Rotational Motion

Rotational Motion

RIGID BODY:-

Rigid body is classified as a system of particles in which distance between each pair of particles remains constant (with respect to time). Remember, inflexible body is a mathematical concept and any system which fulfills the above condition is said to be rigid as long as it fulfills it.

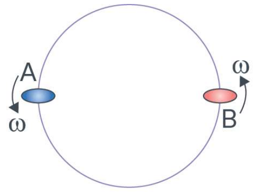
Definitions

Rigid body is defined as a system of particles in which distance between each pair of particles remains constant (with respect to time).

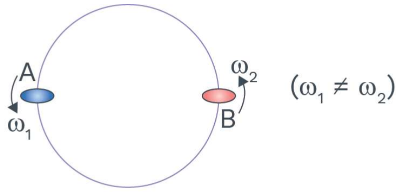
System behaves as a rigid body

System behaves as a non-rigid body

Beads A & B are which move on a circular fixed ring



A + B is rigid body system
but A + B + ring is non-rigid system



A + B is non-rigid body system

If a system is rigid there is no change in the distance between any pair of particles of the system. It means shape and size of system remains constant. Hence, we naturaly feel that while a stone or cricket ball are rigid bodies, a balloon or elastic string is non strick. But any of the above method is rigid as long as comparative distance does not change, whether it is a cricket ball or a balloon. But at the instant when the bat hits the cricket ball or if the balloon is pressed, relative distance changes and now the system acts like a non-rigid system.

Concept Reminder

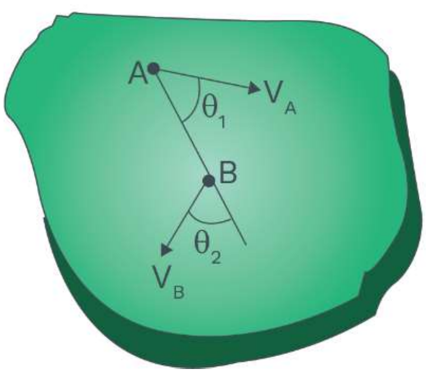
A rigid body is one for which the distances between different particles of the body do not change, even though there are forces on them.

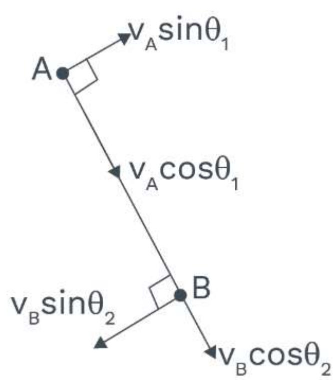
For every pair of elements in a rigid body, there is no velocity of departure or approach between the particles. i.e., any comparative motion of a point B on a rigid body with regard to another point A on the rigid body will be at right angles to line joining A to B, therefore with respect to any particle A of a stiff body the motion of any other particle B of that stiff body is rounded motion.

Let us assume velocities of A and B with respect ground be $\vec{v}_A$ and $\vec{v}_B$ respectively in the figure below.

KEY POINTS

- Rigid body
- Non-rigid body
- Translational motion
- Rotational motion





If the above body is rigid (stiff) $v_A \cos \theta_1 = v_B \cos \theta_2$ (velocity of approach/separation is zero)

v_{BA} = relative velocity of B w.r.t A

$v_{BA} = v_B \sin \theta_2 + v_A \sin \theta_1$ (which is at right angles to line AB)

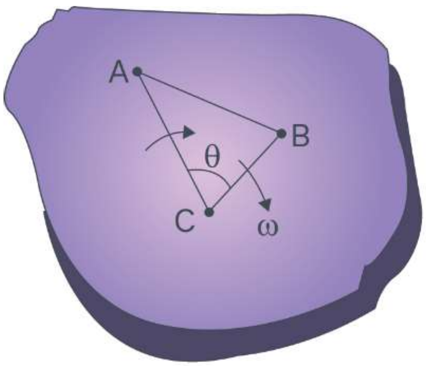
B will be found to move in a circle to an observer placed at A.

w.r.t. any point of the stiff body the angular velocity of all other points of the stiff body is same.

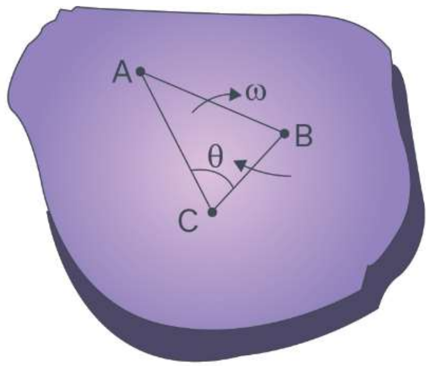
Suppose A, B, C is a stiff system hence during any motion sides AB, BC and CA must rotate through the same angle. Therefore all the sides rotate by the same rate.

Concept Reminder

Types of motion of rigid body:
(a) Pure translational
(b) Pure rotational
(c) Combined translational and rotational



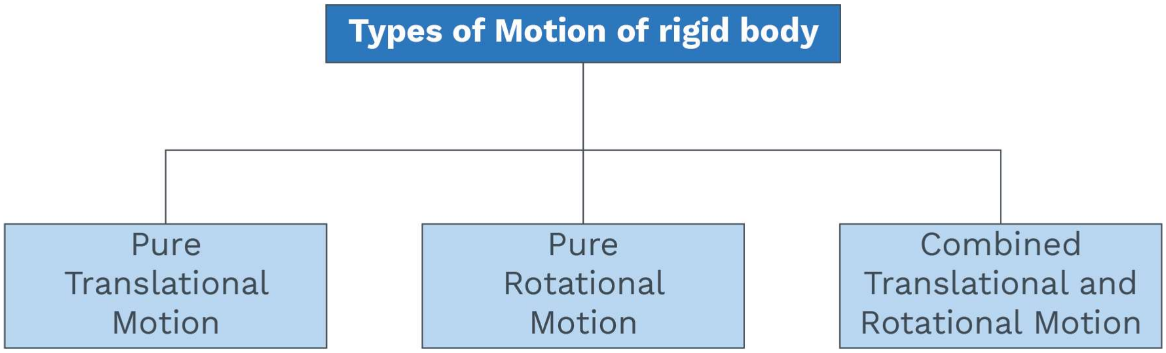
(i)



(ii)

From shape (i) angular velocity of A and B w.r.t. C is ω,

From shape (ii) angular velocity of A and C w.r.t. B is ω



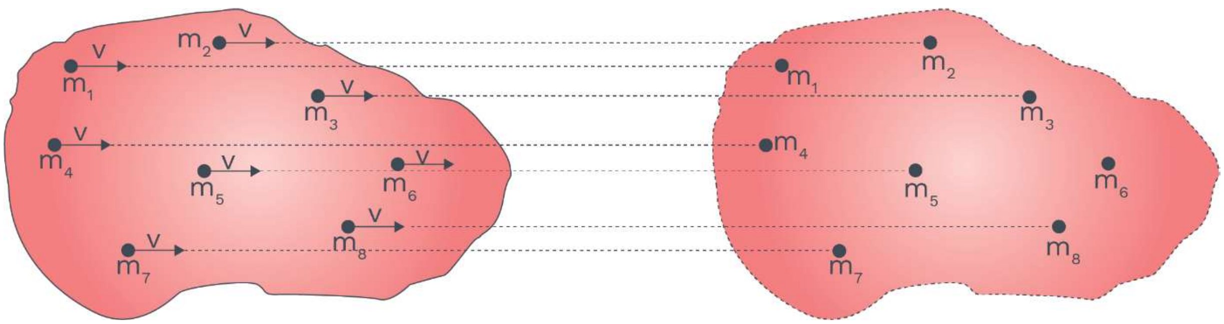
1. Pure Translational Motion:-

A body is said to be in real translational motion, if the displacement of each element of the system is same during any time interval. All through such a motion, all the particles have same displacement ($\vec{s}$), velocity ($\vec{v}$) and acceleration ($\vec{a}$).

think a system of n particle of mass m_1, m_2, m_3, ... m_n undergoing pure translation. Then from above definition of translational motion

Definitions

A body is said to be in pure translational motion, if the displacement of each particle of the system is same during any time interval.



$$\vec{a}_1 = \vec{a}_2 = \vec{a}_3 = \ldots\ldots \vec{a}_n = \vec{a} \text{ (say)}$$

and $$\vec{v}_1 = \vec{v}_2 = \vec{v}_3 = \ldots\ldots \vec{v}_n = \vec{v} \text{ (say)} -$$

From newton's laws for a system

$$\vec{F}_{ext} = m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \ldots\ldots$$

$$\vec{F}_{ext} = M\vec{a}$$

Where M = total mass of the body

$$\vec{P} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \ldots\ldots$$

$$\vec{P} = M\vec{v}$$

Total kinetic energy of body

$$= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \ldots\ldots = \frac{1}{2}Mv^2$$

Concept Reminder

In pure translational motion all the particles of the object have same velocity at any instant of time therefore, the object can considered as a point object.

2. Pure Rotational Motion:-

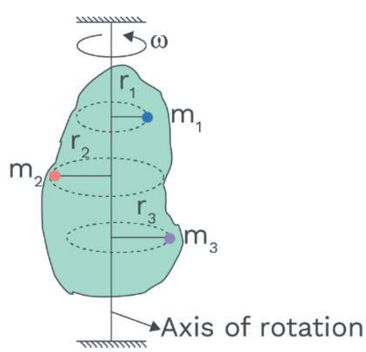


Figure shows a stiff body of arbitrary shape in rotation about a fixed axis, known as the axis of rotation. Each points of the body moves in a circle whose center lies on the axis of rotation, and every point moves all through the same angle

Definitions

A motion in which every point of the body moves in a circle whose center lies on the axis of rotation, and every point moves through the same angle during a particular time interval. Such a motion is called pure rotation.

during a particular time interval. Such a motion is known as pure rotation.

We know that each element has same angular velocity (since the body is rigid.)

So, $v_1 = \omega r_1$, $v_2 = \omega r_2$ $v_3 = \omega r_3 ......v_n = \omega r_n$

Total kinetic energy

$$= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + ...$$

$$= \frac{1}{2}[m_1 r_1^2 + m_2 r_2^2 + ...]\omega^2$$

$$= \frac{1}{2}I\omega^2$$

Where $I = m_1 r_1^2 + m_2 r_2^2 + ...$ (is known as moment of inertia)

ω = angular speed of body.

3. Combined Translational and Rotational Motion:-

A body is thought to be in combined translation and rotational motion if all the point in body rotates about an axis of rotation and axis of rotation moves w.r.t the ground. Any general motion of the rigid body can be viewed as a combined translational and rotational motion.

COMPARISION OF LINEAR MOTION AND ROTATIONAL MOTION:

Linear Motion:

(i) If acceleration is 0, v = constant and s = vt

(ii) If acceleration a = constant, then

(a) $s = \frac{(u+v)}{2}t$

(b) $a = \frac{v-u}{t}$

(c) $v = u + at$

(d) $s = ut + \left(\frac{1}{2}\right)at^2$

(e) $v^2 = u^2 + 2as$

Concept Reminder

Kinetic energy in pure rotation $= \frac{1}{2}I\omega^2$.

Definitions

A body is said to be in combined translation and rotational motion if all point in the body rotates about an axis of rotation and the axis of rotation moves with respect to the ground.

KEY POINTS

- Moment of inertia
- Axis of rotation

(f) $s_{n^{th}} = u + \frac{a(2n-1)}{2}$

(iii) If acceleration is not constant, the above equation will not be applicable. In this case

(a) $v = \frac{dx}{dt}$

(b) $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$

(c) vdv = ads

Rotational Motion:

(i) If acceleration is 0, ω = constant and $\theta = \omega t$

(ii) If acceleration α = constant then

(a) $\theta = \frac{(\omega_1 + \omega_2)}{2} t$

(b) $\alpha = \frac{\omega_2 - \omega_1}{t}$

(c) $\omega_2 = \omega_1 + \alpha t$

(d) $\theta = \omega_1 t + \frac{1}{2} \alpha t^2$

(e) $\omega_2^2 = \omega_1^2 + 2\alpha\theta$

(f) $\theta_{n^{th}} = \theta_1 + \frac{(2n-1)\alpha}{2}$

(iii) If acceleration is not constant, the above equation will not be applicable. In this case

(a) $\omega = \frac{d\theta}{dt}$

(b) $\theta = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$

(c) $\omega d\omega = \alpha d\theta$

Ex. A disc starts turning with constant angular acceleration of $\pi/2$ rad/s^2 about a fixed axis perpendicular to its plane and through its centre.

Concept Reminder

In linear motion:

If acceleration a = constant, then

(a) $s = \frac{(u+v)}{2} t$

(b) $a = \frac{v-u}{t}$

(c) v = u + at

(d) $s = ut + \left(\frac{1}{2}\right) at^2$

(e) $v^2 = u^2 + 2as$

(f) $s_{n^{th}} = u + \frac{a(2n-1)}{2}$

Concept Reminder

In rotational motion:

If acceleration α = constant then

(a) $\theta = \frac{(\omega_1 + \omega_2)}{2} t$

(b) $\alpha = \frac{\omega_2 - \omega_1}{t}$

(c) $\omega_2 = \omega_1 + \alpha t$

(d) $\theta = \omega_1 t + \frac{1}{2} \alpha t^2$

(e) $\omega_2^2 = \omega_1^2 + 2\alpha\theta$

(f) $\theta_{n^{th}} = \theta_1 + \frac{(2n-1)\alpha}{2}$

(a) Find out the angular velocity of the disc after 4 s.

(b) Find out The angular displacement of the disc after 4 s.

(c) Number of turns accomplished by the disc in 4s.

Sol. Here $\alpha = \frac{\pi}{2} \text{ rad/s}^2$

$\omega_0 = 0$

$t = 4$ s

(a) $\omega_{(4s)} = 0 + \left(\frac{\pi}{2} \text{ rad/s}^2\right) \times 4\text{s} = 2\pi \text{ rad/s}$

(b) $\theta_{(4s)} = 0 + \frac{1}{2}\left(\frac{\pi}{2} \text{ rad/s}^2\right) \times (16\,\text{s}^2)$

$= 4\pi$ radian

(c) $n\,2\pi \text{ rad} = \frac{8\pi}{2}\text{rad} = 4\pi$ radian

$n = 2$

Note: For variable angular acceleration we should proceed with differential equation $\frac{d\omega}{dt} = \alpha$.

Ex. A wheel rotates with an angular acceleration given by $\alpha = 4at^3 - 3bt^2$, where t is the time and a and b are constants. If the wheel has original angular speed ω_0, write the equations for the:

(a) Angular speed

(b) Angular displacement

Sol. (a) $\alpha = \frac{d\omega}{dt} \Rightarrow d\omega = \alpha dt$

$$\Rightarrow \int_{\omega_0}^{\omega} d\omega = \int_0^t \alpha\, dt = \int_0^t (4at^3 - 3bt^2)dt$$

$$\Rightarrow \omega = \omega_0 + at^4 - bt^3$$

(b) Further,

$$\omega = \frac{d\theta}{dt} \Rightarrow d\theta = \omega\, dt$$

Concept Reminder

In rotational motion, if angular acceleration is not constant then:

(a) $\omega = \frac{d\theta}{dt}$

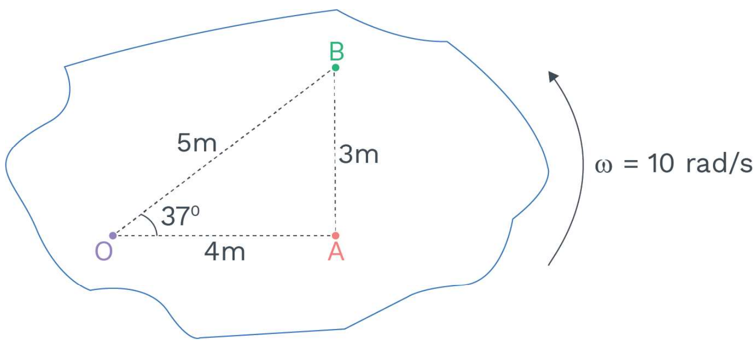
(b) $\theta = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$

(c) $\omega d\omega = \alpha d\theta$

$$\Rightarrow \quad \int_0^\theta d\theta = \int_0^t \omega\, dt = \int_0^t (\omega_0 + at^4 - bt^3)dt$$

$$\Rightarrow \quad \theta = \omega_0 t + \frac{at^5}{5} - \frac{bt^4}{4}$$

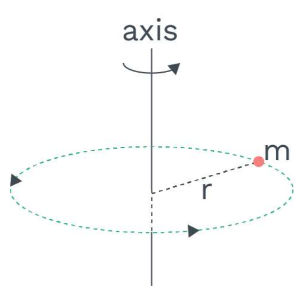
Ex. A rigid lamina is turning about an axis passing perpendicular to its plane through point O as shown in the figure.



The angular velocity of point B w.r.t. A is ?

Sol. In a rigid body, angular velocity of any point w.r.t. any other point is constant and is equal to the angular velocity of the rigid body. Therefore, angular velocity of B w.r.t A is 10 rad/s.

MOMENT OF INERTIA:-



Definitions

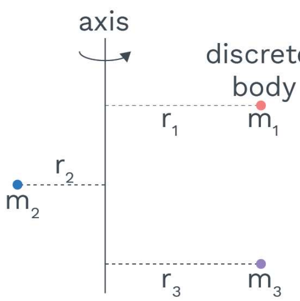
The virtue by which a body revolving about an axis opposes the change in rotational motion is known as moment of inertia.

- The virtue by which a body rotating about an axis opposes the change in rotational motion is called moment of inertia.

- The MOI of a body with respect to an axis of rotation is equal to the product of mass of the body and square of perpendicular distance from rotational axis.

  $I = mr^2$

  r = perpendicular distance from axis of rotation
- Moment of inertia of system of particle



$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + ...$$

$$I = \Sigma mr^2$$

- Moment of inertia depends on:-

  (a) Mass of the block

  (b) Mass distribution of block ⇒ its shape, size and density

  (c) Position of axis of rotation
- Moment of inertia does not depend on-

  (a) Angular velocity

  (b) Angular acceleration

  (c) Torque

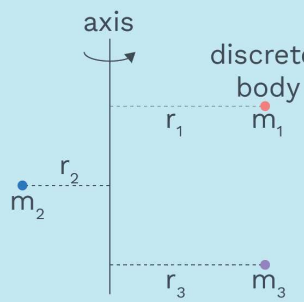
  (d) Angular momentum

  UNIT: S.I.: kg-m^2, C.G.S.: g-cm^2

  As the distance of mass rises from the rotational axis, the moment of inertia (M.I.) increases.
- MOI is a tensor quantity, but for fixed axis rotation it can be considered as a scalar quantity.

Concept Reminder

Moment of inertia of system of particle:

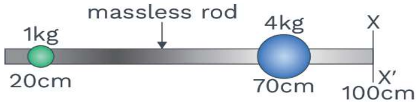


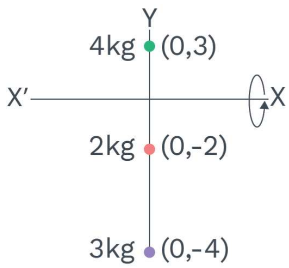
$I = \Sigma mr^2$

Concept Reminder

As the mass of body is the measure of its inertia in linear motion, the moment of inertia about a given axis of rotation resists a change in its rotational motion.

Ex. Calculate moment of inertia w.r.t. rotating axis XX' in following figures.

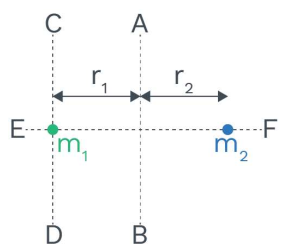




Sol. (a) $I_{XX'} = 4 \times (0.3)^2 + 1 \times (0.8)^2 = 1 \text{ kg} - m^2$

(b) $I_{XX'} = 4 \times (3)^2 + 2 \times (2)^2 + 3 \times (4)^2 = 99 \text{ kg} - m^2$

Ex. Two thick bodies having masses 'm_1' & 'm_2' are situated in a plane perpendicular to line AB at a gap of r_1 and r_2 respectively.



