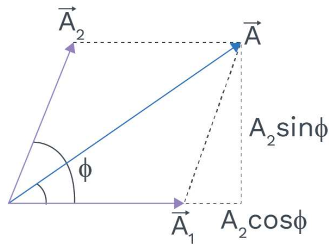
In Same direction:

(a) Having same Frequencies:

Suppose the two individual motions are represented by,

$x_1 = A_1 \sin wt$ and $x_2 = A_2 \sin (wt + \phi)$

Both the simple harmonic motions have same angular frequency ω.



$$x = x_1 + x_2 = A_1 \sin wt + A_2 \sin (wt + \phi)$$
$$= A \sin (wt + \alpha)$$

Here, $A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2 \cos\phi}$

and $\tan\alpha = \dfrac{A_2 \sin\phi}{A_1 + A_2 \cos\phi}$

Thus, we can see that this is similar to the vector addition. The same method of vector addition can be applied to the combination of more than two simple harmonic motion.

Important points to remember before solving the questions:

1. Convert all the trigonometric ratios into sine form and ensure that ωt term is with +ve sign.
2. Make the sign between two term +ve.

Concept Reminder

In our daily life we encounter phenomena which involve resonance. Your experience with swings is a good example of resonance. You might have realised that the skill in swinging to greater heights lies in the synchronisation of the rhythm of pushing against the ground with the natural frequency of the swing.

Rack your Brain

A linear harmonic oscillator of force constant 2×10^6 N/m and amplitude 0.01 m has a total mechanical energy of 160 J. Its:

(1) Maximum P.E. is 160 J
(2) Maximum P.E. is zero
(3) Maximum P.E. is 100 J
(4) Maximum P.E. is 120 J

3. A_1 is the amplitude of that S.H.M whose phase is small.

4. Then resultant x = A_{net} sin (phase of A_1 + α)

Where A_{net} is the vector sum of A_1 & A_2 with angle between them is the phase difference between two S.H.M.

Ex. $x_1 = 3 \sin \omega t$; $x_2 = 4 \cos \omega t$.

Find:

(i) Amplitude of resultant SHM.

(ii) Equation of the resultant SHM.

Sol. First right all SHM's in terms of sine functions with positive amplitude. Keep "ωt" with positive sign.

∴ $x_1 = 3 \sin \omega t$

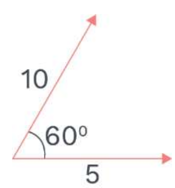
$$x_2 = 4\sin\left(\omega t + \frac{\pi}{2}\right) \qquad A = \sqrt{3^2 + 4^2 + 2 \times 3 \times 4 \cos\frac{\pi}{2}}$$

$$= \sqrt{9+16} = \sqrt{25} = 5$$

$$\tan\phi = \frac{4\sin\frac{\pi}{2}}{3 + 4\cos\frac{\pi}{2}} = \frac{4}{3} \quad (\phi = 53°)$$

Equation $x = 5\sin(\omega t + 53°)$.

Ex. $x_1 = 5 \sin (\omega t + 30°)$; $x_2 = 10 \cos (\omega t)$. Find amplitude of resultant SHM.



Sol. $x_1 = 5\sin(\omega t + 30°)$

$$x_2 = 10\sin\left(\omega t + \frac{\pi}{2}\right)$$

$$A = \sqrt{5^2 + 10^2 + 2 \times 5 \times 10 \cos 60°}$$

$$= \sqrt{25 + 100 + 50} = \sqrt{175} = 5\sqrt{7}.$$

Ex. A object is subjected to two simple harmonic motions $x_1 = A_1 \sin \omega t$ and $x_2 = A_2 \sin(\omega t + \pi/3)$.

Find:

(a) The displacement at $t = 0$
(b) The maximum speed of the object and
(c) The maximum acceleration of the object.

Sol. (a) At $t = 0$,

$x_1 = A_1 \sin \omega t = 0$

a n d

$$x_2 = A_2 \sin\left(\omega t + \frac{\pi}{3}\right) = A_2 \sin\left(\frac{\pi}{3}\right) = \frac{A_2\sqrt{3}}{2}$$

Thus, the resultant displacement at $t = 0$ is

$$x = x_1 + x_2 = A_2 \frac{\sqrt{3}}{2}$$

(b) The resultant of the two motion is a simple harmonic motion of the same angular frequency ω. The amplitude of the resultant motion is

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2 \cos\left(\frac{\pi}{3}\right)}$$

$$= \sqrt{A_1^2 + A_2^2 + A_1A_2}$$

The maximum speed is

$$u_{max} = A\omega = \omega\sqrt{A_1^2 + A_2^2 + A_1A_2}$$

(c) The maximum acceleration is

$$a_{max} = A\omega^2 = \omega^2\sqrt{A_1^2 + A_2^2 + A_1A_2}$$

(b) Having different frequencies

$x_1 = A_1 \sin \omega_1 T$

$x_2 = A_2 \sin \omega_2 t$

Then resultant displacement

$x = x_1 + x_2 = A_1 \sin \omega_1 t + A_2 \sin \omega_2 t$

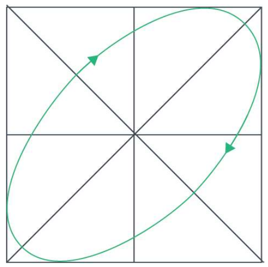
This resultant motion is not SHM.