(i) Find out the moment of inertia of the system about axis AB?

(ii) Find out the moment of inertia of the system about an axis going through m_1 and perpendicular to the line joining m_1 and m_2 ?

(iii) Find out s the moment of inertia of the system about an axis passing through m_1 and m_2?

Sol. (i) Moment of inertia of body on left is $I_1 = m_1 r_1^2$.
Moment of Inertia of body on right is $I_2 = m_2 r_2^2$
MOI of the system about AB is given as $I = I_1 + I_2 = m_1 r_1^2 + m_2 r_2^2$

(ii) Moment of inertia of body on left is $I_1 = 0$. Moment of Inertia of the system about CD is given as
$I = I_1 + I_2 = m_2(r_1 + r_2)^2$

(iii) MOI of body on left is $I_1 = 0$ Moment of inertia of body on right is $I_2 = 0$ Moment of Inertia of system about EF is
$I = I_1 + I_2 = 0 + 0$

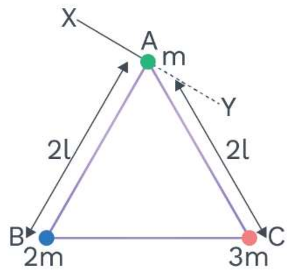
Ex. Three light rods, each rod of length 2ℓ, are joined entered to form a triangle. Three small particles A, B, C of masses m, 2m, 3m are attached to the vertices of the triangle. Find out the MOI of the resulting body about-

(a) An axis through A at right angles to the plane ABC

(b) An axis going through A and the midpoint of BC.

Sol. (a) B is at a far away 2ℓ from the axis XY so the moment of inertia of 'B' (I_B) about XY is $2\ m\ (2\ell)^2$

likewise I_c about XY is $3m\ (2\ell)^2$ and I_A about XY is $m(0)^2$



Therefore the MOI of the body about XY is $2m\ (2\ell)^2 + 3\ m(2\ell)^2 + m(0)^2 = 20\ ml^2$

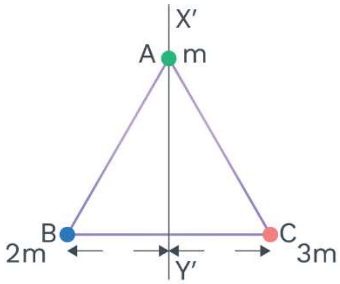
(b) I_A about X' Y' = $m(0)^2$

I_B about X' Y' = $2m\ (\ell)^2$

I_C about X' Y' = $3m\ (\ell)^2$

Therefore the MOI of the body about X' Y' is given as

$m(0)^2 + 2m(\ell)^2 + 3m(\ell)^2 = 5\ ml^2$

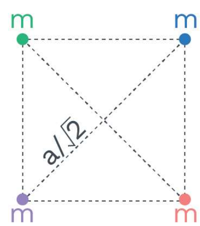


Ex. Four particles each of mass 'm' are kept at the four corners of a square of edge a. Find out the moment of inertia of system about a line perpendicular to the plane of square and passing through the centre of the square.

Sol. The perpendicular distance of each particles from the given line is $\frac{a}{\sqrt{2}}$. The MOI of one particle is, Therefore, $m\left(\frac{a}{\sqrt{2}}\right)^2 = \frac{1}{2}ma^2$

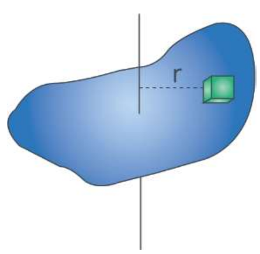
The MOI of the system is,

Therefore, $4 \times \frac{1}{2}ma^2 = 2ma^2$



MOMENT OF INERTIA OF RIGID BODIES:-

For a continuous mass distribution such as observed in a rigid body, we replace the summation of $I = \sum_i m_i r_i^2$ by an integral. If the system is divided into minuscule element of mass dm and if 'r' is the distance from the mass element to the axis of rotation, the moment of inertia is,



$I = \int r^2 dm$

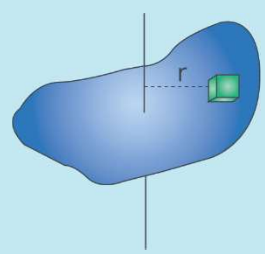
Where the integral is carried over the system.

(A) Uniform rod about a perpendicular bisector:-

Assume a uniform rod of mass 'M' and length l (figure) and assume the moment of inertia is to be determined about the bisector AB. Take up the origin at the middle point O of the rod. think about the element of the rod between a seperate x and x + dx from the origin. As the rod is uniform.

Concept Reminder

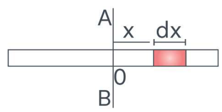
For a continuous mass distribution such as found in a rigid body.



$I = \int r^2 dm$

Rack your Brain

A light rod of length l has two masses m_1 and m_2 attached to its two ends. Find the moment of inertia of the system about an axis perpendicular to rod and passing through the centre of mass.



Mass per unit length of the rod = M/l

So that the mass of the element = (M/l)dx

The perpendicular seperation of the element from the line AB is x. The moment of inertia of this particle about AB is

$$dI = \frac{M}{l} dx\, x^2$$

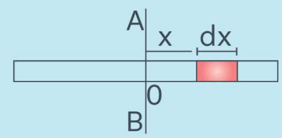
When x = – l/2, the particle is at the left side of the rod. As x is changes from – l/2 to l/2, the particles cover the whole rod.

Thus, the MOI of the entire rod about AB is

$$I = \int_{-l/2}^{l/2} \frac{M}{l} x^2\, dx = \left[\frac{Mx^3}{l\,3}\right]_{-l/2}^{l/2} = \frac{Ml^2}{12}$$

Concept Reminder

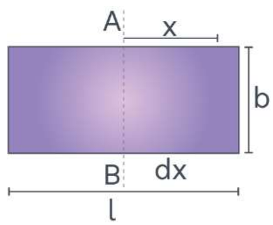
The moment of inertia of the entire rod about AB is.



$$I = \frac{Ml^2}{12}$$

(b) MOI of a rectangular plate about a line parallel to an edge and passing through the centre:-

The situation is shown in diagram. take a line parallel to AB at a distance 'x' from it and one more at a distance x + dx. We can get the strip enclosed between the two lines as the small element.

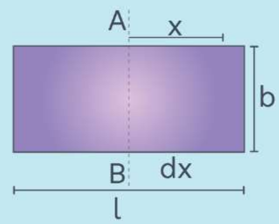


It is ‘small’ because the perpendiculars from different points of the strip to 'AB' differ by not more than dx. As the rectangular plate is uniform,

Its mass per unit area is $\frac{M}{bl}$.

Concept Reminder

Moment of inertia of a rectangular plate about a line parallel to an edge and passing through the centre:



$$I_{AB} = \frac{Ml^2}{12}$$

Mass of the strip is $\frac{M}{bl} b\,dx = \frac{M}{l} dx$

The perpendicular sepration of the strip from AB line = x.

The MOI of the strip about AB = dI = $\frac{M}{l} dx\, x^2$. The MOI of the given plate is, therefore,

$$I = \int_{-l/2}^{l/2} \frac{M}{l} x^2\, dx = \frac{Ml^2}{12}$$

The MOI of the plate about the line parallel to the other edge and going through the centre may be obtained from above formula by replacing l by b and therefore,

$$I = \frac{Mb^2}{12}$$

(c) MOI of a circular ring about its axis (the line perpendicular to the plane of the ring through its centre):

Assume the radius of the ring is R and its mass is M. As all the particles of the ring are at the same perpendicular distance R from the axis, the MOI of the ring is

$$I = \int r^2 dm = \int R^2 dm = R^2 \int dm = MR^2$$

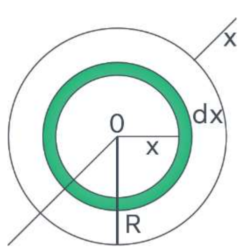
Concept Reminder

Moment of inertia of a circular ring about its axis (the line perpendicular to the plane of the ring through its centre):

$I = MR^2$

(d) MOI of a uniform circular plate about its axis:

Assume the mass of the plate be M and its radii R. The centre is at O and the axis-OX is perpendicular to the plane of the plate.



Concept Reminder

Moment of inertia of a uniform circular plate about its axis:

$$I = \frac{MR^2}{2}$$

Take two concentric circles of radii x and x + dx, both centred at O and think about the area of the plate in between the two circles.

This part of the plate may be thought to be a circular ring of radii x. As the periphery of the ring is '$2\pi x$' and its width is 'dx', the area of this elementary ring is $2\pi\, xdx$. The area of the plate is πR^2. As the plate is uniform,

Its mass per unit area $= \dfrac{M}{\pi R^2}$

Mass of the ring $= \dfrac{M}{\pi R^2} 2\pi \times xdx = \dfrac{2Mxdx}{R^2}$

Using the result achived above for a circular ring, the MOI of the elementary ring about OX is,

$$dI = \left[\frac{2Mxdx}{R^2}\right]x^2$$

The MOI of the plate about OX is

$$I = \int_0^R \frac{2M}{R^2} x^3 dx = \frac{MR^2}{2}$$

(e) Moment of inertia (MOI) of a hollow cylinder about its axis:-

Assume the radius of the cylinder is R and its mass is M. As each element of this cylinder is at same perpendicular distance 'R' from the axis, the MOI of the hollow cylinder about its axis is,

$$I = \int r^2 dm = R^2 \int dm = MR^2$$

Concept Reminder

Moment of inertia of a hollow cylinder about its axis:

$I = MR^2$

(f) MOI of a uniform solid cylinder about its axis:-

Assume the mass of the cylinder be M and its radius R. Draw two cylindrical surface of radii 'x' and x + dx coaxial with the given cylinder. Think about the part of the cylinder in between the two plane. This part of the cylinder may be believed to be a hollow cylinder of radii x. The cross-section

Concept Reminder

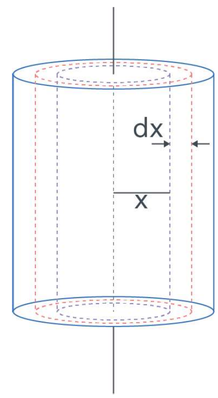
Moment of inertia of a uniform solid cylinder about its axis:

$$I = \frac{MR^2}{2}$$

area of the wall of this hollow cylinder is $2\pi x dx$. If the length of the cylinder is 'l', the volume of the material of this elementary hollow cylinder is $2\pi x dx l$.

The volume of the solid cylinder is $\pi R^2 l$ and it is uniform, hence its mass per unit volume is

$$\rho = \frac{M}{\pi R^2 l}$$



The hollow cylinder mass considered is

$$\frac{M}{\pi R^2 l} 2\pi\, x dx\, l = \frac{2M}{R^2} x\, dx$$

As its radius is x, its MOI about the given axis is

$$dI = \left[\frac{2M}{R^2} x\, dx\right] x^2$$

The MOI of the solid cylinder is, therefore,

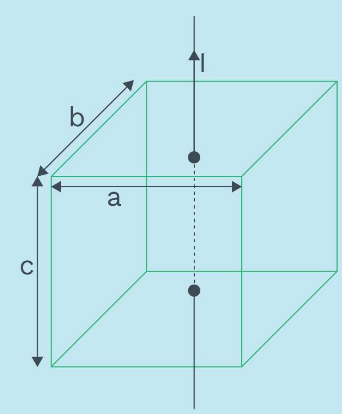
$$I = \int_0^R \frac{2M}{R^2} x^3 dx = \frac{MR^2}{2}$$

Note: that the formula does independent on the length of the cylinder.

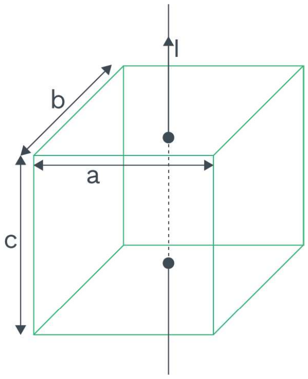
Note:- The moment of Inertia of a cuboid along the axis as shown in the figure is $I = \frac{M(a^2 + b^2)}{12}$

Concept Reminder

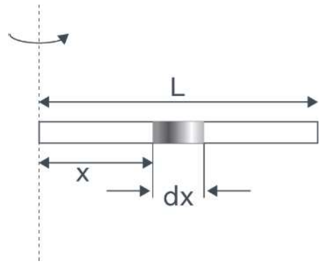
The moment of Inertia of a cuboid along the axis:



$$I = \frac{M(a^2 + b^2)}{12}$$



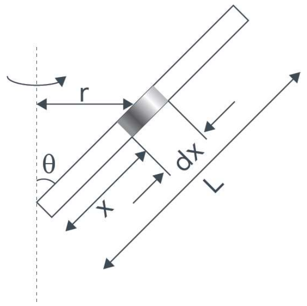
Ex. MOI of a rod about an axis passing through its end and perpendicular to length. If the rod mass is M & mass of element is dm.



Sol. $I = \int r^2 dm = \int_0^L x^2 \frac{M}{L} dx$

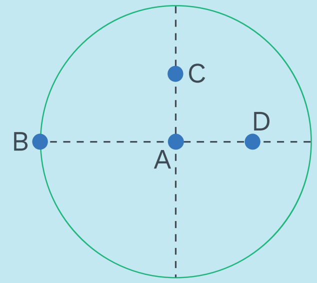
$$= \frac{M}{L}\left[\frac{x^3}{3}\right]_0^L = \frac{ML^2}{3}$$

Ex. MOI of a rod about an axis inclined at an angle θ with the rod & passing through one end.



Concept Reminder

The moment of inertia of uniform circular disc is maximum about an axis perpendicular to the disc and passing through:



(1) B (2) C
(3) D (4) A

Sol. $r = x \sin\theta$

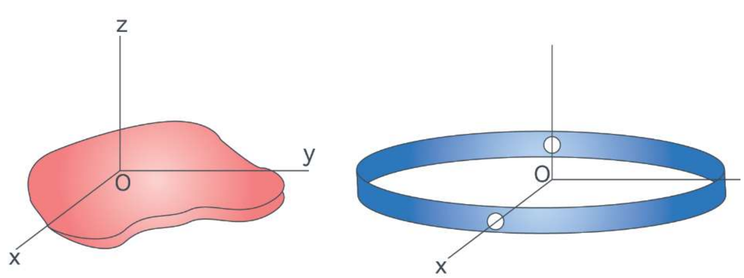
$$\text{So, } I = \int r^2 dm = \int_0^L x^2 \sin^2\theta \frac{M}{L} dx$$

$$= \frac{M}{L}\sin^2\theta\left[\frac{x^3}{3}\right]_0^L = \frac{ML^2\sin^2\theta}{3}$$

THEOREMS OF MOMENT OF INERTIA

Theorem of perpendicular axes (applicable only for two dimensional bodies or plane laminas): -

MOI of a plane lamina about the axis perpendicular to its plane is equal to the sum of moments of inertia of lamina about any two mutually perpendicular axes in its own plane intersecting each other at point through which perpendicular axis passes.



$I_z = I_x + I_y$

Where,

I_x = MI of the body about X-axis

I_y = MI of the body about Y-axis

I_z = MI of the body about Z-axis

$I_Z = I_X + I_Y$ (X - Y Plane)

$I_X = I_Y + I_Z$ (Y - Z Plane)

$I_Y = I_X + I_Z$ (X - Z Plane)

Note: Applicable only for two dimensional bodies and cannot be used for three dimensional bodies.

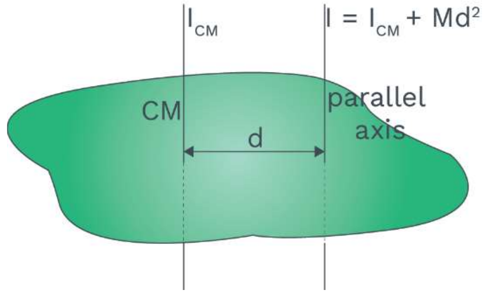
Definitions

The moment of inertia of a plane lamina about an axis perpendicular to its plane is equal to the sum of the moments of inertia of the lamina about any two mutually perpendicular axes in its own plane intersecting each other at the point through which the perpendicular axis passes.

Concept Reminder

Theorem of perpendicular axes is applicable only for two dimensional bodies or plane laminas.

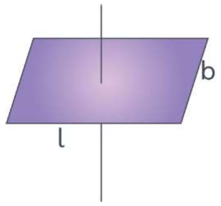
Theorem of parallel axes (for all type of bodies):
MOI of a body about any axis is equal to the moment of inertia about a parallel axis passing through the centre of mass plus product of mass of the body and the square of perpendicular distance between these two parallel axes.



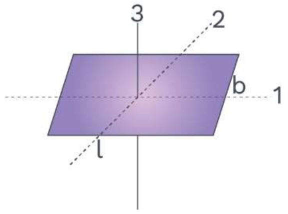
$$I = I_{CM} + Md^2$$

I_{CM} = MOI about the axis passing through centre of mass applicable for bodies of any type and shape.

Ex. Find the MOI of a uniform rectangular plate of mass M, edges of length 'l' and 'b' about its axis passing through the centre and perpendicular to it.

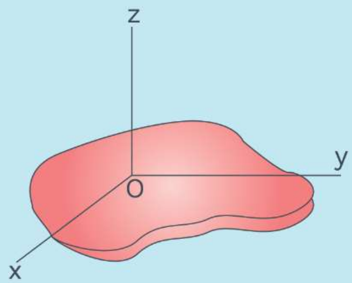


Sol. Using perpendicular axis theorem $I_3 = I_1 + I_2$



$$I_1 = \frac{Mb^2}{12},\quad I_2 = \frac{M\ell^2}{12},\quad I_3 = \frac{M(\ell^2 + b^2)}{12}$$

Concept Reminder

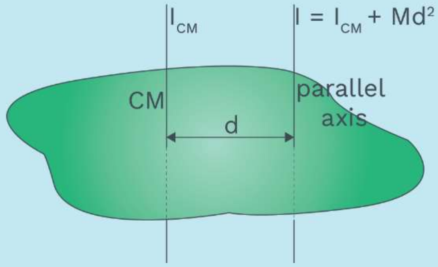


$$I_z = I_x + I_y$$

Definitions

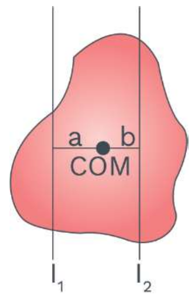
Theorem of parallel axes (for all type of bodies):
Moment of inertia of a body about any axis is equal to the moment of inertia about a parallel axis passing through the centre of mass plus product of mass of the body and the square of perpendicular distance between these two parallel axes.

Concept Reminder



$$I = I_{CM} + Md^2$$

Ex. Find the relation between I_1 and I_2. I_1 and I_2 moment of inertia of the rigid body mass 'm' about an axis as shown in figure.



Sol. Using parallel axis theorem

$I_1 = I_c + ma^2$ ...(i)

$I_2 = I_c + mb^2$ ...(ii)

From equation (i) and (ii)

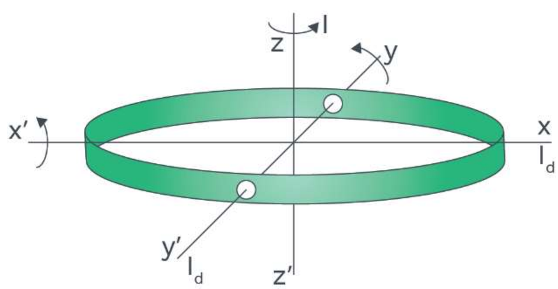
$I_1 - I_2 = m(a^2 - b^2)$

Rack your Brain

Four identical thin rods each of mass M and length l, from a square frame. Find out moment of inertia of this frame about an axis through the centre of square and perpendicular to its plane.

MOI about the diameter of the ring: -

Assume moment of inertia of the ring about each



diameter = I_d (i.e., XX' and YY'). Both diameters are perpendicular to the axis ZZ' which is passing through centre of ring and perpendicular to its plane, by theorem of perpendicular axes.

$I_{xx'} + I_{yy'} = I_{zz'}$

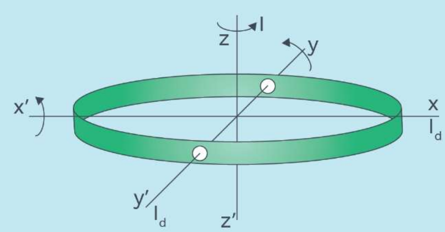
or $I_d + I_d = I_z$

$\Rightarrow$ $2 I_d = MR^2$

$\Rightarrow$ $\boxed{I_d = \frac{1}{2} MR^2}$

Concept Reminder

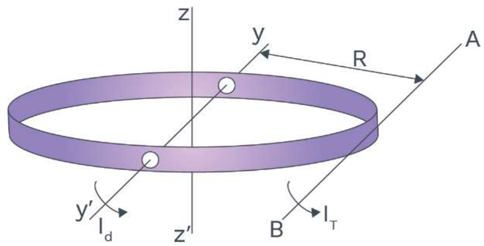
MOI about the diameter of the ring:



$I_d = \frac{MR^2}{2}$

MOI about an axis tangential and parallel to the diameter of the ring: -

Assume moment of inertia of the ring about the tangent AB parallel to the diameter YY' of the ring = I_T



Applying theorem of parallel axes

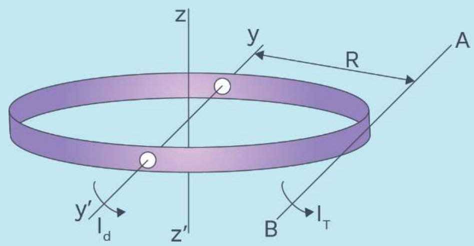
I_T = moment of inertia of ring about diameter YY' + Md^2 (here d = R)

$$I_T = \frac{1}{2}MR^2 + MR^2$$

$$\Rightarrow \boxed{I_T = \frac{3}{2}MR^2}$$

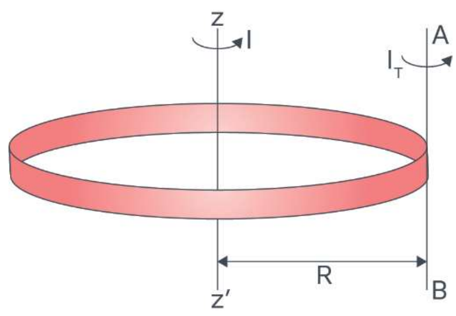
Concept Reminder

MOI about an axis tangential and parallel to the diameter of the ring:



$$I_T = \frac{3}{2}MR^2$$

MOI about the tangent parallel to the axis passing through the centre of ring and perpendicular to its plane:

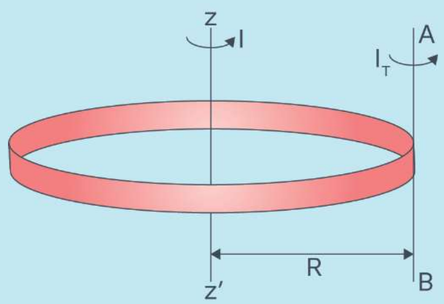


Let M.I. of the ring about tangent (AB) parallel to an axis passing through centre of the ring and perpendicular to its plane = I_T

Applying theorem of parallel axes

Concept Reminder

MOI about the tangent parallel to the axis passing through the centre of ring and perpendicular to its plane:

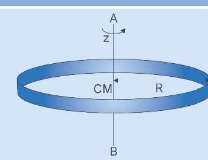
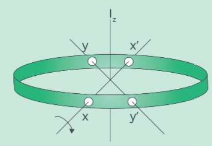
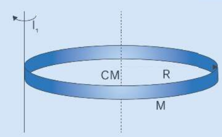
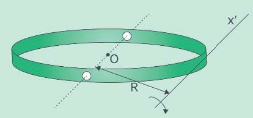
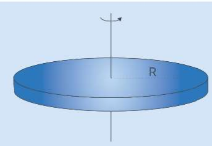
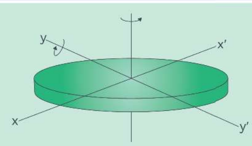
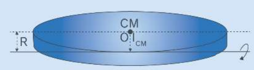
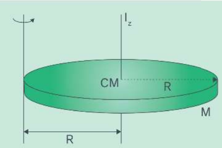


$$I'_T = 2MR^2$$

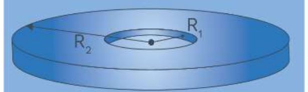
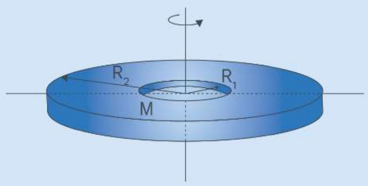
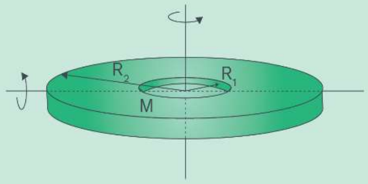
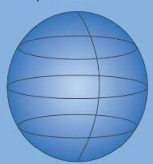
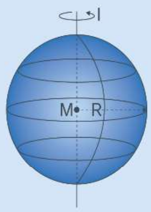
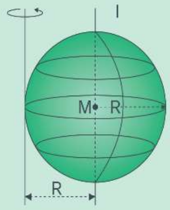
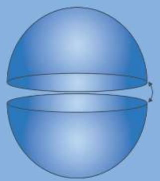
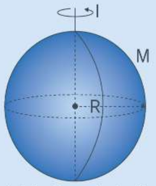
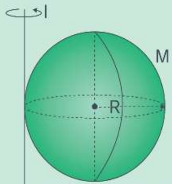
I_T = moment of inertia about ZZ' + MR^2 = MR^2 + MR^2

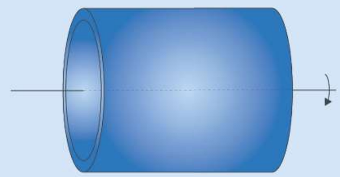
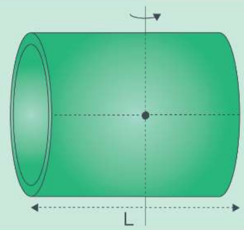
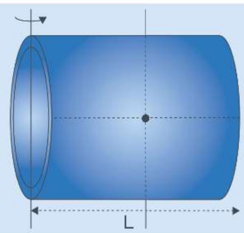
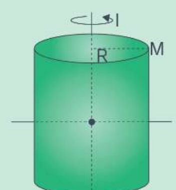
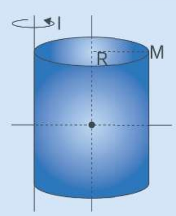
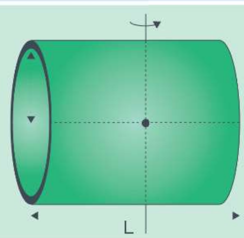
$$\boxed{I_T = 2MR^2}$$

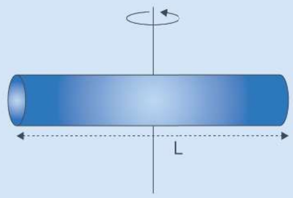
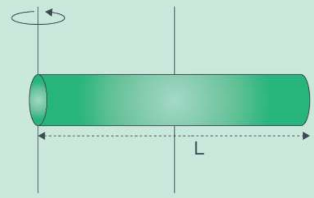
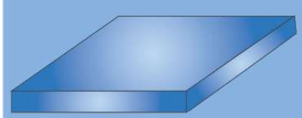
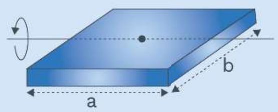
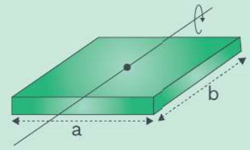
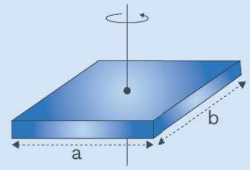
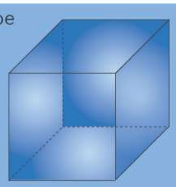
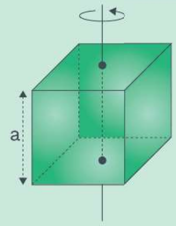
So, by using integration, we can find MOI of some more regular bodies. Here we memorize only the formulas.

MOMENT OF INERTIA OF SOME REGULAR BODIES			
SHAPE OF THE BODY	**POSITION OF THE AXIS OF ROTATION**	**FIGURE**	**MOMENT OF INERTIA (I)**
(1) Circular Ring 	(a) About an axis perpendicular to the plane and passes through the centre		MR^2
	(b) About the diametric axis		$\frac{1}{2}MR^2$
	(c) About an axis tangential to the rim and perpendicular to the plane of the ring		$2MR^2$
	(d) About an axis tangential to the rim and lying in the plane of ring		$\frac{3}{2}MR^2$
(1) Circular Disc  M = Mass R = Radius	(a) About an axis passing through the centre and perpendicular to the plane of disc		$\frac{1}{2}MR^2$
	(b) About a diametric axis		$\frac{MR^2}{4}$
	(c) About an axis tangential to the rim and lying in the plane of the disc		$\frac{5}{4}MR^2$
	(d) About an axis tangential to the rim & perpendicular to the plane of disc		$\frac{3}{2}MR^2$

Rotational Motion

MOMENT OF INERTIA OF SOME REGULAR BODIES			
SHAPE OF THE BODY	**POSITION OF THE AXIS OF ROTATION**	**FIGURE**	**MOMENT OF INERTIA (I)**
(3) Annular disc  M = Mass R_1 = Internal Radius R_2 = Outer Radius	(a) About an axis passing through the centre and perpendicular to the plane of disc		$\frac{M}{2}[R_1^2 + R_2^2]$
	(b) About a diameteric axis		$\frac{M}{4}[R_1^2 + R_2^2]$
(4) Solid Sphere  M = Mass R = Radius	(a) About its diametric axis which passes through its centre of mass		$\frac{2}{5}MR^2$
	(b) About a tangent to the Sphere		$\frac{7}{5}MR^2$
5) Hollow Sphere (Thin spherical Shell)  M = Mass R = Radius (Thickness negligible)	(a) About diametric axis passing through centre of mass	 table tennis ball	$\frac{2}{3}MR^2$
	(b) About a tangent to the surface	 table tennis ball	$\frac{5}{3}MR^2$

MOMENT OF INERTIA OF SOME REGULAR BODIES			
SHAPE OF THE BODY	**POSITION OF THE AXIS OF ROTATION**	**FIGURE**	**MOMENT OF INERTIA (I)**
(6) Hollow Cylinder  M = Mass R = Radius L = Length	(a) About its geometrical axis which is parallel to its length		MR^2
	(b) About an axis which is perpendicular to its length and passes through its centre of mass		$\frac{MR^2}{2} + \frac{ML^2}{12}$
	(c) About an axis perpendicular to its length and passing through one end of the cylinder		$\frac{MR^2}{2} + \frac{ML^2}{3}$
(7) Solid Cylinder M = Mass R = Radius L = Length 	(a) About its geometrical axis, which is along its length		$\frac{MR^2}{2}$
	(b) About an axis tangential to the cylinderical surface and parallel to its geometrical axis		$\frac{3}{2} MR^2$
	(c) About an axis passing through the centre of mass and perpendicular to its length		$\frac{ML^2}{12} + \frac{MR^2}{4}$

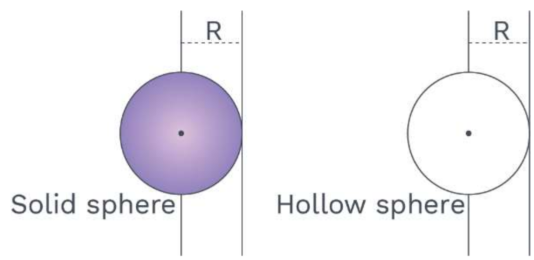
MOMENT OF INERTIA OF SOME REGULAR BODIES			
SHAPE OF THE BODY	**POSITION OF THE AXIS OF ROTATION**	**FIGURE**	**MOMENT OF INERTIA (I)**
(8) Thin Rod  Thickness is negligible w.r.t. length	(a) About an axis passing through centre of mass and perpendicular to its length		$\frac{ML^2}{12}$
	(b) About an axis passing through one end and perpendicular to length of the rod		$\frac{ML^2}{3}$
(9) Rectangular Plate  M = Mass a = Length b = Breadth	(a) About an axis passing through centre of mass and perpendicular to side b in its plane		$\frac{Mb^2}{12}$
	(b) About an axis passing through centre of mass and perpendicular to side a in its plane.		$\frac{Ma^2}{12}$
	(c) About an axis passing throught centre of mass and perpendicular to plane		$\frac{M(a^2 + b^2)}{12}$
(10) Cube  Side = a	About an axis passes through centre of mass and perpendicular to face		$\frac{Ma^2}{6}$

Ex. Find out moment of inertia of a uniform sphere of mass m and radius R about a tangent if the spheres:

(i) Solid (ii) Hollow

Sol. Using parallel axis theorem

$I = I_{CM} + md^2$



For solid sphere

$$I_{CM} = \frac{2}{5}mR^2, \quad d = R$$

$$I = \frac{7}{5}mR^2$$

(ii) Using parallel axis theorem

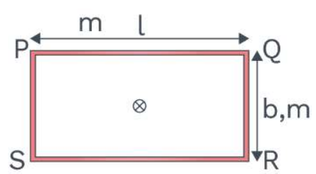
$I = I_{CM} + md^2$

For hollow sphere

$$I_{CM} = \frac{2}{3}mR^2, \quad d = R$$

$$I = \frac{5}{3}mR^2$$

Ex. Find out moment of inertia of a rectangular frame formed by uniform rods having mass 'm' each as shown about an axis passing through its centre and perpendicular to plane of frame? Also find out moment of inertia about an axis passing through PQ?



Sol. MOI about an axis passing through its centre and perpendicular to the plane of frame

$I_C = I_1 + I_2 + I_3 + I_4$

$I_1 = I_3,\ I_2 = I_4$

$I_c = 2\,I_1 + 2\,I_2$

$$I_1 = \frac{m\ell^2}{12} + m\left(\frac{b}{2}\right)^2, \quad I_2 = \frac{mb^2}{12} + m\left(\frac{\ell}{2}\right)^2$$

So, $I_c = \dfrac{2m}{3}(\ell^2 + b^2)$

M.I. about axis PQ of the rod PQ $I_1 = 0$

M.I. about axis PQ of the rod PS

$$I_2 = \frac{mb^2}{3}$$

M.I. about axis PQ of the rod QR

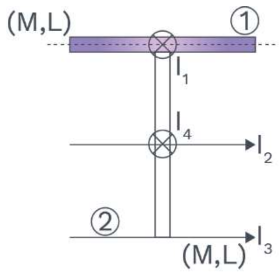
$$I_3 = \frac{mb^2}{3}$$

M.I. about axis PQ of the rod SR

$I_4 = mb^2$

$$I = I_1 + I_2 + I_3 + I_4 = \frac{5\,mb^2}{3}$$

Ex. Two similar rods are arranged in given manner. Find I_1, I_2, I_3, I_4.



Sol. $I_1 = I_{R-1} + I_{R-2} = \dfrac{ML^2}{12} + \dfrac{ML^2}{3}$

$$\boxed{I_1 = \frac{5\,ML^2}{12}}$$

$$I_2 = I_{R-1} + I_{R-2} = 0 + \frac{ML^2}{12} + \frac{ML^2}{4}$$

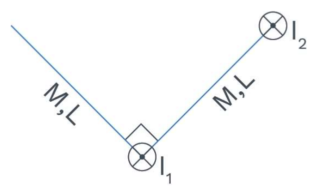
$$I_3 = I_{R-1} + I_{R-2} = 0 + ML^2 + \frac{ML^2}{3}$$

$$\boxed{I_3 = \frac{4ML^2}{3}}$$

$$I_4 = \frac{ML^2}{12} + \frac{ML^2}{4} + \frac{ML^2}{12} = \frac{2ML^2}{12} + \frac{3ML^2}{12}$$

$$\boxed{I_4 = \frac{5ML^2}{12}}$$

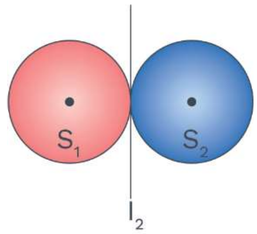
Ex. For given arrangement, Find M.I. about I_1 and I_2.



Sol. $I_1 = \frac{ML^2}{3} \times 2 = \frac{2ML^2}{3}$

$$I_2 = \frac{2ML^2}{3} + ML^2 = \frac{5ML^2}{3}$$

Ex. Two similar solid sphere touch each other. Find M.I. about I_2?



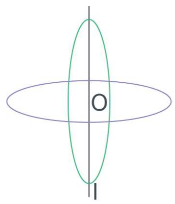
Sol. $I_2 = \left(\frac{2}{5}MR^2 + MR^2\right) \times 2$

$$= \left(\frac{7MR^2}{5}\right) \times 2 = \frac{14MR^2}{5}$$

Ex. Two rings having the same radius and mass are placed such that their centres are at a common point and their planes are at 90° to each other. Find out moment of inertia of the system about an axis passing through the centre and perpendicular to the plane of one of the rings. (M = mass of each ring and R = radius).

Sol. Similar co-centric rings, MOI passing through common centre is

$$I = \frac{MR^2}{2} + MR^2$$



$$I = \frac{3MR^2}{2}$$

Ex. A wire of mass m and length l is bent into a shape of circular loop then its MOI about geometrical axis will be.

Sol. $2\pi r = \ell,\ \ r = \frac{\ell}{2\pi}$

$I = mr^2 \Rightarrow I = \frac{M\ell^2}{4\pi^2}$

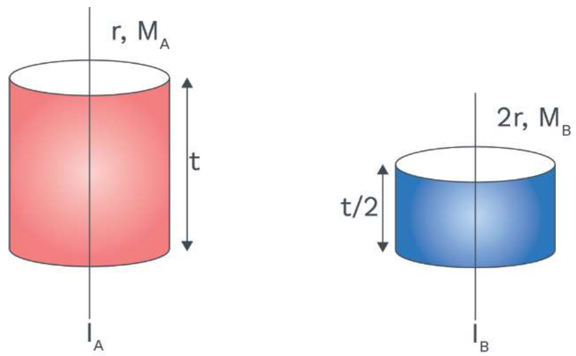
Ex. A disc of radius 'r' is made from an iron plate of thickness 't'. Another disc B of radius '2r' is made from the same material of thickness t/2. Then find ratio of MOI (density is same)

Sol. $\rho = \frac{M_A}{\pi r^2 t}$

$M_A = \rho\pi r^2 t$

$M_B = \rho\pi 4r^2 \frac{t}{2} = \rho\pi\, 2r^2 t$

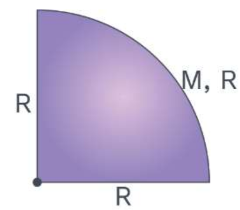
$$\frac{I_A}{I_B} = \frac{\frac{M_A (r)^2}{2}}{\frac{M_B (2r)^2}{2}}$$



$$\frac{I_A}{I_B} = \frac{1}{8}$$

Rotational Motion

Ex. A segment of disc having mass M and radius R is given then find its MOI about the axis shown in figure.



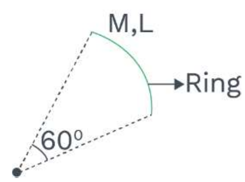
Sol. For a complete disc → mass = 4M

MOI for new disc = $\frac{4MR^2}{2}$

For one segment = $\frac{4MR^2}{2} \times \frac{1}{4}$

One segment = $\frac{MR^2}{2}$

Ex. Find MOI for given figure.



Sol. For complete ring → mass = 6M

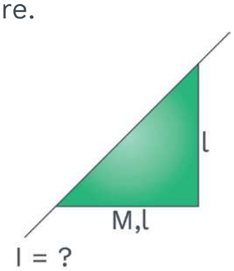
MOI for one segment = $6MR^2 \frac{1}{6}$

$\theta = \frac{L}{R}$ (one segment $= MR^2$)

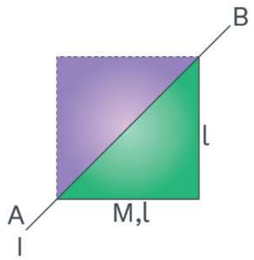
$60 \times \frac{\pi}{180} = \frac{L}{R}$

$R = \frac{3L}{\pi}$, $I = \frac{9ML^2}{\pi^2}$

Ex. Find MOI for given figure.