In two perpendicular directions:

$x = A_1 \sin \omega t$ ...(i)

$y = A_2 \sin (\omega t + \phi)$ ...(ii)

The amplitudes A_1 & A_2 may be different and Phase difference ϕ & ω is same.
Thus, equation of the path may be obtained by eliminating t from (i) and (ii)



$$\sin\omega t = \frac{x}{A_1} \quad \text{...(iii)}$$

$$\cos\omega t = \sqrt{1-\frac{x^2}{A_1^2}} \quad \text{...(iv)}$$

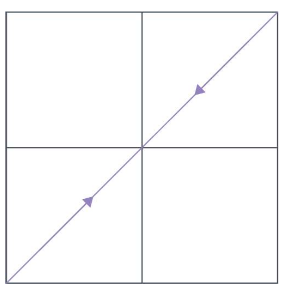
On rearranging we get

$$\frac{x^2}{A_1^2}+\frac{y^2}{A_2^2}-\frac{2xy\cos\phi}{A_1A_2} = \sin^2\phi \quad \text{...(v)}$$

(General equation of ellipse)

Special case:

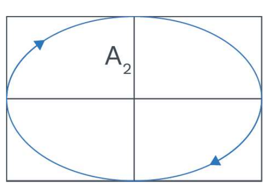
(1) If $\phi = 0°$



$$\therefore \quad \frac{x^2}{A_1^2}+\frac{y^2}{A_2^2}-\frac{2xy}{A_1A_2} = 0$$

$$\therefore \quad y = \frac{A_2}{A_1}\cdot x \quad \text{(equation of straight line)}$$

(2) If $\phi = 90°$



Rack your Brain

A wave has S.H.M. whose period is 4 s while another wave which also possess S.H.M. has its period 3 s. If both are combined, then the resultant wave will have the period equal to:

(1) 4 s (2) 5 s

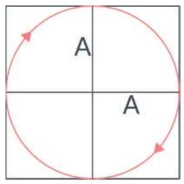
(3) 12 s (4) 3 s

Concept Reminder

Any mechanical structure, like a building, a bridge, or an aircraft may have several possible natural frequencies. An external periodic force or disturbance will set the system in forced oscillation. If, accidentally, the forced frequency ω_d happens to be close to one of the natural frequencies of the system, the amplitude of oscillation will shoot up(resonance), resulting in possible damage.

$$\Rightarrow \quad \frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} = 1 \quad \text{(equation of ellipse)}$$

(3) If $\phi = 90°$ & $A_1 = A_2 = A$



then,

$x^2 + y^2 = A^2$ (equation of circle)

TYPES OF OSCILLATIONS

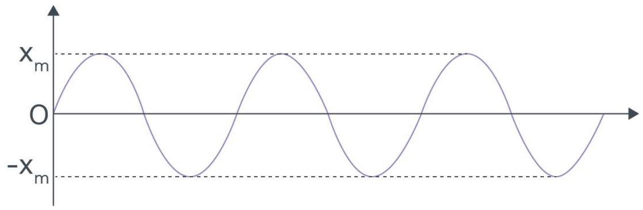
Free, Damped, Forced oscillations and Resonance:

(a) Free oscillation

(i) The oscillations of a particle with fundamental frequency under the influence of restoring force are defined as free oscillations.

(ii) The amplitude, frequency and energy of oscillations remain constant.

(iii) The oscillator which keeps on oscillating with constant amplitude for infinite time is known as free oscillator.



Definitions

The oscillations of a particle with fundamental frequency under the influence of restoring force are defined as free oscillations.

(b) Damped oscillations:

- The amplitude decreases continuously with the passing of time are called damped oscillations.
- In many real systems, non-conservative forces such as friction retard the motion. Therefore the mechanical energy of the system diminishes in time, and the motion is said to be damped.

Definitions

The oscillations in which the amplitude decreases gradually with the passage of time are called damped oscillations.

The lost mechanical energy is transformed into internal energy of the object & the retarding medium.

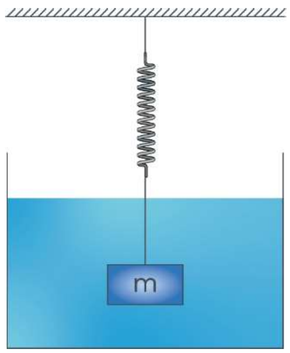
- The retarting force can be expressed as $\vec{F} = -b\vec{v}$ (where b is a constant called the damping coefficient) and restoring force on the system is −kx, we can write Newton's second law as

$$\Sigma f_x = -kx - bv = ma$$

$$-kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

This is the differential equation of damped oscillation, solution of this equation is given by

$$x = Ae^{(-b/2m)t} \cos(\omega' t + \phi)$$



Where angular frequency of oscillation is

$$\omega' = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} = \sqrt{\omega^2 - \left(\frac{b}{2m}\right)^2}$$

Where $\omega = \sqrt{\frac{k}{m}}$ represents the angular frequency in the absence of retarting force (the undamped oscillator) & is called natural frequency.

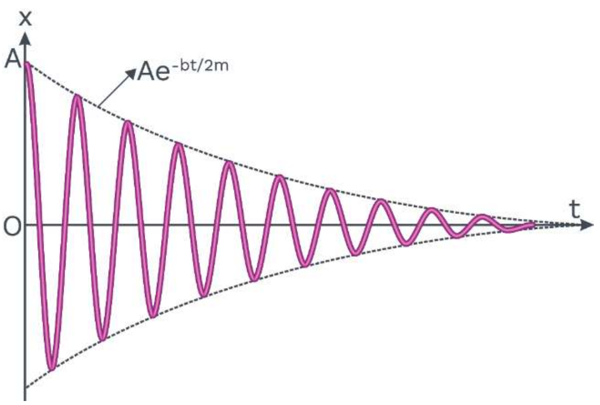
(c) Effect of damped oscillation:

- When the retarding force is small, the oscillatory character of the motion is preserved but amplitude decreases in time, and it decays exponentially with time.