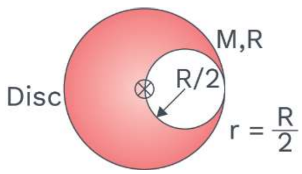


Sol. M.I. of Complete square = 2I



$$2I = \frac{2M(L^2)}{12} \Rightarrow I = \frac{ML^2}{12}$$

Ex. A hole of radii R/2 is cut from a thin circular plate of radii R and mass M as shown in the diagram. Then find out MOI of the plate about an axis through O perpendicular to plane.



Sol. $I_{remaining} = I_{complete} - I_{removed}$ about same axis

$$= \frac{MR^2}{2} - \left(\frac{mr^2}{2} + mr^2\right)$$

$$= \frac{MR^2}{2} - \frac{3}{2}mr^2$$

$$= \frac{MR^2}{2} - \frac{3}{2}\left(\frac{M}{4}\right)\left(\frac{R}{2}\right)^2$$

$$= \frac{MR^2}{2} - \frac{3MR^2}{32} = \frac{13MR^2}{32}$$

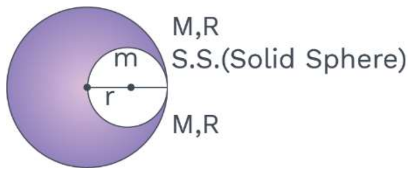
Note:- $\pi r^2 \rightarrow M$

$$1 \rightarrow \frac{M}{\pi r^2}$$

$$\pi\left(\frac{R}{2}\right)^2 \rightarrow \frac{M}{\pi R^2} \times \frac{\pi R^2}{4}$$

$$m = \frac{M}{4}$$

Ex. A hole of radii R/2 is cut from a solid sphere of radius R and mass M as shown in the figure. Then find moment of inertia of the sphere about an axis passing through centre and perpendicular to plane.



Sol. $\frac{4}{3}\pi R^3 \rightarrow M,\ r = \frac{R}{2}$

$$1 \rightarrow \frac{3M}{4\pi R^3}$$

$$\frac{4\pi r^3}{3} \rightarrow \frac{3M}{4\pi R^3} \times \frac{4\pi r^3}{3}$$

$$m = \frac{M}{R^3} \times r^3 \Rightarrow m = \frac{M}{8}$$

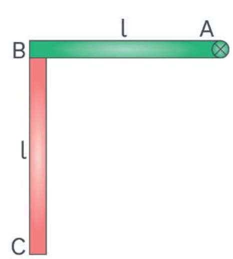
$$I = \frac{2}{5}MR^2 - \left(\frac{2}{5}mr^2 + mr^2\right)$$

$$= \frac{2}{5}MR^2 - \left(\frac{7mr^2}{5}\right) = \frac{2}{5}MR^2 - \frac{7}{5}\left(\frac{M}{8}\right)\left(\frac{R^2}{4}\right)$$

$$= MR^2\left(\frac{2}{5} - \frac{7}{160}\right)$$

$$I = \frac{57MR^2}{160}$$

Ex. Two rods each having length 'l' and mass 'm' joined together at point B as shown in diagram. Then find out moment of inertia about axis passing through A and perpendicular to the plane of page as shown in diagram.



Sol. We find the resultant MOI I by dividing in two parts such as

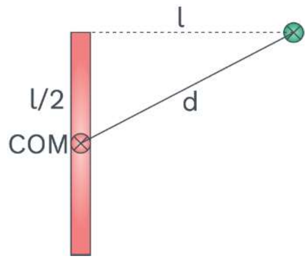
I = MOI of rod AB about A +M.I of rod BC about A

$I = I_1 + I_2$ ...(i)

Calculation of I_1

$$I_1 = \frac{m\ell^2}{3} \qquad \text{...(ii)}$$

Calculation of I_2



Use parallel axis theorem

$I_2 = I_{CM} + md^2$

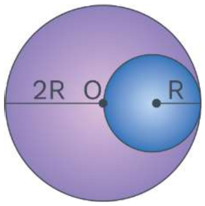
$$= \frac{m\ell^2}{12} + m\left(\frac{\ell^2}{4} + \ell^2\right) = \frac{m\ell^2}{12} + \frac{5\,\ell^2}{4}m \qquad \text{...(iii)}$$

Put value from eq. (ii) & (iii) into (i)

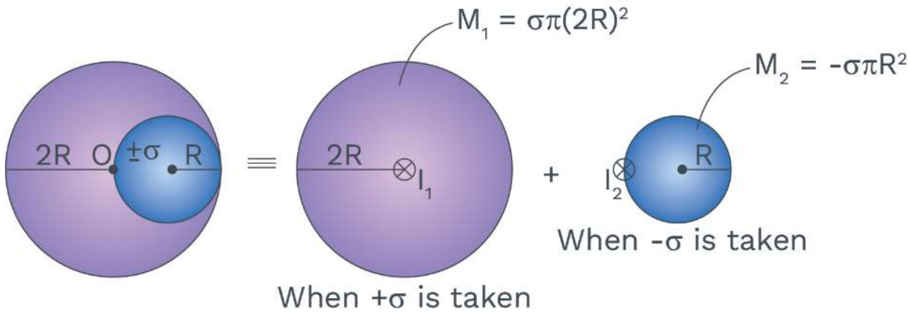
$$\Rightarrow \quad I = \frac{m\ell^2}{3} + \frac{m\ell^2}{12} + \frac{5\ell^2 m}{4}$$

$$I = \frac{m\ell^2}{12}(4 + 1 + 15) \Rightarrow I = \frac{5\,m\ell^2}{3}$$

Ex. A uniform disc having radii 2R and mass density σ as shown in figure. If a small disc of radii R is cut from the disc as shown. Then find the MOI of remaining disc around the axis that passes through O and is perpendicular to the plane of the page.



Sol. We assume that in remaining part a disc of radii R and mass density ± σ is placed. Then



Total Moment of Inertia $I = I_1 + I_2$

$$I_1 = \frac{M_1(2R)^2}{2}$$

$$I_1 = \frac{\sigma\pi 4R^2 \cdot 4R^2}{2} = 8\pi\sigma R^4$$

To determine I_2 we use parallel axis theorem

$$I_2 = I_{CM} + M_2R^2$$

$$I_2 = \frac{M_2R^2}{2} + M_2R^2$$

$$I_2 = \frac{3}{2}M_2R^2 = \frac{3}{2}(-\sigma\pi R^2)R^2$$

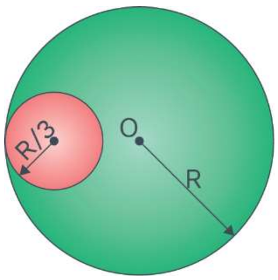
$$I_2 = -\frac{3}{2}\sigma\pi R^4$$

Now $I = I_1 + I_2$

$$I = 8\pi\sigma R^2 - \frac{3}{2}\sigma\pi R^4$$

$$I = \frac{13}{2}\sigma\pi R^4$$

Ex. A uniform disc of radii R has a round disc of radii R/3 cut as shown in Fig. The mass of the remaining portion (shaded) of the disc equals M. Find out the MOI of such a disc relative to the axis going through geometrical centre of original disc and at right angles to the plane of the disc.



Sol. Assume the mass per unit area of the material of disc be σ. Now the empty space can be belived as having density –σ and σ.

Now $I_0 = I_\sigma + I_{-\sigma}$

$$I_\sigma = \frac{(\sigma\pi R^2)R^2}{2} = \text{MI of } \sigma \text{ about O}$$

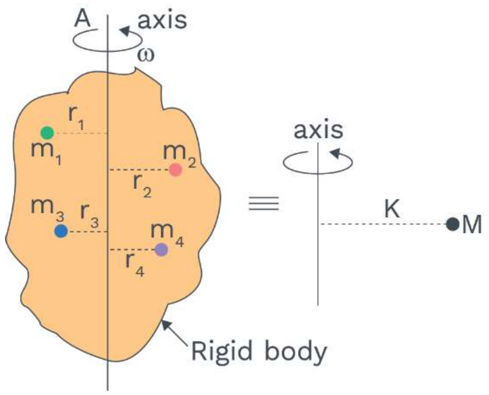
$$I_{-\sigma} = \frac{-\sigma\pi\left(\frac{R}{2}\right)^2\left(\frac{R}{3}\right)^2}{2} + \left[-\sigma\pi\left(\frac{R}{3}\right)^2\right]\left(\frac{2R}{3}\right)^2$$

= MI of –σ about 0

$$\therefore \quad I_0 = \frac{4}{9}\sigma\pi R^4$$

RADIUS OF GYRATION (K)

The radius of gyration of a body is the distance from axis of rotation, the square of this distance when multiplied by the mass of body then it gives the moment of inertia of the body ($I = MK^2$) about same axis of rotation.



Definitions

The radius of gyration of a body is the distance from axis of rotation, the square of this distance when multiplied by the mass of body then it gives the moment of inertia of the body ($I = MK^2$) about same axis of rotation.

$$I = MK^2 \quad \Rightarrow \quad I = \sum mr^2$$

$$MK^2 = \sum mr^2$$

$$K^2 = \frac{m_1 r_1^2 + m_2 r_2^2 + ... + m_n r_n^2}{M}$$

$$K = \sqrt{\frac{m_1 r_1^2 + m_2 r_2^2 + ... + m_n r_n^2}{m_1 + m_2 + ... + m_n}}$$

If $m_1 = m_2 = m_3 = m$ then, $M = mn$

$K = \sqrt{\frac{r_1^2 + r_2^2 + ... + r_n^2}{n}}$ (n = total number of particles)

Radius of gyration $K = \sqrt{\frac{I}{M}}$

(K has no sense without axis of rotation and K is scalar quantity)

Radius of gyration depends on

(i) Axis of rotation

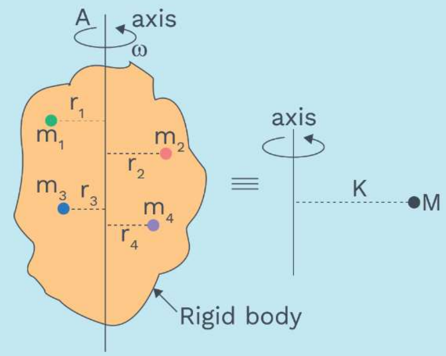
(ii) Distribution of mass of body

Radius of gyration does not depend on

(i) Mass of the body

(ii) Angular quantities (angular displacement, angular velocity etc.)

Concept Reminder

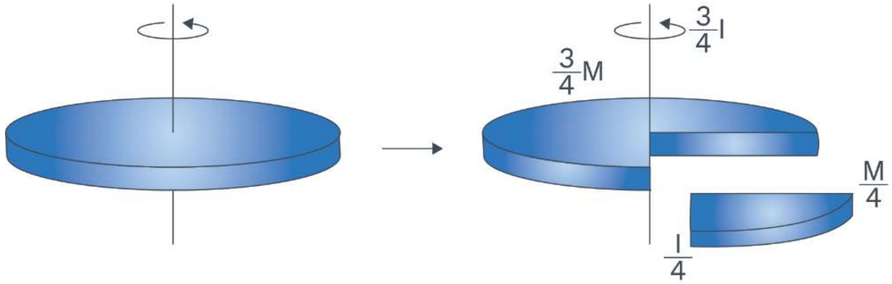


$I = MK^2$

Rack your Brain

Find the ratio of the radii of gyration of a circular disc about a tangential axis in the plane of disc and of circular ring of same radius and mass about a tangential axis in the plane of ring.

Symmetrical separation



M = Mass of disc

R = Radius of disc

If $\frac{M}{4}$ part is separated.

Remaining mass $= \frac{3}{4}M = M'$

MI of remaining part $I' = \frac{3}{4}I$

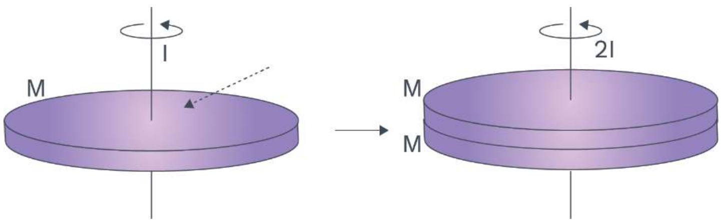
Radius of gyration of remaining part

$$K' = \sqrt{\frac{\frac{3I}{4}}{\frac{3M}{4}}} = \sqrt{\frac{I}{M}} = K \Rightarrow K \text{ remains unchanged.}$$

Concept Reminder

For symmetrical separation and symmetrical attachment, radius of gyration remains unchanged.

Symmetrical attachment:

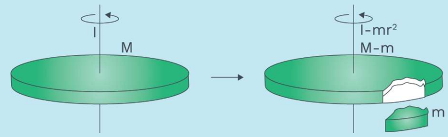


$$K = \sqrt{\frac{I}{M}}$$

$$K' = \sqrt{\frac{2I}{2M}} = K$$

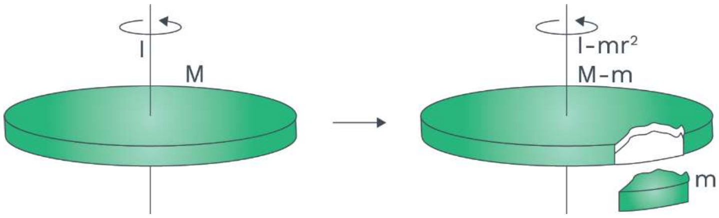
Concept Reminder

Unsymmetrical separation:



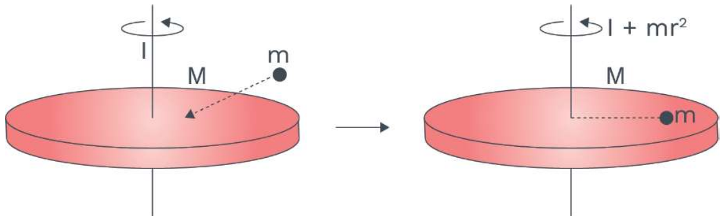
$$K = \sqrt{\frac{I - mr^2}{M - m}}$$

Unsymmetrical separation:



$$K = \sqrt{\frac{I - mr^2}{M - m}}$$

Unsymmetrical attachment:



$$K = \sqrt{\frac{I + mr^2}{M + m}}$$

Ex. Find out the radius of gyration of a solid uniform sphere of radius R about its tangent.

Sol. $I = \frac{2}{5}mR^2 + mR^2 = \frac{7}{5}mR^2 = mK^2$

$$\Rightarrow \quad K = \sqrt{\frac{7}{5}}R$$

Ex. Find out the radius of gyration of a hollow uniform sphere of radius R about its tangent.

Sol. $mK^2 = \sqrt{\frac{5}{3}}mR^2$

$$K = \sqrt{\frac{5}{3}}R$$

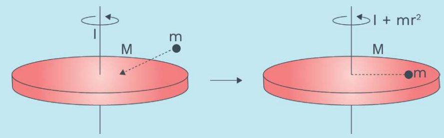
TORQUE

It is the physical agency which is responsible for change in state of rotation. Torque is essential for producing turning / toppling phenomena.

For producing torque the force is required and it is product of force and perpendicular distance of line of action of force (liver arm from axis).

Concept Reminder

Unsymmetrical attachment:



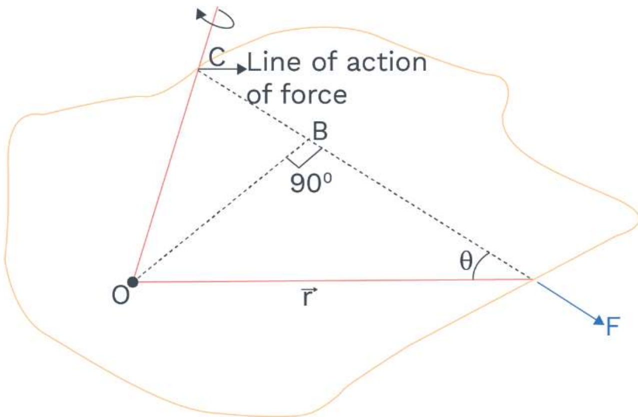
$$K = \sqrt{\frac{I + mr^2}{M + m}}$$

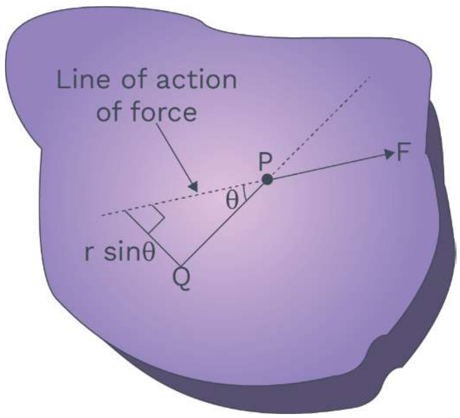
KEY POINTS

- Radius of gyration
- Torque
- Toppling
- Liver arm

Definitions

It is the physical agency which is responsible for change in state of rotation. Torque is essential for producing turning / toppling phenomena.





In figure OB is the perpendicular distance of line of action from axis which is r sin θ.

So, τ = Fr sin θ

Vector form, $\vec{\tau} = \vec{r} \times \vec{F}$

Where $\vec{F}$ = force applied

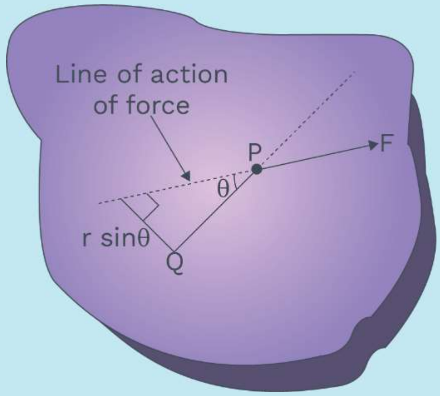
$\vec{r}$ = position vector of the application point of force w.r.t the point about which we want to determine the torque.

It is an axial vector, i.e. Its direction is always perpendicular to the plane containing vector $\vec{r}$ and $\vec{F}$.

Its direction is determined by right hand screw rule.

Positive sign to all torques acting to turn a body anti-clockwise and a minus to all torques tending to turn it clockwise.

Concept Reminder



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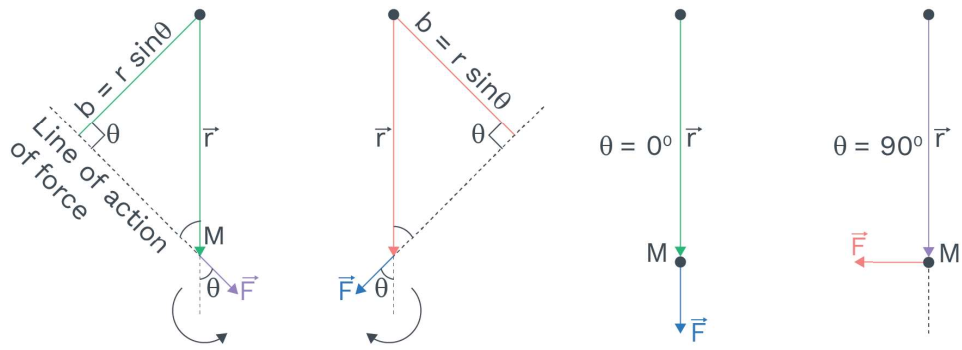


Although force is required for producing the torque yet every force is not capable to produce torque. If the line of action of force passes through axis then torque about the axis is zero. Torque is also defined as moment of force.

Minimum torque

When $|\sin\theta|_{min} = 0$

i.e., $\theta = 0°$ or $180°$

$\tau_{min} = 0$

Maximum torque

When $|\sin\theta|_{max} = 1$

i.e., $\theta = 90°$

$\tau_{max} = Fr$

Rotation of a door about a hinge, rotation of grinding wheel about a pivot or unbolting a nut by a pipe- wrench can be cited as examples of torque. In these for producing a desired rotational effect.

τ = constant

$\Rightarrow$ $Fr\sin\theta$ = constant

$$\Rightarrow \quad F = \frac{\text{constant}}{r\sin\theta}$$

Longer the arm and greater the sin θ, lesser will be the force required for producing desired rotational effect. Therefore, it is much easier to rotate a body about a given axis when the force is applied at maximum distance from the axis of rotation and normal to the arm.

Eg : Easy sharpening of long pencil.

Concept Reminder

Torque is a rotational analogue of force or turning effect of force. It is the measure of the tendency of a force to rotate an object about some axis.

Comparison between force and torque:

Torque is the effect of rotatory motion and in revolving motion it plays same part as force plays in the translatory motion, i.e., the torque is rotational analogue of force.

Rotatory motion	Translatory motion
$\vec{\tau} = I\vec{\alpha}$	$\vec{F} = m\vec{a}$
$W = \int \vec{\tau} \cdot d\vec{\theta}$	$W = \int \vec{F} \cdot d\vec{s}$
$P = \vec{\tau} \cdot \vec{\omega}$	$P = \vec{F} \cdot \vec{v}$
$\vec{\tau} = \frac{d\vec{L}}{dt}$	$\vec{F} = \frac{d\vec{P}}{dt}$

Ex. Force $\vec{F} = -2\hat{i} + 2\hat{j} + 2\hat{k}$ is acting on a point whose position vector is given as $\hat{i} + 2\hat{j} - \hat{k}$ then find out torque about origin.

Sol. $\vec{\tau} = \vec{r} \times \vec{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix}$$

$$\vec{\tau} = (4+2)\hat{i} - 2(2-2)\hat{j} + (2+4)\hat{k}$$

$$= 6\hat{i} - 0\hat{j} + 6\hat{k}$$

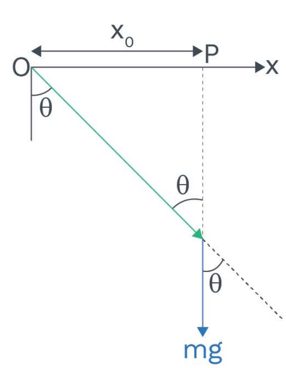
$$\vec{\tau} = 6\hat{i} + 6\hat{k}$$

$$|\vec{\tau}| = 6\sqrt{2} \text{ Nm}$$

Rack your Brain

What is the torque of force $\vec{F} = 2\hat{i} - 3\hat{j} + 4\hat{k}$ N acting at the point $\vec{r} = 3\hat{i} + 2\hat{j} + 3\hat{k}$ m about origin?

Ex. A particle of mass 'M' is released in the vertical plane from a point 'P' at x = x_0 on the x-axis it falls vertically along y-axis. Find out the torque τ acting on the particle at a time t about origin?

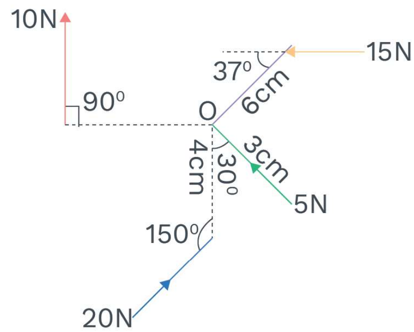


Sol. Torque is created by the force of gravity

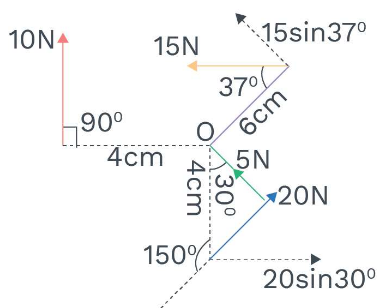
$$\vec{\tau} = rF\sin\theta\,\hat{k}$$

or $\tau = r_{\perp}F = x_0 mg$

Ex. Find out the total torque acting on the body shown in figure about the point O.



Sol.



$$\tau_0 = 15\sin 37 \times 6 \odot + 20\sin 30° \times 4 \odot - 10 \times 4 \odot$$

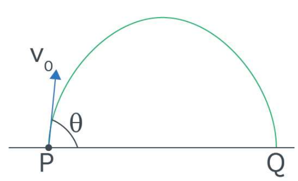
$= 54 + 40 - 40 = 54$ N-cm

$\tau_0 = 0.54$ N-m

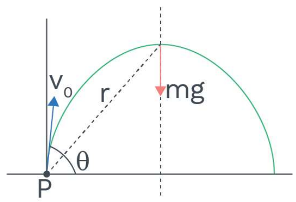
Ex. A particle having mass 'm' is projected with a velocity 'v_0' from a point P on a horizontal ground making an angle 'θ' with horizontal. Find the torque about the point of projection acting on the particle-

(a) When it is at its maximum height ?

(b) When it is just about to hit the ground back ?



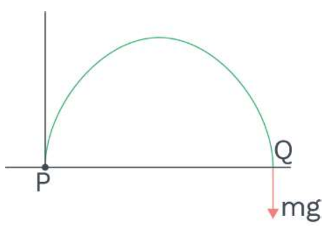
Sol. (a) When particle is at maximum height then τ about point P is $\tau_p = r_\perp F$



$$\tau_p = \frac{R}{2}mg = mg \times \frac{v_0^2 \sin 2\theta}{2g}$$

$$\tau_p = \frac{mv_0^2 \sin 2\theta}{2}$$

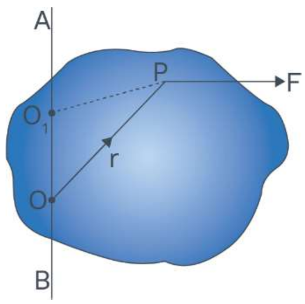
(b) when particle is at point 'Q' then 'τ' about point P is $\tau'_p = r_\perp F$



$r_\perp = R$; $F = mg$

$$\tau'_p = mgR = mg\frac{v_0^2 \sin 2\theta}{g}$$

Torque about an axis:



The torque of a force '$\vec{F}$' about an axis AB is defined as the component of torque of $\vec{F}$ about any point O on the axis AB, along the axis AB.

In the given figure torque of $\vec{F}$ about O is

$$\vec{\tau_0} = \vec{r} \times \vec{F}$$

Rack your Brain

A thin rod of length L and mass M is bent at its mid-point into two halves so that angle between them is 90°. the moment of inertia of the bend rod about an axis passing through bending point and perpendicular to the plane is:

(1) $\frac{ML^2}{6}$ (2) $\frac{\sqrt{2}ML^2}{24}$

(3) $\frac{ML^2}{24}$ (4) $\frac{ML^2}{12}$

The torque of $\vec{F}$ about AB, τ_{AB} is component of $\overrightarrow{\tau_0}$ along line AB.

There are four cases of the torque of a force about an axis.

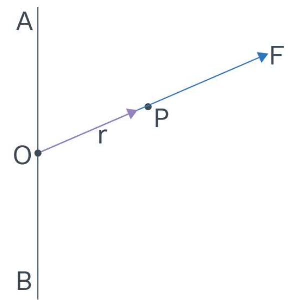
Case-I: Force is parallel to the axis of rotation,

$\vec{F} \parallel \overline{AB}$

AB is axis of rotation about which the torque is required $\vec{r} \times \vec{F}$ is perpendicular to $\vec{F}$, but $\vec{F} \parallel \overrightarrow{AB}$, hence $\vec{r} \times \vec{F}$ is perpendicular to $\overline{AB}$.

The component of $\vec{r} \times \vec{F}$ and $\overrightarrow{AB}$ is, therefore, zero.

Case-II: The line of force intersects the axis of rotation (F intersect AB)

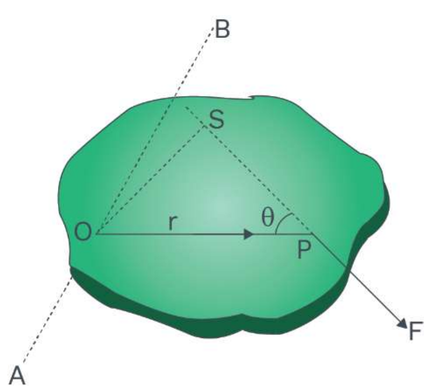


$\vec{F}$ intersects AB along $\vec{r}$ then $\vec{F}$ and $\vec{r}$ are along the same line. The torque about O is

$\vec{r} \times \vec{F} = 0$

Hence component of this torque along line AB is also zero.

Case-III: $\vec{F}$ perpendicular to $\overrightarrow{AB}$ but $\vec{F}$ and AB don't intersect.



In the 3-D, two lines may be perpendicular without intersecting each other. Two nonparallel and nonintersecting lines are known as skew lines.

Diagram shows the plane through the point of application of force 'P' that is perpendicular to axis of rotation AB. Assume that the plane intersects the axis at the point 'O'. The force 'F' is in this plane (since F is perpendicular to AB). Taking the origin at O,

Torque = $\vec{r} \times \vec{F} = \overrightarrow{OP} \times \vec{F}$

Thus, torque = rF sin θ = F(OS)

Where 'OS' is the perpendicular from 'O' to the line of action of the force $\vec{F}$. The line OS is also perpendicular to axis of rotation. It is thus the length of common perpendicular to the force and the axis of rotation.

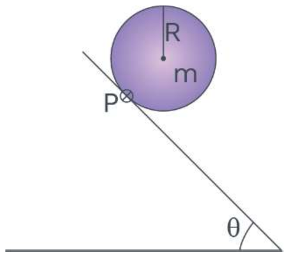
The direction of $\vec{\tau} = \overrightarrow{OP} \times \vec{F}$ is along the axis AB because $\overrightarrow{AB} \perp \overrightarrow{OP}$ and $\overrightarrow{AB} \perp \vec{F}$. The torque about AB is, therefore, equal to magnitude of $\vec{\tau}$ that is F.(OS).

Thus, the torque of F about AB = magnitude of the force F × length of common perpendicular to force and the axis. The common perpendicular OS is known as the lever arm or moment arm of this torque.

Case-IV: $\vec{F}$ and $\overrightarrow{AB}$ are skew but not perpendicular.

Here we resolve $\vec{F}$ into two components, one is parallel to the axis and other is perpendicular to the axis. Torque of parallel part is zero and that of perpendicular part may be found, by using the result of case (III).

Ex. Find torque of weight about the axis passing through point 'P'.



Sol. $\vec{\tau} = \vec{r} \times \vec{F}$, $\vec{r} = R$, $\vec{F} = mg\sin\theta$

'r' and 'F' both are at perpendicular so torque about point

τ = mg R sin θ

BODY IS IN EQUILIBRIUM

We can say rigid body is in equilibrium when it is in

(a) Translational equilibrium

i.e., $\vec{F}_{net} = 0$

$F_{net\ x} = 0$ and $F_{net\ y} = 0$

(b) Rotational equilibrium

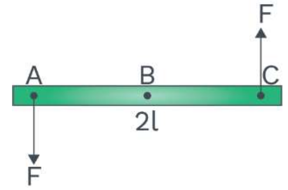
$\vec{\tau}_{net} = 0$

i.e., torque about every point is zero.

Note: (i) If total force on the block is zero then net torque of the forces may or may not be zero.

Ex.-

A pair of forces each of equal magnitude and acting in opposite direction on the rod.



(ii) If net force on the body is zero then torque of the forces about each and every point is same

τ about B

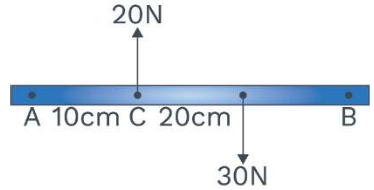
$$\tau_B = F\ell + F\ell \odot$$

$$\tau_B = 2F\ell \odot$$

τ about C

$$\tau_C = 2F\ell \odot$$

Ex. Calculate the point of application of third force for which body is in equilibrium when forces of 30 N & 20 N are acting on rod as shown in diagram.



Sol. Assume the magnitude of third force is F, is applied in upward direction then the body is in the equilibrium when

(i) $\vec{F}_{net} = 0$ (translational equilibrium)

$\Rightarrow$ 20 + F = 30

$\Rightarrow$ F = 10 N

Therefore, the body is in translational equilibrium condition when 10 N force act on it in upward direction.

KEY POINTS

- Translational equilibrium
- Rotational equilibrium

Concept Reminder

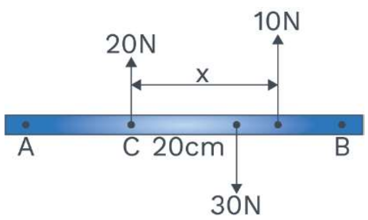
Translational equilibrium:

$\vec{F}_{net} = 0$

Rotational equilibrium:

$\vec{\tau}_{net} = 0$

(ii) Let us assume that this 10 N force act. Then keep the body in rotational equilibrium. So, Torque about C = 0



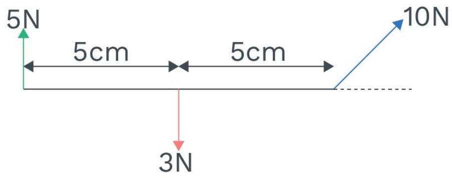
i.e. $\tau_c = 0$

$\Rightarrow$ $30 \times 20 = 10\,x$

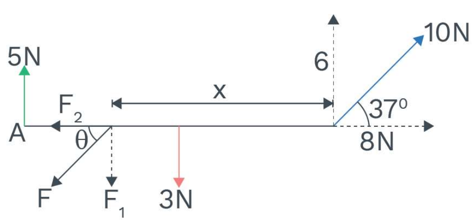
$x = 60$ cm

So 10 N force is used at 70 cm from point A to keep the body in equilibrium.

Ex. Calculate the point of application of force, when forces are acting on the rod as shown in figure.



Sol. Since the body is in equilibrium so we conclude $\vec{F}_{net} = 0$ and torque about every point is zero i.e., $\vec{\tau}_{net} = 0$.



Assume that we apply force F downward at A angle 'θ' from the horizontal, at x distance from B

From $\vec{F}_{net} = 0$

$\Rightarrow$ $F_{net\ x} = 0$ (gives)

$F_2 = 8$ Newton

From $F_{net\ y} = 0$; then $\Rightarrow 5 + 6 = F_1 + 3$

$\Rightarrow$ $F_1 = 8$ Newton

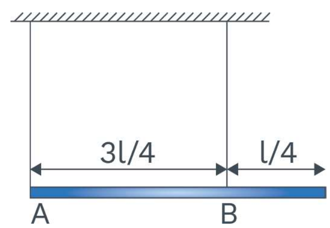
If body is in equilibrium condition then torque about point B is zero.

$\Rightarrow \quad 3 \times 5 + F_1 . x - 5 \times 10 = 0$

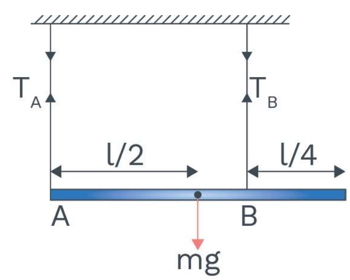
$\Rightarrow \quad 15 + 8x - 50 = 0$

$x = \frac{35}{9} \quad \Rightarrow x = 4.375 \text{ cm}$

Ex. A uniform rod length 'l', mass 'm' is hung from two strings of same length from a ceiling as shown in figure. calculate the tensions in the strings ?



Sol. Assume us assume that tension in right and left string is T_B and T_A respectively. Then



Rod is in equilibrium then

$\vec{F}_{net} = 0$ and $\vec{\tau}_{net} = 0$

From $\vec{F}_{net} = 0$

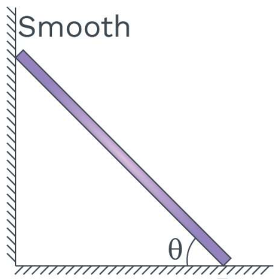
$mg = T_A + T_B \quad$ ...(i)

From $\vec{\tau}_{net} = 0$ about A

$mg\frac{\ell}{2} - \frac{3\ell}{4}T_B = 0$

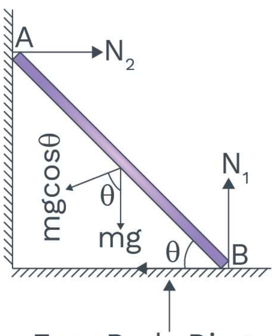
$\Rightarrow \quad T_B = \frac{2mg}{3}$

Ex. A stationary uniform rod of mass m, length l leans against a soft vertical wall making an angle θ with rough horizontal floor. Find out the frictional force & normal force that is exerted by the floor on the rod?



Sol. If rod is stationary so the linear acceleration and angular acceleration of rod is zero. i.e.,

$a_{cm} = 0; \alpha = 0$



Free Body Diagram

$$\left.\begin{aligned} N_2 &= f \\ N_1 &= mg \end{aligned}\right\} \qquad \because a_{cm} = 0$$

Torque about every point of the rod should also be zero

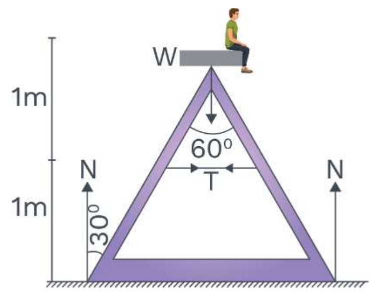
$\therefore \quad a = 0$

$$\tau_A = 0 \Rightarrow mg\cos\theta\frac{\ell}{2} + f\ell\sin\theta = N_1 \cos\theta \,.\, l$$

$$N_1 \cos\theta = \sin\theta f + \frac{mg\cos\theta}{2}$$

$$f = \frac{mg\cos\theta}{2\sin\theta} = \frac{mg\cot\theta}{2}$$

Ex. The ladder shown in diagram has negligible mass and rests on a frictionless floor. The crossbar attaches the two legs of the ladder at the halfway point. Angle between the two legs is 60°. A boy sitting on the ladder has a mass of 80 kg. Find out the contact force applied by the floor on each leg and the tension in the crossbar.



Sol. The forces acting on different elements are shown in diagram. Consider the vertical equilibrium of 'the ladder plus the boy' system. The forces applied on this system are its weight 80 g and the contact force N + N = 2 N due to the floor.

Thus, 2N = 80 g

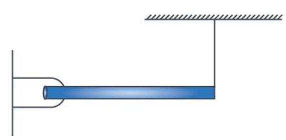
or N = 40 × 9.8 = 392 N

Now assume the equilibrium condition of the left leg of the ladder. Taking torques of the forces appling on it about the upper end,

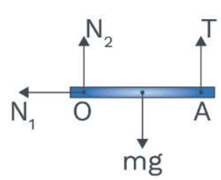
N(2m) tan 30° = T (1 m)

or $T = N\frac{2}{\sqrt{3}} = (392\ \text{N}) \times \frac{2}{\sqrt{3}} = 450\ \text{N}$

Ex. A thin plank of mass 'm' and length ℓ is pivoted at one side and it is held stationary in the horizontal position by means of a light thread as shown in the diagram then find out force on the pivot.



Sol. F.B.D. of the plank is shown in figure.



∴ Plank is in equilibrium condition

So, F_{net} and τ_{net} on the plant is zero.

(i) From $F_{net} = 0$

⇒ $F_{net\ x} = 0$ ⇒ $N_1 = 0$

Now, $F_{net\ y} = 0$

$\Rightarrow \quad N_2 + T = mg \qquad ...(i)$

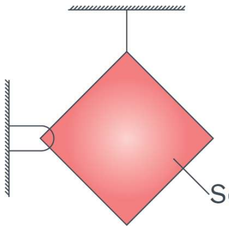
From $\tau_{net} = 0$

$\Rightarrow \quad \tau_{net}$ about point A is zero

So, $N_2 \cdot \ell = mg \cdot \frac{\ell}{2}$

$\Rightarrow \quad N_2 = \frac{mg}{2}$

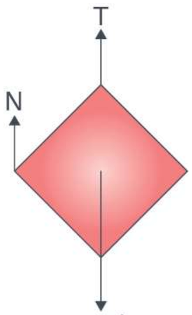
Ex. A plate (square) is hinged as shown in diagram and it is held stationary by means of a light thread as shown in diagram. Then find the force exerted by the hinge.



Sol. FBD

$\because$ Body is in equilibrium and T and mg force passing through one line so from

$\tau_{net} = 0, N = 0$



RELATION BETWEEN TORQUE AND ANGULAR ACCELERATION: -

The angular acceleration of a stiff body is directly proportional to the add of the components of torque along the axis of rotation. The proportionality constant 'α' is the inverse of the MOI about that axis,

or $\alpha = \frac{\Sigma\tau}{I}$

Concept Reminder

A rigid body is in mechanical equilibrium, if it is in translational and rotational equilibrium i.e., if total external force and torque acting on body is zero.