Concept Reminder

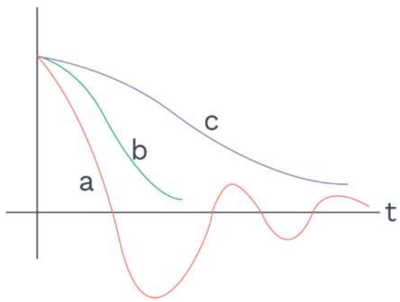
Angular frequency of damped oscillation is

$$\omega' = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} = \sqrt{\omega^2 - \left(\frac{b}{2m}\right)^2}$$



(a) When the magnitude of retarding force is small such that $b/2m < \omega$, the system is said to be underdamped.

(b) When b reaches a critical value bc such that $\frac{b_c}{2m} = \omega$, then system does not oscillate & is said to be critically damped.



(c) When retarding force is large as compared to restoring force,

i.e., $\frac{b}{2m} > \omega$ then system is overdamped.

- Mechanical energy of undamped oscillator is $\frac{1}{2}kA^2$. For a damped oscillator amplitude is not constant but depend on time, so total energy is

$$E(t) = \frac{1}{2}k(Ae^{-bt/2m})^2 = \frac{1}{2}kA^2e^{-bt/m}$$

Rack your Brain

The damping force on an oscillator is directly proportional to the velocity. The units of the constant of proportionality are:

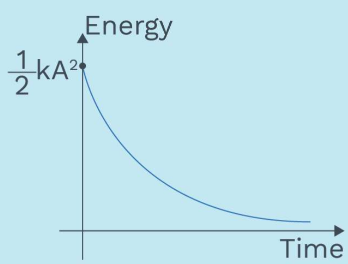
(1) kg ms^{-1} (2) kg ms^{-2}

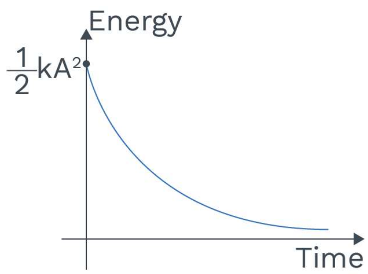
(3) kg s^{-1} (4) kg s

Concept Reminder

For a damped oscillator amplitude is not constant but depend on time, so total energy is

$$E(t) = \frac{1}{2}k(Ae^{-bt/2m})^2 = \frac{1}{2}kA^2e^{-bt/m}$$





(d) Forced oscillations:

- All free oscillations finally die out because of ever present damping force, However, an external agency can maintain these oscillations. These are called forced or driven oscillations.
- Under forced periodic oscillation, system does not oscillate with its natural frequency (ω) but with driven frequency (ω_d).
- Suppose an external force 'F(t)' of amplitude F_0 that varies periodically with time is applied to a damped oscillator.

Such a force is

$$F(t) = F_0 \cos \omega_d t$$

The equation of particle under combined force is

$$ma(t) = -kx - bv + F_0 \cos \omega_d t$$

$$\frac{md^2x}{dt^2} + \frac{bdx}{dt} + kx = F_0 \cos \omega_d t$$

After solving,

$$x = A'\cos(\omega_d t + \phi)$$

Where,

$$A' = \frac{F_0}{m\sqrt{(\omega_d^2 - \omega^2)^2 + \left(\frac{b\omega_d}{m}\right)^2}}$$

$$\tan\phi = \frac{-v_0}{\omega_d x_0}$$

(ω is natural frequency, $\omega = \sqrt{\frac{k}{m}}$)

Definitions

All free oscillations eventually die out because of ever present damping force, However, an external agency can maintain these oscillations. These are called forced or driven oscillations.

Where mass of the particle is m and v_0 and x_0 are the velocity and the displacement of the particle at time t = 0, at the moment when we applied the periodic force.

- The amplitude of driven oscillator decreases due to damping forces but on account of the energy gained from external source (driver) it remains constant.
- The amplitude of forced vibration is calculated by the difference between the frequency of applied force & the natural frequency. If difference between frequency is small then amplitude will be large.

Resonance:

- For small damping, the amplitude is large when the frequency of the driving force is near the natural frequency of oscillation ($\omega_d \approx \omega$).

$\because \quad \omega_d = \omega$

$\therefore \quad \boxed{A = \dfrac{F_0}{b\omega_d}}$ (amplitude of oscillation at resonance)

The increase in amplitude near the natural frequency is called resonance, & the natural frequency ω is also called the resonance frequency of the system.

- In the state of resonance condition, there occurs maximum energy transfer from the driver to the driven.

(a) Small Damping and Driving Frequency far from Natural frequency:-

In this case, $\omega_d b$ will be much smaller than $m(\omega^2 - \omega_d^2)$ and we can neglect that term.

Then equation of amplitude reduces to

$$A' = \frac{F_0}{m(\omega_d^2 - \omega^2)} \qquad \text{...(i)}$$

Concept Reminder

Any mechanical structure, like a building, a bridge, or an aircraft may have several possible natural frequencies. An external periodic force or disturbance will set the system in forced oscillation. If, accidentally, the forced frequency ω_d happens to be close to one of the natural frequencies of the system, the amplitude of oscillation will shoot up(resonance), resulting in possible damage.