Therefore, for a rigid body we have the rotational analog of Newton's second law;

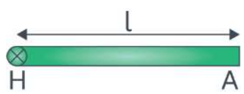
$\Sigma \tau = I\alpha$ ...(i)

Following two points are significint regarding the above mathematical equation.

(i) The above mathematical equation is valid only for stiff bodies. If the body is not stiff like a rotating tank of water, the angular acceleration 'α' is different for different particles.

(ii) The sum $\Sigma\tau$ in the above equation includes only the torques of the external forces, because all the internal torques add to zero.

Ex. A rod (uniform) of mass 'm' and length 'ℓ' can rotate in vertical plane about a smooth horizontal axis hinged at point H.



(i) Find out angular acceleration α of the rod just after it is free from initial horizontal position from rest?

(ii) Determine the acceleration (tangential and radial) of point A at this moment.

Sol. **(i)** $\tau_H = I_H\alpha$

$$mg \cdot \frac{\ell}{2} = \frac{m\ell^2}{2}\alpha \quad \Rightarrow \alpha = \frac{3g}{2\ell}$$

(ii) $$a_{\tau A} = \alpha\ell = \frac{3g}{2\ell} \cdot \ell = \frac{3g}{2}$$

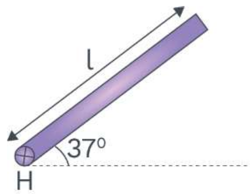
$$a_{CA} = \omega^2 r = 0 \cdot \ell = 0$$

($\because$ $\omega = 0$ just after release)

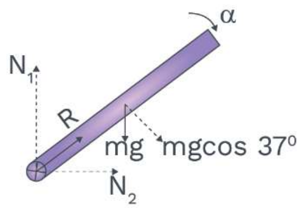
Ex. A rod (uniform) of mass 'm' and length ℓ hinged at point 'H' can rotate in vertical plane about a s'mooth horizontal axis. Find out force exerted by the hinge just after the rod is released from rest condition, from an initial position making angle of 37° with horizontal?

Rack your Brain

A solid cylinder of mass 2 kg and radius 4 cm is rotating about its axis at the rate of 3 rpm. Find the torque required to stop it after 2π revolutions.



Sol. Just After releasing rod at 37° from horizontal F.B.D. of plank



From $\tau_{net} = I\alpha$

τ about point A,

$$\tau_A = mg\cos 37°\frac{\ell}{2} = \frac{m\ell^2}{3}\cdot\alpha$$

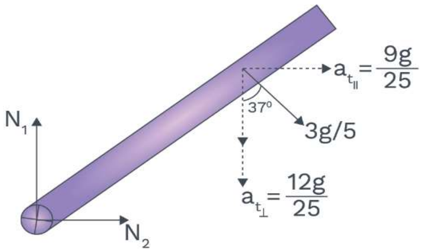
$\Rightarrow$ $\alpha = \frac{6g}{5\ell}$ rad / sec^2

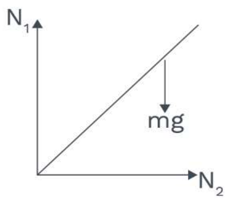
Now tangential acceleration of centre of mass

$$a_\tau = \alpha\cdot\frac{\ell}{2} = \frac{3g}{5}\ \text{m / s}^2$$

Just after release the rod $v_{cm} = 0 \Rightarrow a_r = 0$

Now resolving of 'a_t' in horizontal and vertical direction as shown in diagram





From F_{net} = ma in both vertical and horizontal direction

$$N_2 = m\left(\frac{9g}{25}\right) \Rightarrow N_1 = \frac{13\,mg}{25}$$

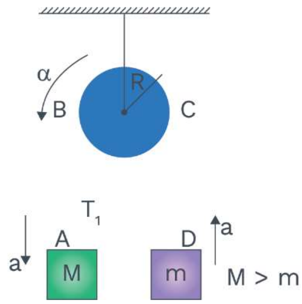
Now, $R = \sqrt{N_1^2 + N_2^2}$

$$R = \frac{mg\sqrt{10}}{5}$$

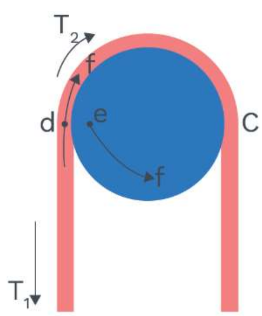
PULLEY BLOCK SYSTEM: -

If there is friction present between pulley and string and pulley have massive then tension is different on two sides of the pulley.

Reason: - To understand this concept we take a pulley block system as shown in figure.



Assume that tension made in part AB of the string is 'T_1' and block 'M' move downward. If friction is present between string and pulley then it opposes the relative slipping between pulley and string, take two-point e and f on pulley and string respectively. If friction is present then due to this, both points needs to move together. So friction force act on d and e in the direction as shown in diagram.



This friction force f acting on point d increases the tension T_1 by a small amount dT.

Then $T_1 = T_2 + dT$

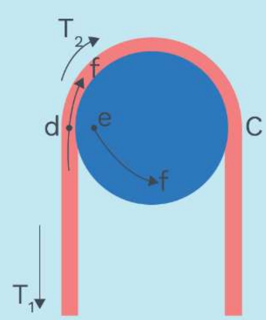
or we can say $T_2 = T_1 - f$

Concept Reminder

At the centre of gravity of a body, the total gravitational torque on the body is zero.

Concept Reminder

If there is friction between pulley and string and pulley have some mass, then tension is different on two sides of the pulley.

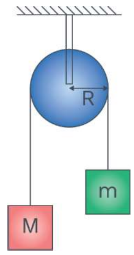


In this way the tension on two side of pulley is different If there is no relative slipping between pulley and string then

$$\alpha = \frac{a_t}{R} = \frac{a}{R}$$

Ex. The pulley shown in diagram has moment of inertia I about its axis and radius R. Find out the acceleration of the two blocks. If the string is light and does not slip on the pulley.

Sol. Assume the tension in the left string is 'T_1' and that in the right string is 'T_2'. Suppose the block of mass 'M' goes down with an acceleration 'a' and the other block moves up with the equal acceleration. This is also the tangential acceleration of the wheel rim as the string does not slip over the rim.



The angular acceleration of the wheel $\alpha = \frac{a}{R}$

The equations of motion for the mass 'M', the mass 'm' and the pulley are as follows:

$Mg - T_1 = Ma$ ...(i)

$T_2 - mg = ma$ ...(ii)

$T_1R - T_2R = I\alpha = \frac{Ia}{R}$ ...(iii)

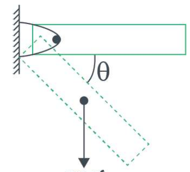
Substituting for T_2 and T_1 from equations (i) and (ii) in equation (iii)

$$[m(g - a) - m(g + a)]R = \frac{Ia}{R}$$

Solving, we get

$$a = \frac{(M - m)gR^2}{I + (M + m)R^2}$$

Ex. Rod of mass 'M' and length 'L' is released then α at this position?

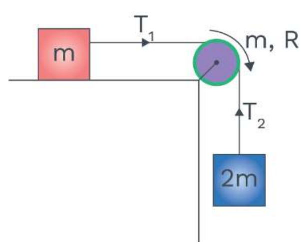


Sol. $\tau = Mg\frac{L}{2}\cos\theta = \alpha \times \frac{ML^2}{3}$

$$\alpha = \frac{Mg\frac{L}{2}\cos\theta}{M\frac{L^2}{3}} = \frac{3g\cos\theta}{2 \times L}$$

$$\alpha = \frac{3g\cos\theta}{2L}$$

Ex. If no slipping is present between pulley and string then find a & T_1 & T_2 as shown in figure.



Sol. $a = \frac{2mg}{2m + m + \frac{mR^2}{2} \times \frac{1}{R^2}} = \frac{4g}{7}\ m/s^2$

$T_1 = ma$ ...(i)

$T_1 = \frac{4mg}{7}N$

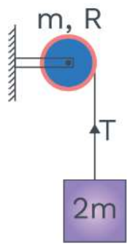
$2mg - T_2 = 2ma$ ...(ii)

$T_2 = 2mg - 2m\left(\frac{4g}{7}\right) = \frac{14mg - 8mg}{7}$

$T_2 = \frac{6mg}{7}N$

Ex. A string (cord) is wrapped around the disc as shown in figure when string gets tight find.

(i) a of block (ii) T in the cord.



Sol. $a = \dfrac{2mg}{2m + \dfrac{m}{2}} = \dfrac{2mg}{\dfrac{5m}{2}} = \dfrac{4mg}{5m} = \dfrac{4g}{5}$

$2mg - T = 2ma$

$T = 2mg - \dfrac{8mg}{5} = \dfrac{2mg}{5}$

FORCE COUPLE

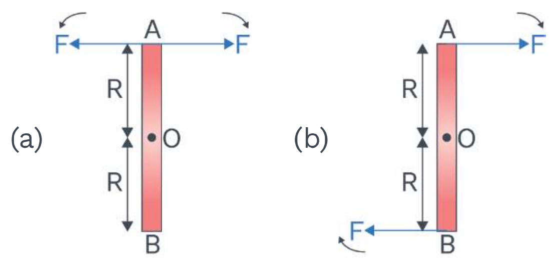
Force couple: A pair of equal and opposite forces with different line of action is known as couple or force couple. A couple produces rotation without translation.

- If on a system 2 forces of equal magnitude acting in opposite direction with different line of action produce force couple.
- This force couple tries to change only rotation state of the body.
- When we open the lid of bottle by turning it our fingers apply a couple to lid.

Concept Reminder

Torque acting on a system of particles or rigid body vanishes if either $\vec{F} = 0$ or $\vec{r} = 0$ or the angle between them is 0° or 180°.

Ex. Check whether the forces applied are acting as couple or not.



Sol. (a) $F_{net} = 0$, $(\tau_{net}) = 0$

Not a force couple (same line of force)

Mathematically, It is defined as

$|\tau_{net}| = F(\perp^r$ distance between line of force)

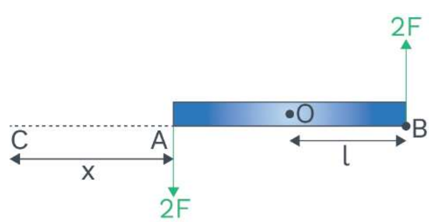
(b) $F_{net} = 0$, $\tau_{net} = -FR - FR$

$= -2FR$

It is a force couple (Different line of force)

Ex. **(i)** Check whether the forces produced are force couple or not.

(ii) Find torque about A, O, B, C



Sol. **(i)** Forces make couple as $F_{net} = 0$ & different Line of force.

(ii) $\tau_B = 0 + 2F(2\ell) = 4F\ell$

$\tau_A = 0 + 2F(2\ell) = 4Fl$

$\tau_0 = 2F(\ell) + 2F(\ell) = 4F\ell$

$\tau_C = -2F(x) + 2F(2\ell) + 2F(x) = 4F\ell$

$\tau_A = \tau_B = \tau_0 = \tau_C = 4F\ell$

Note: For rigid body-

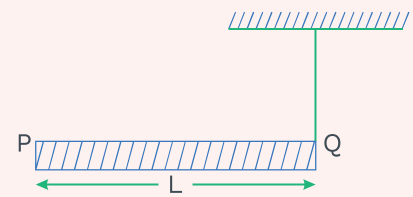
(a) It is possible to have non-zero torque with zero resultant force, eg: Force couple.

(b) It is also possible to have net force with zero resultant torque about COM. eg: Free fall of a body.

(c) If on a system net force is zero then calculation of torque is independent upon the choice of point about which it is to be calculated.

Rack your Brain

A rod PQ of mass M and length L is binged at end P. The rod is kept horizontally by a massless string tied to point Q as shown in figure. When string is cut, the initial angular acceleration of the rod is.



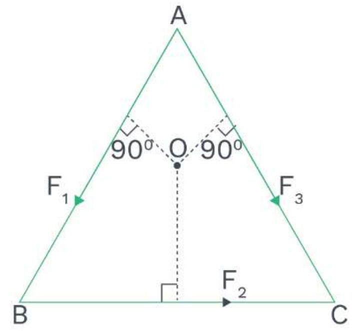
Concept Reminder

In rotational motion, moment of inertia and torque play the same role as mass and force respectively play in linear motion.

Concept Reminder

It is possible to have non-zero torque with zero resultant force, eg: Force couple.

Ex. O is the midpoint of an equilateral triangle ABC. F_1, F_2 and F_3 are three forces acting along the sides AB, BC and AC respectively as shown here. What should be the relation between F_1, F_2 and F_3, so that the total torque about O is zero.



Sol. Assume perpendicular distance of any arm from the centre of equilateral triangle is d.
Torque = (Force × Perpendicular distance of line of act of force from centre of triangle)
Total torque at point O,

$$\tau = F_1 d + F_2 d - F_3 d = 0$$

$\Rightarrow$ $F_3 = F_1 + F_2$

Ex. Two small childs weighing 10 kg and 15 kg are trying to balance a light rigid seesaw of total length 5.0 m, with the fulcrum at the centre. If one of the childs is sitting at an end.

(i) Where should the other sit ?
(ii) Find normal force by the fulcrum.

Sol. (i) It is show that the 10 kg child should sit at the end and the 15 kg child should sit closer to the centre. Assume his distance from the centre is x.
As the childs are in equilibrium, the normal force between a child and the seesaw equals the weight of that child. Considering the rotating equilibrium of the seesaw, the torque of the forces acting on it should add to zero. The forces are

(a) (15 kg) g downward by the 15 kg child,
(b) (10 kg) g downward by the 10 kg child,
(c) the normal force by the fulcrum.

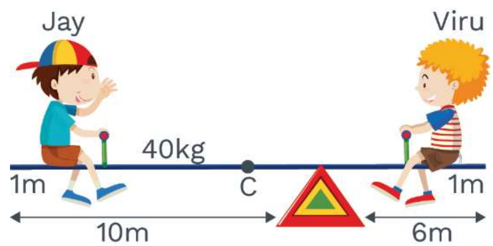
Taking torques about the fulcrum,

$(15\text{ kg})g\, x = (10\text{ kg})g\,(2.5\text{ m})$

or $x = 1.7\text{ m}$

(ii) $F_H = 150 + 100 = 250\text{ N}$

Ex. Jay and viru are trying to balance a rigid seesaw of total length 16 m, if mass of jay is 20kg and weight of seesaw is 400N then what should be mass of viru in order to balance it.



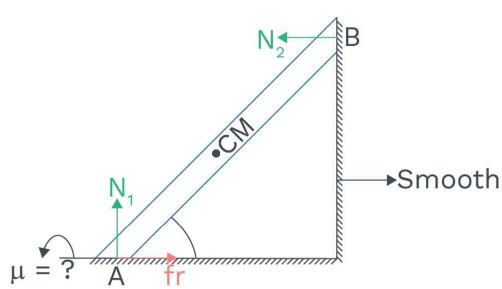
Sol. $\tau_{net} = 200 \times 9 + 400 \times 2 - mg(5) = 0$ (about fulcrum)

$= 180 + 80 - 5\text{ m} = 0$

$5\text{ m} = 260$

$m = 52\text{ kg}$

Ex. A ladder is at rest as shown in figure. what will be the minimum value of m so that ladder will be in equilibrium.

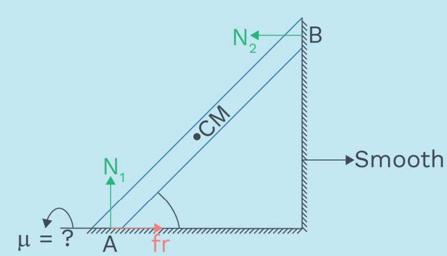


Sol. $-mg\left(\frac{\ell}{2}\cos\theta\right) + N_2\ell\sin\theta = 0$ (torque about A)

$$N_2 = \frac{mg\cos\theta}{2\sin\theta}$$

Concept Reminder

The minimum value of μ so that ladder will be in equilibrium.

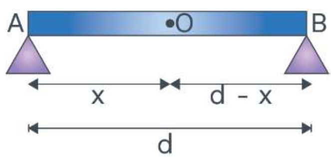


$$\mu_{min} = \frac{\cot\theta}{2}$$

$$\mu\, mg \geq \frac{mg \cot\theta}{2}$$

$$\mu \geq \frac{\cot\theta}{2} \Rightarrow \mu_{min} = \frac{\cot\theta}{2}$$

Ex. A rod of weight 'W' is supported by two parallel knife edges A and B and is in equilibrium in a horizontal position. The knives are at a distance 'd' from each other. The centre of mass (COM) of the rod is at distance x from A. The normal reaction on A is.



Sol. $N_A + N_B = W$

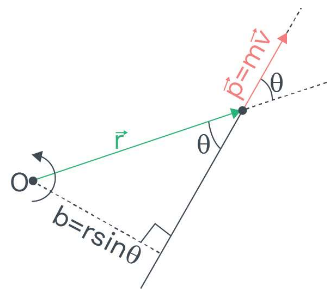
$\vec{\tau}_{net} = W \times (d - x) - N_A(d) = 0$ (about B)

$Wd - Wx = N_A d$

$$N_A = \frac{W(d-x)}{d}$$

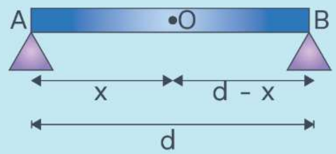
ANGULAR MOMENTUM (MOMENT OF LINEAR MOMENTUM)

Angular momentum of a block about a given axis is the product of its linear momentum and perpendicular distance of the line of action of linear momentum vector from the axis of rotation.



Angular momentum = Linear momentum (P) × Perpendicular distance of line of action of momentum from the axis of rotation ($r_\perp$).

Concept Reminder



$$N_A = \frac{W(d-x)}{d}$$

KEY POINTS

- Angular momentum

Definitions

Angular momentum of a body about a given axis is the product of its linear momentum and perpendicular distance of line of action of linear momentum vector from the axis of rotation.

$$L = mv \times r \sin\theta$$

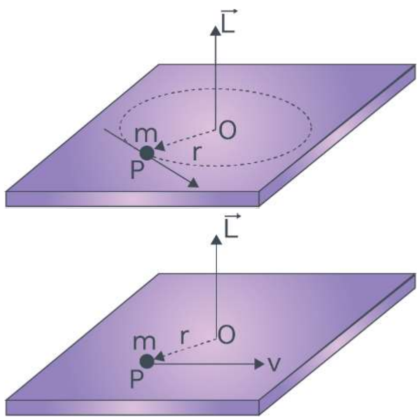
$$\Rightarrow \quad \vec{L} = \vec{r} \times \vec{p}$$

Here $\vec{L}$ is the angular momentum of a moving particle about point O, $\vec{p}$ is the linear momentum of the particle and $\vec{r}$ is the position vector of the particle regarding the point.

Unit: S.I. J-sec or kg-m^2/sec

Dimensions: $[ML^2T^{-1}]$

Angular momentum is an axial vector.



- As torque $(\vec{r} \times \vec{F})$ is defined as the 'moment of force', Angular momentum is also defined as moment of linear momentum.
- In cartesian coordinates angular momentum:

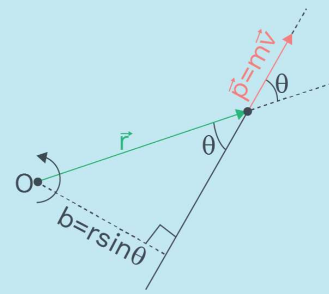
$$\vec{L} = m \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ v_x & v_y & v_z \end{vmatrix}$$

$$\vec{L} = (\vec{r} \times \vec{p}) = m(\vec{r} \times \vec{v}) \quad [\because \vec{p} = m\vec{v}]$$

$$\vec{L} = m[(x\hat{i} + y\hat{j} + z\hat{k}) \times (v_x\hat{i} + v_y\hat{j} + v_z\hat{k})]$$

i.e., $\vec{L} = m[\hat{i}(yv_z - zv_y) - \hat{j}(xv_z - zv_x) + \hat{k}(xv_y - yv_x)]$

Concept Reminder

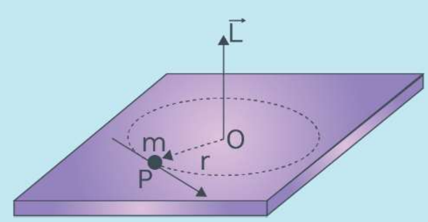


$$\vec{L} = \vec{r} \times \vec{p}$$

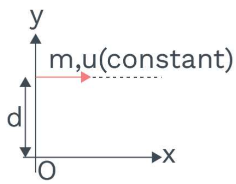
Here $\vec{L}$ is the angular momentum of a moving particle about point O, $\vec{p}$ is the linear momentum of the particle and $\vec{r}$ is the position vector of the particle regarding the point.

Concept Reminder

Angular momentum is an axial vector.



Ex. A ball mass 'm' starts moving from point (0, d) with a constant velocity $u\hat{i}$. Find out its angular momentum about the origin at this moment what will be the answer at the later time?



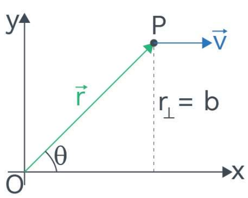
Sol. $\vec{L} = -mdu\hat{k}$ and it will remain same.

Ex. A ball of mass 'm' is moving along the line y = b, z = 0 with speed v (constant). State whether the angular momentum of ball about origin is decreasing, increasing or constant.

Sol. $|\vec{L}| = mvr\sin\theta$

$= mvr_{\perp} = mvb$

$\therefore$ $|\vec{L}|$ = constant as v, b and m all are constants.



Direction of $\vec{r} \times \vec{v}$ also remains the same. Thus, the angular momentum of particle about origin remains constant with due course of time.

Note: In this problem $|\vec{r}|$ is increasing, θ is decreasing but r sin θ, i.e., b remains constant. Hence, the angular momentum remains constant.

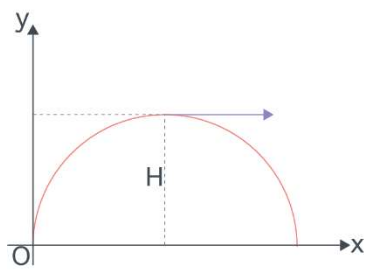
Concept Reminder

In Cartesian coordinate:

$\vec{L} = m[\hat{i}(yv_z - zv_y) - \hat{j}(xv_z - zv_x) + \hat{k}(xv_y - yv_x)]$

Ex. A ball of mass 'm' is projected with velocity v at an angle θ with the horizontal. Find out its angular momentum about the point of the projection when it is at the highest point of its trajectory.

Sol. At highest point it has only horizontal component of velocity $v_x = v\cos\theta$. Length of the right angles to the horizontal velocity from O is the maximum height, where



$$H_{max} = \frac{v^2 \sin^2\theta}{2g}$$

$$\Rightarrow \quad \text{Angular momentum } L = \frac{mv^3 \sin^2\theta \cos\theta}{2g}$$

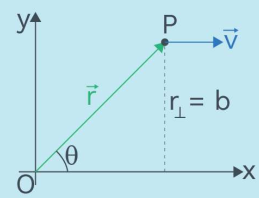
Angular Momentum of a stiff body rotating about a fixed axis:-

Assume a particle P of mass m is going in a circle of radii 'r' and at some instant speed of the particle is 'v'. For find out the angular momentum of the particle about the axis of rotation, the origin may be chosen everywhere on the axis. We select at the centre of the circle. In this case $\vec{r}$ and $\vec{P}$ are at right angles to each other and $\vec{r} \times \vec{P}$ is along the axis.

Therefore, component of $\vec{r} \times \vec{P}$ along the axis is mvr itself. The angular momentum of the whole stiff body about AB is the add of components of all particles, i.e.,

$$L = \sum m_i r_i v_i$$

Concept Reminder



In this case, angular momentum of particle about origin remains constant with due course of time.

Concept Reminder

Angular momentum of projectile about point of projection when it is at topmost point is,

$$L = \frac{mv^3 \sin^2\theta \cos\theta}{2g}$$

Here, $v_i = r_i\omega$

$$\therefore \quad L = \sum_i m_i r_i^2 \omega_i$$

or $L = \omega \sum_i m_i r_i^2$

or $L = I\omega$

Therefore, I is the MOI of the stiff body about AB.

Note: Angular momentum L about axis is the component of '$\vec{\omega}$' along the axis. Almost of the cases angular momentum about axis is $I\omega$.

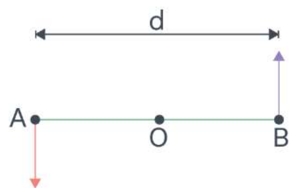
Concept Reminder

Angular Momentum of a rigid body rotating about a fixed axis:

$L = I\omega$

Ex. Two small balls 'A' and 'B', each of mass m, are attached tightly to the ends of a light rod of length d. The structure spins about the perpendicular bisector of the rod at an angular speed ω. Find out the angular momentum of the individual balls and of the system about the axis of rotation.

Sol. Consider the situation shown in figure. The velocity of the ball 'A' with respect to the centre O is $v = \frac{\omega d}{2}$



The angular momentum L of the ball w.r.t. the axis is

$$L_1 = mvr = m\left(\frac{\omega d}{2}\right)\left(\frac{d}{2}\right) = \frac{1}{4}m\omega d^2$$

The equal the angular momentum L_2 of the second ball. The angular momentum of the system is same to sum of these two angular momenta i.e., $L = 1/2\ m\omega d^2$.

CONSERVATION OF ANGULAR MOMENTUM:

The time rate of exchange of angular momentum of a particle about some reference point in an inertial frame of reference is same to the net torques acting on it.

or $\vec{\tau}_{net} = \dfrac{d\vec{L}}{dt}$ ...(i)

Now, assume that $\vec{\tau}_{net} = 0$, then $\dfrac{d\vec{L}}{dt} = 0$

So, that $\vec{L}$ = constant.

When the external resultant torque acting on a system is zero, the net vector angular momentum of the system remains constant. This is the assumption of the conservation of angular momentum.

For a stiff body rotating about an axis (the z-axis, say) that is fixed in an inertial reference frame, we have

$L_z = I\omega$

It is possible for the MOI I of a rotating body to change by relocation of its parts. If no net external torque acts, then 'L_z' must remains constant and if MOI 'I' does change, there must be a compensating change in ω. The principle of angular momentum conservation in this case is stated.

$I\omega$ = constant.

Ex. A wheel of MOI I and radius R is rotating about its axis at an angular speed ω_0. It picks up a stationary ball of mass 'm' at its edge. Find out the new angular speed of the wheel.

Sol. Total external torque on the system is zero. Therefore, angular momentum will remain conserved.

Thus, $I_1\omega_1 = I_2\omega_2$

or $\omega_2 = \dfrac{I_1\omega_1}{I_2}$

Definitions

When the resultant external torque acting on a system is zero, the total vector angular momentum of the system remains constant. This is the principle of the conservation of angular momentum.

Concept Reminder

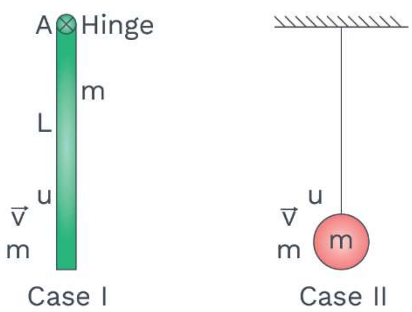
$\vec{\tau}_{net} = \dfrac{d\vec{L}}{dt}$

if $\vec{\tau}_{net} = 0$, then $\vec{L}$ = constant.

Here, $I_1 = I$, $\omega_1 = \omega_0$, $I_2 = I + mR^2$

$$\therefore \quad \omega_2 = \frac{I\omega_0}{I + mR^2}$$

Note:



Comments on Linear Momentum:

In case-I: Linear momentum is not conserved just before and just after collision because during collision hinge force act as an external force.

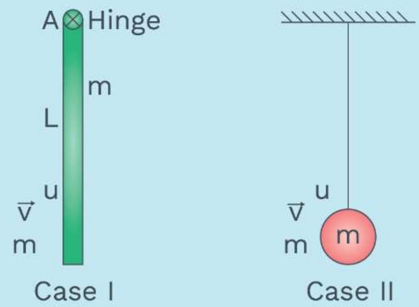
In case-II: Linear momentum is conserved just before and just after collision because no external force on the string.

Comments on Angular Momentum:

In case-I: Hinge force acts at an external force during collision but except point A all the other reference point given $\tau_{net} \neq 0$. So, conservation of angular momentum is only for point A.

In case-II: angular momentum is conserved at all points in the world.

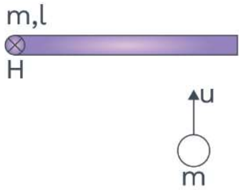
Concept Reminder



In case-I: Linear momentum is not conserved.

In case-II: Linear momentum is conserved.

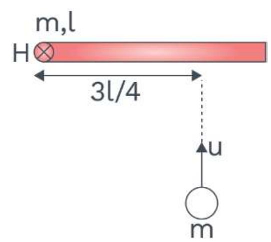
Ex. A rod (uniform) of mass 'm' and length ℓ can rotate freely on a soft horizontal plane about a vertical axis hinged at point 'H'. A point mass having same mass 'm' coming with an initial speed 'u' perpendicular to rod, strikes rod in-elastically at its free end. Calculate the angular velocity of the rod just after collision?



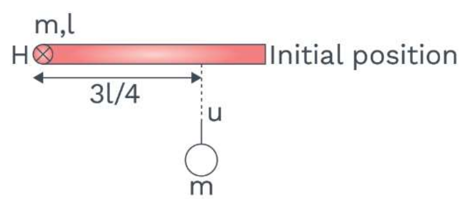
Sol. Conservation of angular momentum is about H because no external force is present in horizontal plane which is producing torque about H.

$$mu\ell = \left(\frac{m\ell^2}{3} + m\ell^2\right)\omega \quad \Rightarrow \omega = \frac{3u}{4\ell}$$

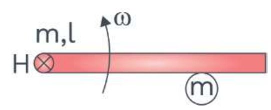
Ex. A rod (uniform) of mass 'm' and length 'l' can rotate freely on a soft horizontal plane about the vertical axis hinged at point 'H'. A point mass having same mass 'm' coming with an initial speed 'u' perpendicular to the rod, strikes rod and sticks to it at a distance of $3\ell/4$ from hinge point. Calculate the angular velocity of the rod just after collision?



Sol. From conservation of angular momentum about H, initial angular momentum = final angular momentum



$$m \cdot u\frac{3\ell}{4} = m\left(\frac{3\ell}{4}\right)^2\omega + \frac{m\ell^2}{3}\omega$$



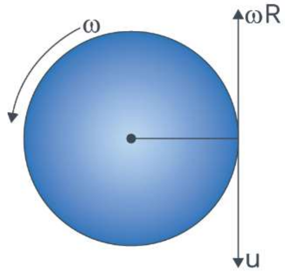
$$\Rightarrow \quad \frac{3mu\ell}{4} = m\ell^2\left[\frac{1}{3} + \frac{9}{16}\right]\omega$$

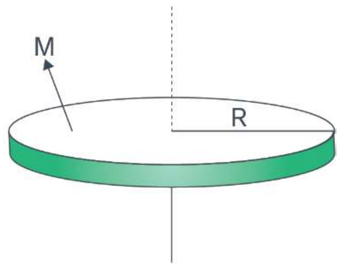
$$\frac{3u}{4\ell} = \left[\frac{16+27}{48}\right]\omega$$

$$\Rightarrow \quad \omega = \frac{36u}{43\ell}$$

Ex. A baby of mass 'm' stands at the edge of a circular platform of radii R and moment of inertia. A platform is at rest initially case. But the platform rotate when the baby jumps off from the platform tangentially with velocity 'u' with respect to platform. Determine the angular velocity of the platform.

Sol. Let the angular velocity of platform is ω. Then the velocity of baby with respect to ground v.





$v_{mD} = v_{mG} - V_{DG}$

$u = v_m + \omega R$

$v_m = u - \omega R$

Now from angular momentum conservation

$L_i = L_f$

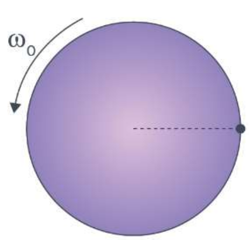
$0 = mv_mR - I\omega$

$\Rightarrow \quad I\omega = m(u - \omega R) \,.\, R$

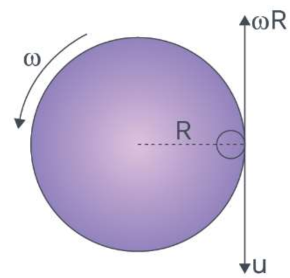
$$\Rightarrow \quad \omega = \frac{muR}{I + mR^2}$$

Ex. Consider the situation of previous example. If the platform is rotating initially with angular velocity ω_0 and then baby jumps off tangentially. Find out the new angular velocity of the platform.

Sol. Let the angular velocity of platform after jumps off the mass is ω. Then velocity of baby.



Initially



$v_m = v_{mp} + v_p$

$v_m = u - \omega R$

From angular momentum conservation

$$(I + mR^2)\omega_0 = I\omega - m(u - \omega R)R$$

$$I\omega_0 + mR^2\omega_0 = I\omega - muR + m\omega R^2$$

$$\Rightarrow \quad \omega = \frac{(I + mR^2)\omega_0 + muR}{(I + mR^2)}$$

ANGULAR IMPULSE:

The angular impulse L of a torque in a given time interval is known as $\int_{t_1}^{t_2} \vec{\tau}\, dt$.

Here, '$\vec{\tau}$' is the resultant torque acting on the body. Therefore,

$$\vec{\tau} = \frac{d\vec{L}}{dt} \qquad \therefore \vec{\tau}\, dt = d\vec{L}$$

or $\int_{t_1}^{t_2} \vec{\tau}\, dt$ = angular impulse $\vec{L}_2 - \vec{L}_1$

Therefore, the angular impulse of the resultant torque is same to the change in angular momentum. Assume few examples based on the angular impulse.

Conservation of Angular Momentum

Newton's 2nd law in rotation:

Where $\vec{\tau}$ and $\vec{L}$ are about the same axis.

Angular momentum of a system or a particle remains constant if τ_{ext} = 0 about axis of rotation.

Even if the net angular momentum is not constant, one of its component about an axis remains constant if the component of torque about that axis is zero.

Impulse of torque: $\int \tau dt = \Delta J$

$\Delta J \rightarrow$ Charge in angular momentum.

- Suppose a ball is tied at one end of a cord whose other end passes through a vertical hollow tube. The tube is held in one hand and the cord in the

Definitions

The angular impulse of a torque in a given time interval is defined as $\int_{t_1}^{t_2} \vec{\tau}\, dt$.

Concept Reminder

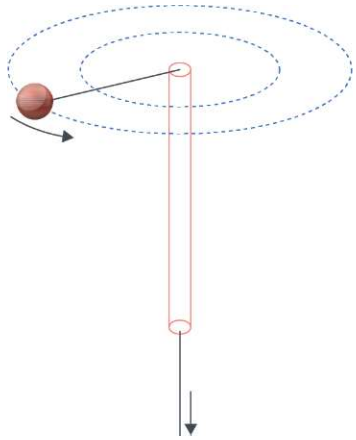
- $\vec{\tau} = \frac{d\vec{L}}{dt} \qquad \therefore \vec{\tau}\, dt = d\vec{L}$.
- $\int_{t_1}^{t_2} \vec{\tau}\, dt$ = angular impulse $\vec{L}_2 - \vec{L}_1$

KEY POINTS

- Angular impulse
- Conservation of angular momentum

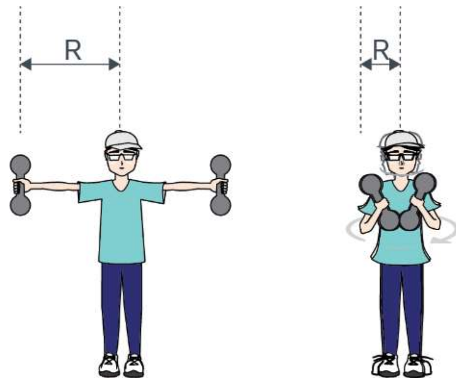
other. The ball is set into rotation in a horizontal circle. If the cord is pulled down, shortening the radius of the circular path of the ball, the ball rotates faster than before. The cause is that by shortening the radius of the circle, the moment of inertia of the ball about the axis of rotation decreases.

Hence, by the law of conservation of angular momentum, the angular velocity of the ball about the axis of rotation increases.



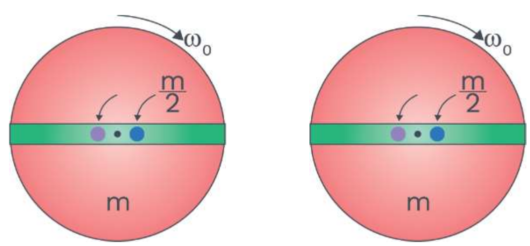
- If when a diver jumps into water from a some height, he doesn't keep his body straight but pulls in his legs and arms toward: the centre of his body. On doing so, the MOI 'I' of his body decreases. But since the angular momentum $I\omega$ remains constant, his angular velocity ω correspondingly increases. Therefore, during jumping he can rotate his body in the air.

- A man with his arms outstretched and holding the heavy dumb bells in each hand, is standing at centre of a rotating table. When the man pulls in his arms, the speed of rotation of the table increases. The reason is that on pulling in the arms, the distance 'R' of the dumbells from the axis of rotation decreases and so moment of inertia of the man decreases. Therefore, by angular momentum conservation, the angular velocity increases.



In the same way, the ballet dancer and ice skater decrease or increase the angular velocity of spin about the vertical axis by pulling or extending out their limbs.

Ex. A uniform disc of mass 'm' and radii R is free to rotate in horizontal plane about a vertical soft fixed axis going through its centre. There is a soft groove along the diameter of the disc and two small balls of mass 'm/2' each are placed in it on either side of the centre of the disc as shown in figure. The disc is given initial angular velocity ω and released, then-



(i) The angular speed of the disc when the balls reach the end of the disc is.

(ii) The speed of each ball relative to ground just after they leave the disc is.

(iii) The net work done by the forces exerted by disc on one of ball for the duration ball remains on disc is.

Sol. (i) Let the angular speed of disc when the balls reach the end be ω. From conservation of angular momentum

$$\frac{1}{2}mR^2\omega_0 = \frac{1}{2}mR^2\omega + \frac{m}{2}R^2\omega + \frac{m}{2}R^2\omega$$

or $\omega = \frac{\omega_0}{3}$

(ii) The angular speed of the disc just after the balls leave the disc is $\omega = \frac{\omega_0}{3}$

Assume the speed of each ball just after they leave the disc be v. From conservation of energy.

$$\frac{1}{2}\left(\frac{1}{2}mR^2\right)\omega_0^2 = \frac{1}{2}\left(\frac{1}{2}mR^2\right)\omega^2 + \frac{1}{2}\left(\frac{m}{2}\right)v^2 + \frac{1}{2}\left(\frac{m}{2}\right)v^2$$

Solving we get

$$v = \frac{2R\omega_0}{3}$$

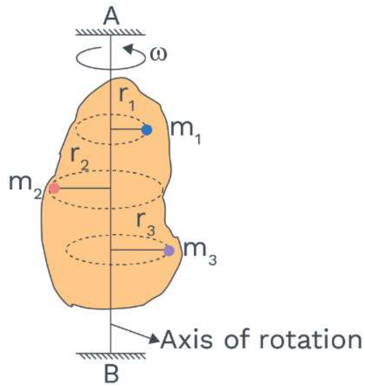
Note: $v = \sqrt{(\omega R)^2 + v_r^2}$, v_r = radian velocity of the ball.