Concept Reminder

Condition of resonance is

$\omega_d = \omega$

where,

ω = natural frequency

ω_d = driven frequency

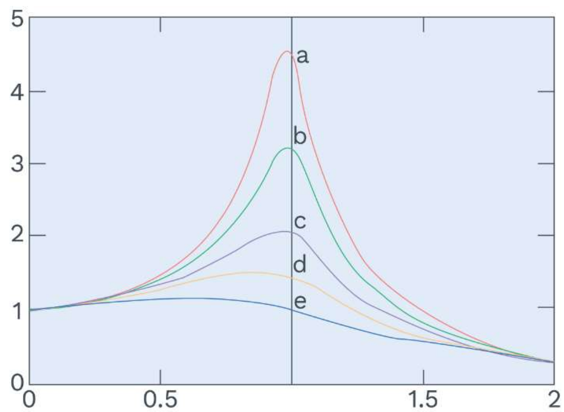


Figure shows the dependence of the displacement amplitude of an oscillator on the angular frequency of the driving force for different amounts of damping present in the system. It may be noted that in all the cases the amplitude is greatest when $\frac{\omega}{\omega_d} = 1$. The curves show that figure show that smaller the damping, the taller and narrower is the resonance peak.

If we changing the driving frequency, the amplitude tends to infinity when it same the natural frequency. But this is an ideal case of zero damping, a case which never arises in a actual system as the damping is never perfectly zero. We must have experienced in a swing that when the timing of your push exactly matches with the time period of the swing, our swing gets the maximum amplitude. This amplitude is more, but not infinity, because there is always some damping in your swing. This will become clear in the figure (b).

(b) Driving Frequency Close to Natural Frequency:-

If ω_d is very close (nearest) to ω, $m(\omega^2 - \omega_d^2)$ would be much less than $\omega_d b$, for any reasonable value of b, then equation of amplitude reduces to

$$A = \frac{F_0}{\omega_d b} \qquad \text{...(ii)}$$

Concept Reminder

The mechanical energy in a real oscillating system decreases during oscillations because external forces, such as drag, inhibit the oscillations and transfer mechanical energy to thermal energy. The real oscillator and its motion are then said to be damped.

Rack your Brain

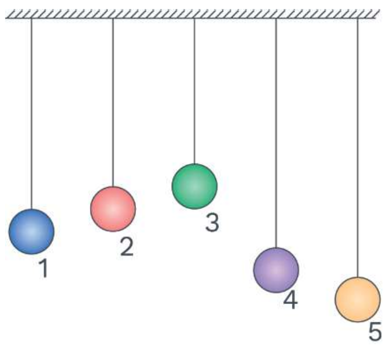
The equations of two S.H.M.'s is given as $x = a\cos(\omega t + \delta)$ and $y = a\cos(\omega t + \alpha)$, where $\delta = \alpha + \frac{\pi}{2}$, the resultant of the two S.H.M.'s represents:

(1) A hyperbola
(2) A circle
(3) An ellipse
(4) None of these

This makes it clear that maximum possible amplitude for a given driving frequency is governed by the driving frequency and the damping, and is never infinity. The phenomenon of rise in amplitude when the driving force is close to the natural frequency of the oscillator is known resonance condition.

In our daily life we encounter event which involve resonance. Your experience with swings is a better example of resonance. We might have realised that the skill in swinging to greater heights lies in the synchronisation of the rhythm of pushing against the ground with the natural frequency of the swing.

To illustrate this point further, Assume a set of 5 simple pendulums of assorted lengths suspended from a common rope as shown in given. The pendulums '1' and '4' have the same lengths and the others have different lengths. Now let us set pendulum '1' into motion. The energy from this simple pendulum gets transferred to other simple pendulums through the connecting rope and they start oscillating. The driving force is support through the connecting rope (string). The frequency of this force is the frequency with which simple pendulum '1' oscillates. If we observe the response of simple pendulums 2, 3 and 5, they first start oscillating with their natural frequencies of oscillations and different amplitudes, but this motion is gradually damped and not sustained. Their frequencies of oscillation gradually changes and ultimately they oscillate with the frequency of simple pendulum 1, i.e. the frequency of the driving force but with different amplitudes. They oscillate with small (less) amplitudes. The response of pendulum '4' is in contrast to this set of pendulums. It oscillates with the equal frequency as that of pendulum '1' and its amplitude gradually picks up and becomes very large value. A resonance-like response is seen in figure. This happens because in this the resonance condition is satisfied, i.e. the natural frequency of the system coexist with that of driving force.



We have so far considered oscillating systems which have just one natural frequency. Generally, a system may have many natural frequencies. We will see examples of such systems (air columns, vibrating strings etc.) in the next chapter. Any mechanical structure, an aircraft, like a building or a bridge may have several possible natural frequencies. An external periodic force or disturbance will set the system in forced oscillation. If, accidentally, the forced frequency ω_d happens to be close to one of the natural frequencies of the system, the amplitude of oscillation will shoot up resulting in possible damages. This is why soldiers go out of step while cross a bridge. For the same condition, an earthquake will not cause uniform damage to all building in an affected area, even if they are built with the same strength and materials. The natural frequencies depends on a building own height, and other size parameters, and the nature of building materials. The one with its natural frequency near to the frequency of seismic wave in likely to be damaged more.

Important Points

- When a tunning fork is struck against a rubber pad, the prongs begin to execute free vibration.
- When the stem of vibrating tunning fork is pressed against the top of tabla, then the tabla will suffer forced vibration.
- Soldiers are request to break step while crossing a bridge, If the soldiers march in step, there is a possibility that the frequency of the foot steps may match the natural frequency of the bridge. Due to resonance, the bridge may start oscillating violently, thereby damaging itself.
- The oscillations of simple pendulum in air are damped oscillations.

Ex. In damped oscillations, the amplitude after 50 oscillations is 0.8 a_0, where a_0 is the initial amplitude, then determine amplitude after 150 oscillations.

Sol. The amplitude, a, at time t is given by

$$a = a_0 \exp(-\alpha t)$$

$$a_{50} = a_0 \exp(-\alpha \times 50\,T) = 0.80\,a_0$$

Where T is the period of oscillation

$$a_{150} = a_0 \exp(-\alpha \times 150\,T) = a_0 (0.8)^3 = 0.512\,a_0 .$$

Ex. A body of mass 600 gm is attached to a spring of spring constant k = 100 N/m and it is performing damped oscillations. If damping constant is 0.2 and driving force is F = F_0 cos(ωt), where F_0 = 20 N. Find the amplitude of oscillation at resonance.

Sol. For damped oscillations,

$$A = \frac{F_0 / m}{\sqrt{(\omega^2 - \omega_0^2)^2 + \left(\frac{b\omega}{m}\right)^2}}$$

At resonance,

$$\omega = \omega_0 \text{ so } A = \frac{F_0}{b\omega}$$

Here, $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{100}{0.6}} = 13 \text{ rad / s}$

So, $A = \frac{20}{(0.2)13} = 7.7 \text{ m}$

EXAMPLES

Q1 **The equation of motion of a particle which started at t = 0 is given by x = 5 sin (20t + π/3) where x is in centimetre and t in second. When does the particle.**

(a) First come to rest?

(b) First have zero acceleration?

(c) First have maximum speed?

Sol: $x = 5\sin\left(20t + \frac{\pi}{3}\right)$

(a) When particle comes to rest its velocity = 0

$$v = 100\cos\left(20t + \frac{\pi}{3}\right)$$

$$0 = 100\cos\left(20t + \frac{\pi}{3}\right)$$

$$\Rightarrow \quad 20t + \frac{\pi}{3} = \frac{\pi}{2} \quad \Rightarrow \quad t = \frac{\pi}{120}$$

(b) $a = -2000\sin\left(20t + \frac{\pi}{3}\right)$

$$0 = 2000\sin\left(20t + \frac{\pi}{3}\right)$$

$$\Rightarrow \quad 20t + \frac{\pi}{3} = \pi$$

(c) $t = \frac{\pi}{30}$

$$v = 100\cos\left(20t + \frac{\pi}{3}\right)$$

For, v_{max} $\quad 20t + \frac{\pi}{3} = \pi$

$$\Rightarrow \quad t = \frac{\pi}{3}$$

$$20t + \frac{\pi}{3} = \pi$$

$$\Rightarrow \quad t = \frac{\pi}{3}.$$

Q2 **At an instant a particle in S.H.M. located at distance 2 cm from mean position, have magnitudes of velocity and acceleration 1 m/s and 10 m/s² respectively. Calculate the amplitude and the time period of the motion.**

Sol: $x = 2$ cm, $v = 100$ cm/s, $a = 1000$ m/s²

$a = -\omega^2 x$

$|a| = \omega^2 |x|$

$1000 = \omega^2 2$

$\omega^2 = 500$

$\omega = 10\sqrt{5}$

$v^2 = \omega^2 (A^2 - x^2)$

$100^2 = 500 (A^2 - 4)$

$A^2 - 4 = 20$

$A^2 = 24$

$A = \sqrt{24}$ cm

$$T = \frac{2\pi}{\omega} = \frac{\pi}{5\sqrt{5}} = 0.285\,.$$

Q3 **A particle performing SHM with amplitude 10cm. Calculate distance from mean position the kinetic energy of the particle is thrice of its potential energy?**

Sol: According to question

$$\frac{1}{2}mv^2 = 3 \times \frac{1}{2}m\omega^2 x^2$$

$$\Rightarrow \frac{1}{2}m\omega^2\left(A^2 - x^2\right) = \frac{3}{2}m\omega^2 x^2$$

$$\Rightarrow A^2 = 4x^2 \Rightarrow x = \pm\frac{A}{2} = \pm\frac{10\text{ cm}}{2} = \pm 5\text{ cm.}$$

Q4 **A vertical spring-mass system with lower end of spring is fixed, made to undergo small oscillations. If the spring is stretched by '25 cm', energy stored in the spring is 5J. Find the mass of the block if it makes '5' oscillations each second.**

Sol: $\frac{1}{2}Kx^2 = 5$

$\frac{1}{2}K \times \left(\frac{25}{100}\right)^2 = 5$

K = 160

$\omega = 10\pi = \sqrt{\frac{K}{m}}$

$m = \frac{K}{100\pi^2}$

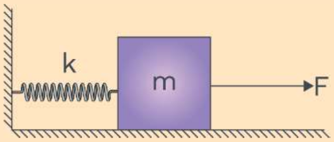
$m = \frac{160}{100\pi^2} = \frac{16}{10\pi^2} = 0.16 \text{ kg.}$

Q5 **A spring mass system is shown in diagram. Spring is initially unstretched. A man starts pulling the body with constant force F. Find**

(a) The time period of motion and the amplitude of the body.

(b) The K.E. of the body at mean position.

(c) The energy stored in the spring when the body passes through the mean position.

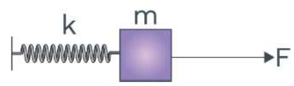


Sol: (a) Initial position of body is an extreme position.

At equilibrium, F = KA $\Rightarrow$ A = F/K

Time period

$T = 2\pi\sqrt{\frac{m}{k}}$.



(b) Kinetic energy at mean position

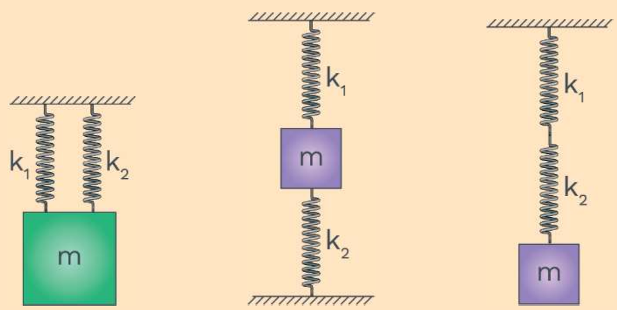
$$K.E. = \frac{1}{2} K A^2$$

$$= \frac{1}{2} K \left(\frac{F}{K}\right)^2 = \frac{F^2}{2K}.$$

(c) Total energy of the body

$$\frac{1}{2} K A^2 = \frac{1}{2} K \left(\frac{F}{K}\right)^2 = \frac{F^2}{2K}.$$

Q6 **Three spring mass systems are shown in diagram. Assuming gravity free space, find out the time period of oscillations in each case. What should be the solution if space is not gravity free?**



Sol: (a) When spring is stretched by x then restoring force.

$F = K_1 x + K_2 x$

$F = K_{eq} x$

$K_{eq} x = K_1 x + K_2 x$

$K_{eq} = K_1 + K_2$

$$T = 2\pi \sqrt{\frac{m}{K_{eq}}} = 2\pi \sqrt{\frac{m}{K_1 + K_2}}$$

(b) When bock is displaced by x from mean position then restoring force.

$F = K_1 x + K_2 x$

$K_{eq} x = K_1 x + K_2 x$

$K_{eq} = K_1 + K_2$

$$T = 2\pi\sqrt{\frac{m}{K_{eq}}} = 2\pi\sqrt{\frac{m}{K_1 + K_2}}$$

(c) When block is displaced by x and extension in upper spring is x_1, extension in lower spring is x_2

then, $F = K_1 x_1 \Rightarrow x_1 = \frac{F}{K_1}$

$$F = K_2 x_2 \Rightarrow x_2 = \frac{F}{K_2}$$

$$F = K_{eq} x \Rightarrow x = \frac{F}{K_{eq}}$$

$$x = x_1 + x_2 \Rightarrow \frac{F}{K_{eq}} = \frac{F}{K_1} + \frac{F}{K_2}$$

$$\Rightarrow K_{eq} = \frac{K_1K_2}{K_1 + K_2}$$

$$T = 2\pi\sqrt{\frac{m}{K_{eq}}} = 2\pi\sqrt{\frac{m\left(K_1 + K_2\right)}{K_1K_2}}.$$

When space is not gravity free then answers do not change as time period of spring mass system is independent of gravity.

Q7 **A block of mass 1 kg hanging from a vertical spring executes SHM of amplitude 0.2 m and time period π/10 sec. Find the maximum force exerted by the spring on the block.**

Sol: We have

$$T = 2\pi\sqrt{\frac{m}{k}}$$

So, $k = \frac{4\pi^2 m}{T^2}$

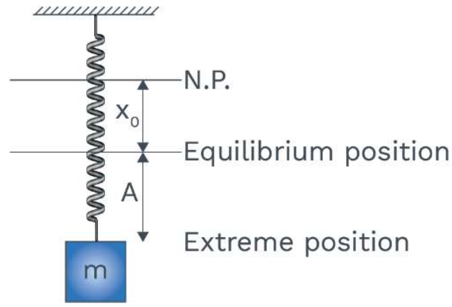
$x_0 = \frac{mg}{k}$ at equilibrium

Maximum force applied by spring,

$$= k(x_0 + A) = k\left(\frac{mg}{k} + A\right) = mg + kA$$

$$= 1 \times 10 + \frac{4\pi^2 m}{T^2} \times A$$

$$= 10 + \frac{4 \times 10 \times 1}{(\pi / 10)^2} \times 0.2 = 90 \text{ N.}$$



Q8 **A pendulum is suspended in a lift and its period of oscillation is T_0 when the lift is stationary.**

(i) What will be the period 'T' of oscillation of pendulum, if the lift begins to accelerate downwards with an acceleration equal to $\frac{3g}{4}$?

(ii) What must be the acceleration of the lift for the period of oscillation of the pendulum to be $\frac{T_0}{2}$?

Sol: (i) $T_0 = 2\pi\sqrt{\frac{\ell}{g}} \Rightarrow T = 2\pi\sqrt{\frac{\ell}{g_{eff}}}$

$$g_{eff} = g - \frac{3g}{4} = \frac{g}{4}$$

So, time period

$$T = 2\pi\sqrt{\frac{\ell}{\frac{g}{4}}} \Rightarrow T = 2T_0$$

(ii) $\frac{T_0}{2} = 2\pi\sqrt{\frac{\ell}{g_{eff}}}$

$$\Rightarrow \frac{2\pi}{2}\sqrt{\frac{\ell}{g}} = 2\pi\sqrt{\frac{\ell}{g_{eff}}}$$

$$g_{eff} = 4g \Rightarrow a = 3g \text{ upwards.}$$

Q9 **Compound pendulums are made of :**

(a) A rod of length 'l' suspended through a point located at distance l/4 from centre of rod.

(b) A ring of mass 'm' and radius 'r' suspended through a point on its periphery.

(c) A uniform square plate of edge a suspended through a corner of plate.

(d) A uniform disc of mass 'm' and radius 'r' suspended through a point r/2 away from the centre.

Find the time period in each case.

Sol: (a) Time period of compound pendulum is

$$T = 2\pi\sqrt{\frac{I_{cm} + md^2}{mg\ell}}$$

For rod of length ℓ,

$$T = 2\pi\sqrt{\frac{\frac{m\ell^2}{12} + m\left(\frac{\ell}{4}\right)^2}{mg\ell/4}} = 2\pi\sqrt{\frac{\frac{m\ell^2(4+3)}{48}}{mg\ell/4}}$$

$$T = 2\pi\sqrt{\frac{7\ell}{12g}}.$$

(b) $$T = 2\pi\sqrt{\frac{mr^2 + mr^2}{mgr}}$$

$$T = 2\pi\sqrt{\frac{2r}{g}}.$$

(c) $$T = 2\pi\sqrt{\frac{\frac{ma^2}{6} + m\left(\frac{a}{\sqrt{2}}\right)^2}{mg\frac{a}{\sqrt{2}}}}$$

$$T = 2\pi\sqrt{\frac{\sqrt{8}\,a}{3g}}.$$

(d) $$T = 2\pi\sqrt{\frac{\frac{mr^2}{2} + m\left(\frac{r}{2}\right)^2}{mg\frac{r}{2}}}$$

$$T = 2\pi\sqrt{\frac{3r}{2g}}.$$

Q10 **A particle is subjected to two SHM's simultaneously**

$X_1 = a_1 \sin\omega t$ and $X_2 = a_2 \sin(\omega t + \phi)$

Where a_1 = 3.0 cm, a_2 = 4.0 cm.

Find Resultant amplitude if the phase difference ϕ has values:

(a) 0° **(b) 60°** **(c) 90°**

Sol: $A_r = A_1^2 + A_2^2 + 2A_1A_2\cos\theta$

(a) $A_r = \sqrt{3^2 + 4^2 + 2\times 3\times 4\times \cos 0}$ = 7 cm

(b) $A_r = \sqrt{3^2 + 4^2 + 2\times 3\times 4\times \cos 60} = \sqrt{37}$ cm = 6.1 cm

(c) $A_r = \sqrt{3^2 + 4^2 + 2\times 3\times 4\times \cos 90}$ = 5 cm.

Q11 **A body of mass 2 kg suspended through a vertical spring executes simple harmonic motion of period 4s. If the oscillations are stopped and the body hangs in equilibrium, find the potential energy stored in the spring.**

Sol: $K = m\omega^2 \Rightarrow K = m\left(\frac{2\pi}{T}\right)^2$

$K = \frac{4\pi^2 m}{T^2}$

mg = Kx equilibrium

$x = \frac{mg}{K} \Rightarrow U = \frac{1}{2}Kx^2$

$= \frac{1}{2}K\frac{m^2g^2}{K^2} = \frac{m^2g^2T^2}{2\times 4\pi^2 m} = \frac{2^2\times 10^2\times 4^2}{2\times 4\times 10\times 2}$ = 40 J.

Q12 **An object of mass 0.2 kg executes simple harmonic oscillations along the x–axis with a frequency of (25 /π) Hz. At the position x = 0.04, the object has kinetic energy of 0.5 J and potential energy 0.4 J. Find the amplitude of oscillations**

Sol: Total energy, $E = \frac{1}{2} m\omega^2 A^2$

$\Rightarrow \; 0.5 + 0.4 = \frac{1}{2} \times 0.2 \left(2\pi \times \frac{25}{\pi}\right)^2 A^2$

$\Rightarrow \; 9 = (50)^2 A^2 \quad \Rightarrow \quad A = 0.06$ m.

Q13 **A particle is executing SHM. Find the positions of the particle where its speed is 8 cm/s, If maximum magnitudes of its velocity and acceleration are 10 cm/s and 50 cm/s² respectively.**

Sol: $|v| = |A|\,\omega \quad \Rightarrow \quad 10 = |A|\,\omega$

$|a| = |A|\,\omega^2 \quad \Rightarrow \quad 50 = |A|\,\omega^2$

$\Rightarrow \; \omega = 5 \quad \Rightarrow \quad |A| = 2$

$v^2 = \omega^2 (A^2 - x^2)$

$8^2 = 5^2 (2^2 - x^2)$

$4 - x^2 = \frac{64}{25} \quad \Rightarrow \quad x^2 = \frac{36}{25} \quad \Rightarrow \quad x = \pm\frac{6}{5}$ cm.

Q14 **A object executes simple harmonic motion with an amplitude of '10 cm' and time period '6 s'. At t = 0 it is at position x = 5 cm from mean position and going towards positive x-direction. Write the equation for the displacement x at time t. Find the magnitude of the acceleration of the object at t = 4 s.**

Sol: A = 10 cm, T = 6 sec

So, $x = A\sin\left(\frac{2\pi}{T}t + \phi\right)$

$= (10 \text{ cm}) \sin\left(\frac{2\pi}{6}t + \phi\right)$

$= 10 \text{ cm} \sin\left(\frac{\pi}{3}t + \phi\right)$

at t = 0, x = 5 cm.

$\Rightarrow \; 5 \text{ cm} = 10 \text{ cm} \sin(0 + \phi)$

$\Rightarrow \; \frac{1}{2} = \sin(\phi); \; \phi = \frac{\pi}{6}$

So, x = (10 cm) $\sin\left(\frac{\pi}{3}t+\frac{\pi}{6}\right)$

Acceleration, a = $-\omega^2$ (10 cm) $\sin\left(\frac{\pi}{3}t+\frac{\pi}{6}\right) = -\left(\frac{\pi}{3}\right)\times(10\text{ cm})\sin\left(\frac{\pi}{3}\times 4+\frac{\pi}{6}\right)$

$$= \frac{\pi^2\times 10}{9} \simeq 11\text{ cm/s}^2.$$

Q15 **A simple harmonic motion has an amplitude A and time period T. Find the time required by it to travel directly from**

(A) x = 0 to x = A/2 **(B)** $x=-\frac{A}{\sqrt{2}}$ **to** $x=\frac{A}{\sqrt{2}}$

Sol: (A) $x = A\sin\left(\frac{2\pi}{T}t+\phi\right)$

$0 = A\sin\left(\frac{2\pi}{T}0+\phi\right)$

$\Rightarrow \frac{2\pi}{T}0+\phi=0 \quad \Rightarrow \quad \phi=0$

$x = A\sin\left(\frac{2\pi t}{T}\right) \quad \Rightarrow \quad \frac{2\pi t}{T}=\frac{\pi}{6}$

$t=\frac{T}{12}$

(B) $x = A\sin\left[\frac{2\pi t}{T}+\phi\right]$

$\frac{-A}{\sqrt{2}} = A\sin\left[\frac{2\pi 0}{T}+\phi\right]$

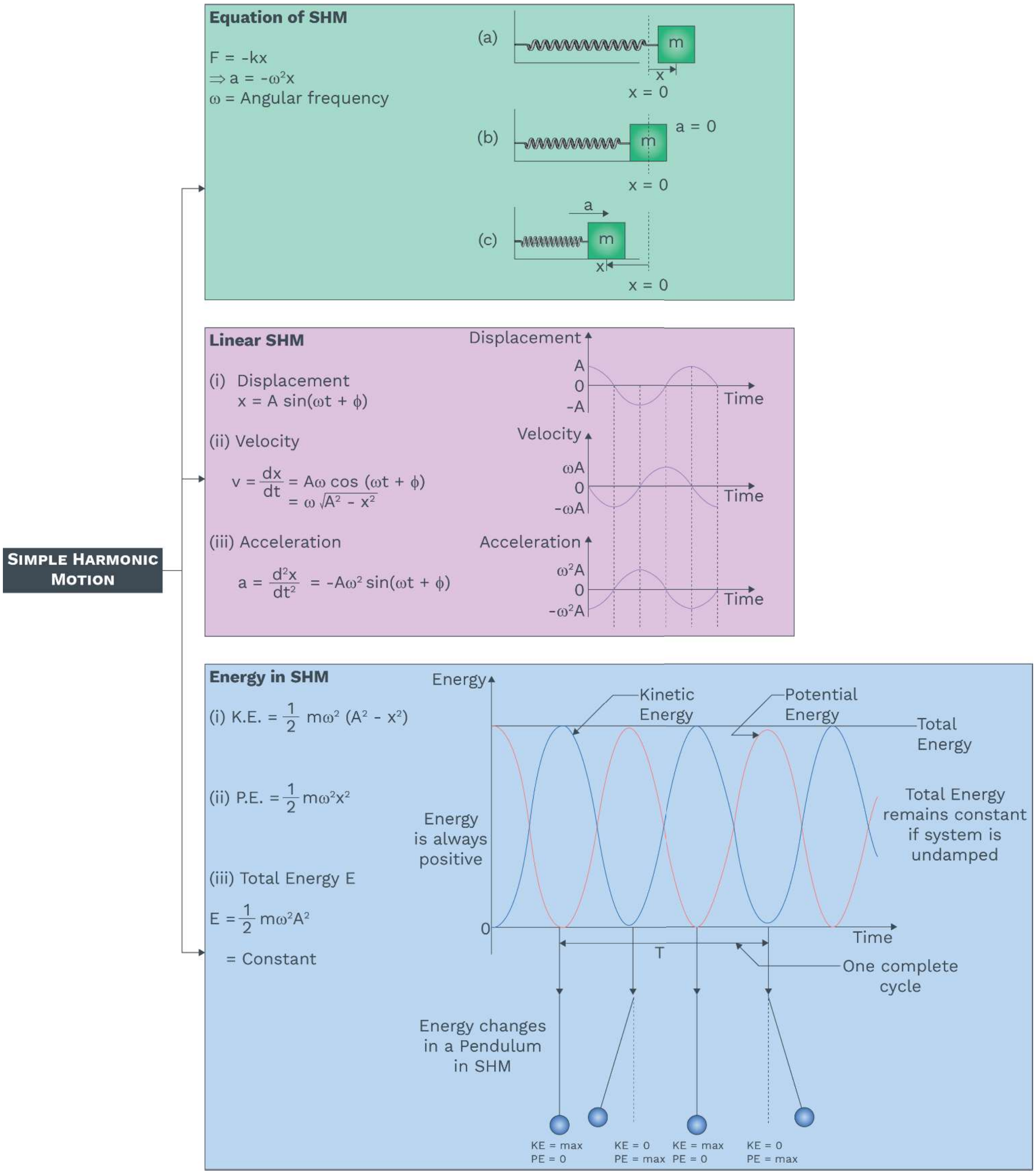
$\phi=\frac{-\pi}{4}$

$x = A\sin\left[\frac{2\pi t}{T}-\frac{\pi}{4}\right]$

$\frac{A}{\sqrt{2}} = A\sin\left[\frac{2\pi t}{T}-\frac{\pi}{4}\right]$

$\frac{2\pi t}{T}-\frac{\pi}{4}=\frac{\pi}{4} \quad \Rightarrow \quad t=\frac{T}{4}.$

Mind Map



Simple Harmonic Motion
Equation of SHM
F = -kx
⇒ a = -ω²x
ω = Angular frequency
(a)
m
x
x = 0
(b)
m
a = 0
x = 0
(c)
a
m
x
x = 0
Linear SHM
(i) Displacement
x = A sin(ωt + ϕ)
(ii) Velocity
v = dx/dt = Aω cos (ωt + ϕ)
= ω √(A² - x²)
(iii) Acceleration
a = d²x/dt² = -Aω² sin(ωt + ϕ)
Displacement
A
0
-A
Time
Velocity
ωA
0
-ωA
Time
Acceleration
ω²A
0
-ω²A
Time
Energy in SHM
(i) K.E. = 1/2 mω² (A² - x²)
(ii) P.E. = 1/2 mω²x²
(iii) Total Energy E
E = 1/2 mω²A²
= Constant
Energy
Kinetic Energy
Potential Energy
Total Energy
Energy is always positive
Total Energy remains constant if system is undamped
0
Time
T
One complete cycle
Energy changes in a Pendulum in SHM
KE = max
PE = 0
KE = 0
PE = max
KE = max
PE = 0
KE = 0
PE = max