(iii) Work done by all forces equal

$$K_f - K_i = \frac{1}{2}\left(\frac{m}{2}\right)v^2 = \frac{mR^2\omega^2}{9}$$

KINETIC ENERGY OF AN OBJECT IN PURE ROTATION:

For i^th particle



$$(KE)_i = \frac{1}{2}m_i v_i^2$$

$$(KE)_{body} = \Sigma\frac{1}{2}m_i(\omega R_i)^2$$

$$= \frac{1}{2}\omega^2\Sigma m_i R_i^2$$

$$\boxed{(KE)_{body} = \frac{1}{2}I\omega^2}$$

Rack your Brain

A circular platform is mounted on a frictionless vertical axle. Its radius R = 2m and its moment of inertia about the axle is 200 kg m^2. It is initially at rest. A 50 kg man stands on the edge of the platform and begins to walk along the edge at the speed of 1 ms^{-1} relative to ground. Find the time taken by the man to complete one revolution.

Relation between $\vec{L}$ and KE :

$$(KE) = \frac{1}{2}I\omega^2 \times \frac{I}{I} = \frac{1}{2}\frac{(I\omega)^2}{I} \quad \text{(If L = constant)}$$

$$\boxed{(KE) = \frac{L^2}{2I}}$$

$$\boxed{KE = \frac{\text{constant}}{I}}$$

Ex. If L is increased by 10% then what will be the % change in KE.

Sol. $K = \frac{L^2}{2I}$

$$L_2 = L_1 + \frac{10}{100}L_1 = 1.1\,L_1$$

$$\frac{K_1}{K_2} = \frac{\frac{L_1^2}{2I}}{\frac{(1.1\,L_1)^2}{2}} \Rightarrow \frac{K_1}{K_2} = \frac{L_1^2}{1.21\,L_1^2}$$

$K_2 = 1.21\,K_1$

Percentage change

$$= \frac{f-i}{i} \times 100 = \frac{1.21K_1 - K_1}{K_1} \times 100$$

Percentage change = 21%.

Ex. KE is reduced by 64% of it's previous value then % change in L will be.

Sol. $K_2 = K_1 - \frac{64}{100}K_1 = 0.36K_1$

$$\frac{1}{0.36} = \frac{L_1^2}{L_2^2}$$

$L_2^2 = L_1^2(0.36)$

$L_2 = 0.6\,L_1$

$$\text{Percentage change} = \frac{0.6L_1 - L_1}{L_1} \times 100$$

= 0.4 × 100 = 40% (decrease)

Concept Reminder

In pure rotation motion,

$$W_{rot} = \frac{1}{2}I(\omega_f^2 - \omega_i^2)$$

Concept Reminder

Rotational work done by a force about a fixed axis is defined as $W_{rot} = \int \vec{\tau} \cdot d\vec{\theta}$.

Concept Reminder

In rotational motion,

$$KE = \frac{1}{2}I\omega^2 = \frac{L^2}{2I}$$

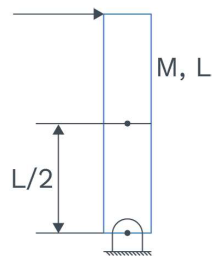
WORK, ENERGY THEORM IN ROTATION

Work Energy Theorem

$W_{all} = KE_{Final} - KE_{Initial}$

Here in rotation $W_{rot} = \frac{1}{2}I(\omega_f^2 - \omega_i^2)$

Ex. A gentle push occurs, then find angular speed of rod, when it becomes horizontal.



Sol. $W_H + W_{mg} = \frac{1}{2}I(\omega_f^2 - \omega_i^2)$ $(\because W_H = 0)$

$$Mg\frac{L}{2} = \frac{1}{2}I(\omega)^2$$

$$Mg\frac{L}{2} = \frac{ML^2}{3\times 2}(\omega^2)$$

$$\boxed{\omega = \sqrt{\frac{3g}{L}}}$$

Ex. System is released from rest. When rod becomes vertical. Find (i) angular speed of rod (ii) Velocity of particle B. (Given: mass of rod is 4 m).

Sol. (i) $W_{all} = \Delta K$

$$(W_{mg})_{rod} + (W_{mg})_{particles} = \frac{1}{2}I(\omega_f)^2 - 0$$

$$\frac{4\,mg\ell}{2} + \frac{mg\ell}{4} + \frac{mg3\ell}{4} + mg\ell = \left(\frac{142m\ell^2}{48}\right)\omega^2$$

$$\boxed{\omega = \sqrt{\frac{192\,g}{71\ell}}}$$

(ii) $v_B = \sqrt{\frac{192\,g}{71\ell}} \times \frac{3\,\ell}{4}$

$\vec{\tau}_{net} = \frac{d\vec{L}_{sys}}{dt} \rightarrow$ Rotational form of Newton's 2nd law.

If $\vec{\tau}_{net} = 0$ (about any axis or point)

$$d\vec{L}_{sys} = 0$$

$$d\vec{L}_{sys} = \text{constant}$$

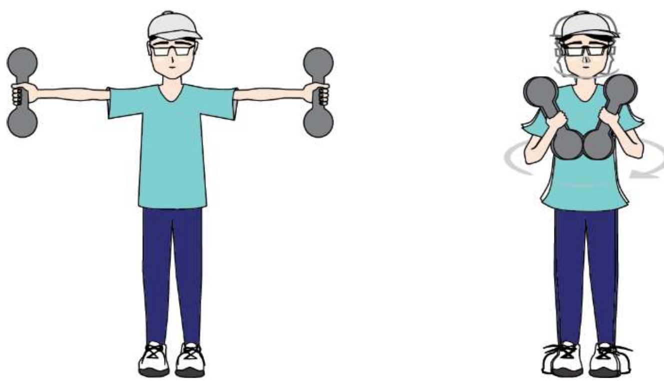
$$I\omega = \text{constant}$$

$$\omega \propto \frac{1}{I}; \quad \vec{L}_i = \vec{L}_f$$

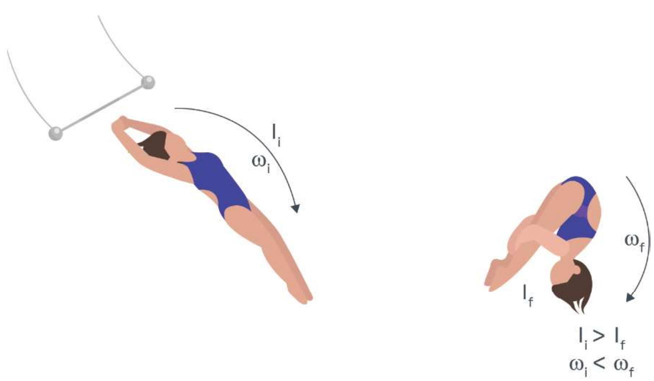
If a system is isolated from its surrounding i.e. any internal interaction between part of the system can not alter its total angular momentum.

Examples Based on Conservation of Angular Momentum:

- If a man skating on ice folds his arms then his MOI decreases and 'ω' increases.



- A diver jumping from a some height folds his arms and legs (MOI 'I' decrease) in order to increase number of rotation in air by increasing 'w'.



- If a person moves towards the centre of rotating platform then 'I' decrease and 'ω' increase.

Note:

- The angular speed of a planet around the sun increases when it comes near the sun.
- The speed of the inside layers of the whirlwind in a tornado is alarmingly high.
- If external torque on the system is zero, then the angular momentum is conserved. However the rotational kinetic energy is not conserved.

$$I_1\omega_1 = I_2\omega_2$$

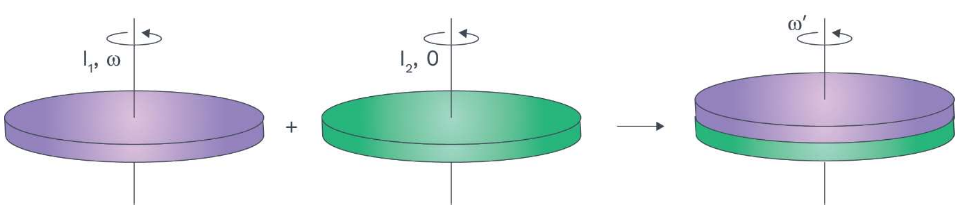
$$\Rightarrow I_1 \times \frac{1}{2} I_1 \omega_1^2 = I_2 \times \frac{1}{2} I_2 \omega_2^2$$

$$\Rightarrow I_1 K_{r_1} = I_2 K_{r_2}$$

If $I_1 > I_2$ then $K_{r_1} < K_{r_2}$

So, if the moment of inertia decreases, the rotational kinetic energy increases and vice versa.

Ex. A disc is rotating about Geometrical axis with MOI I_1 and another disc is at rest whose MOI about the same axis is I_2. If the other one is placed on the first disc co-axially then new ω of the system will be.



Sol. $I_1\omega + 0 = (I_1 + I_2)\omega'$

$$\omega' = \left(\frac{I_1}{I_1 + I_2}\right)\omega$$

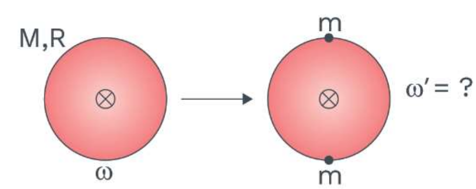
Ex. A dancer is standing on a heavy rotating platform. If she stretch her arms then angular momentum of dancer.

Sol. $(\vec{L}_P + \vec{L}_D)_i = (\vec{L}_P + \vec{L}_D)_f$

$I_P \omega_i + I_D \omega_i = I_P \omega_f + I_D \omega_f$

In system I increases so, new ω decreases. It means L of dancer will increases.

Ex. If two small particles of mass m sticks on a ring of mass M and radius R as shown in figure new angular speed of system will be ?



Sol. $I = MR^2$, $\quad I = MR^2 + 2mR^2$

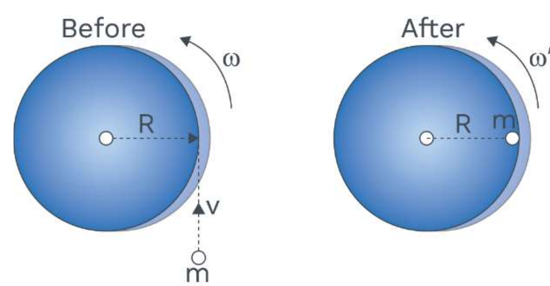
$I\omega = I'\omega'$

$$\omega' = \left(\frac{M}{M+2m}\right)\omega$$

Ex. A cylinder (solid) of mass 'M' and radius 'R' is rotating along its axis with angular velocity ω without friction. A particle of mass 'm' moving with velocity v collides against the cylinder and sticks to its rim. After the impact calculate angular velocity of cylinder.

Sol. Initial angular momentum of cylinder = $I\omega$

Initial angular momentum of particle = mvR



Before collision the total angular momentum

$L_1 = I\omega + mvR$

After collision the total angular momentum

$L_2 = (I + mR^2)\omega'$

$L_1 = L_2$

$\Rightarrow \quad (I + mR^2)\omega' = I\omega + mvR$

New angular velocity $\omega' = \dfrac{I\omega + mvR}{I + mR^2}$

Note: Initial kinetic energy of the system

$$= \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2$$

Final kinetic energy of the system

$$= \frac{1}{2}(I + mR^2)\omega'^2$$

Ex. Two discs of equal moment of inertia rotating about their regular axis going through centre and perpendicular to plane of disc with angular speeds ω_1 and ω_2. They are take into contact face to face coinciding the axis of rotation. then find-

(i) Angular speed of the system

(ii) KE_i of the system

(iii) KE_f of the system

(iv) Loss in KE of the system

Sol. (i) $(\vec{L}_i)_{D_1} + (\vec{L}_i)_{D_2} = (\vec{L}_f)_{D_2} + (\vec{L}_f)_{D_1}$

$$I\omega_1 + I\omega_2 = (I + I)\omega$$

$$\omega = \frac{I(\omega_1 + \omega_2)}{2I}$$

$$\omega = \frac{\omega_1 + \omega_2}{2}$$

(ii) $KE_i = \frac{1}{2}I\omega_1^2 + \frac{1}{2}I\omega_2^2 = \frac{1}{2}I(\omega_1^2 + \omega_2^2)$

(iii) $KE_f = \frac{1}{2}(I + I)\omega^2$

$$= \frac{1}{2} \times 2I \times \left(\frac{\omega_1 + \omega_2}{2}\right)^2 = \frac{I(\omega_1 + \omega_2)^2}{4}$$

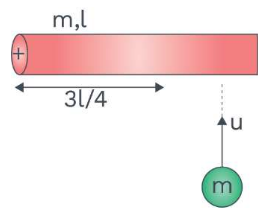
(iv) $KE_{loss} = KE_i - KE_f$

$$= \frac{I}{2}(\omega_1^2 + \omega_2^2) - \frac{I}{4}(\omega_1 + \omega_2)^2$$

$$KE_{loss} = \frac{I}{4}(\omega_1 - \omega_2)^2$$

Ex. A rod (uniform) of mass m and length l can rotate freely on a soft horizontal plane about a vertical axis hinged at point 'H'. A point mass having equal mass m coming with an initial speed 'u' perpendicular to rod, strikes rod and sticks to it at the distance of 3l/4 from hinge point. Find the angular speed of rod just after collision?

Sol. Angular Momentum about hinge



$$L_i = L_f$$

$$mu\left(\frac{3\ell}{4}\right) = \left(\frac{m\ell^2}{3} + m\left(\frac{3\ell}{4}\right)^2\right)\omega$$

$$\omega = \frac{36u}{43\ell}$$

Note: For Fixed Axis Rotation:

- If question is asking about acceleration then use $\vec{\tau}_{net} = I\vec{\alpha}$.
- If question is asking about angular speed (ω) then first through is WET (Work Energy Theorem)
- If question discusses about relation between Iv/sw then first thought is COAM (conservation of angular momentum).

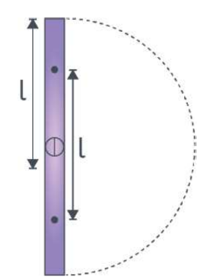
Ex. A rod (uniform) of mass m and length ℓ is kept vertical with the lower side clamped. It is slightly pushed to let it drop down under gravity. Find out its angular speed when the rod is going through its lowest position. Neglect any friction at the clamp. Calculate the linear speed of the free end at this instant?

Sol. If the rod reaches its lowest position, the centre of mass is lowered by a distance ℓ. Its gravitational potential energy is decreased by $mg\ell$. As no energy is lost due to friction, this should be same to the increase in the kinetic energy. If the rotation occurs about the horizontal axis through the clamped end, the MOI is $I = ml^2/3$. Thus,

$$\frac{1}{2}I\omega^2 = mg\ell$$

$$\Rightarrow \quad \frac{1}{2}\left(\frac{m\ell^2}{3}\right)\omega^2 = mg\ell$$

or $$\omega = \sqrt{\frac{6g}{\ell}}$$



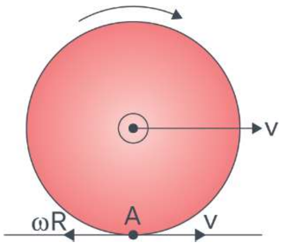
The linear speed of the free side is

$$v = \ell\omega = \sqrt{6g\ell}$$

COMBINED ROTATIONAL AND TRANSLATIONAL MOTION OF A RIGID BODY:

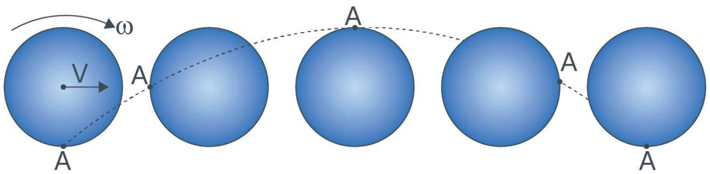
We have already study about translational motion affected by a force and rotational motion about a fixed axis caused by a torque. Now we are going to consider a motion in which body undergoes rotational as well as translational motion. Rolling is an example of such type motion. If axis of rotation is moving then motion is combined translational and rotational motion.

To understand concept of combined translational and rotational motion we consider a uniform disc rolling on the horizontal surface. Velocity of its COM is v_{com} and its angular speed is ω as shown in diagram.



Assume a point A on the disc and concentrate on its motion.

Path length of point A with respect to ground will be a cycloid as shown in diagram.



Motion of point 'A' with respect to center of mass (COM) is pure rotational while COM itself is moving in a straight line path. Therefore, for the analysis of rolling motion we deal translational motion separately and rotational motion separately and then we combine result to analyse the overall motion.

Concept Reminder

Conservation of angular momentum can also help you distinguish between a hard boiled egg and a raw egg. The egg which spins at a slower rate shall be raw egg.

Rack your Brain

A thin circular ring of mass M and radius r is rotating about its axis with constant angular velocity ω. Two objects each of mass m are attached gently to opposite end of diameter. Find final angular velocity of ring.

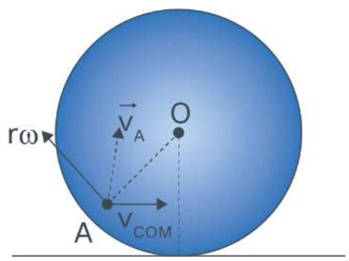
Any point 'A' velocity on the stiff body can be expresses as-

$$\vec{v}_A = \vec{v}_{A\,COM} + \vec{v}_{COM}$$

$$v = |\vec{v}_{COM}|$$

$|\vec{v}_{A\,COM}| = r\omega$ in the direction perpendicular ($\perp$) to line OA.

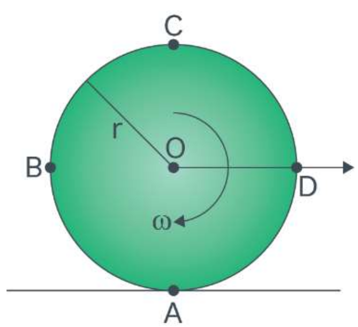
Therefore, the velocity of point 'A' is the vector addition of as $\vec{v}_{COM}$ and $\vec{v}_{B\,COM}$ shown in figure.



Important points in combined Rotational and translation motion:

- Velocity of any point of the stiff body in combined R + T motion is the vector sum of 'v' (velocity of centre of mass) and 'rω'.

For example- A disc of radius r has linear velocity v and angular velocity ω as shown below then find velocity of point A. B, C, D on the disc



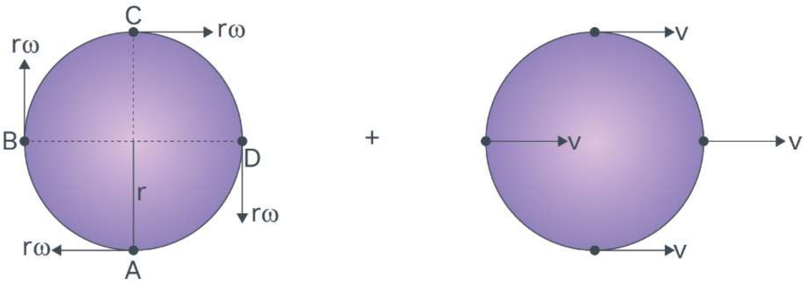
We divide our problem in two parts:

(1) Pure Rotational about centre of mass.

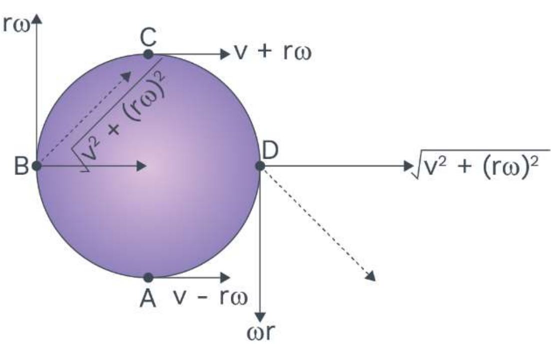
(2) Pure Translational

Concept Reminder

Velocity of any point of the rigid body in combined R + T motion is the vector sum of v(velocity of centre of mass) and rω.



Then combine the result of above both

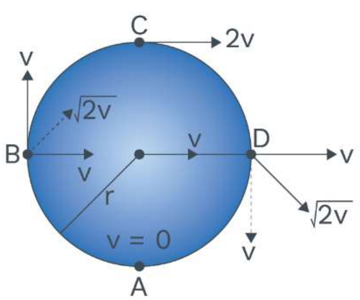


In combined translational and rotational motion angular velocity of any point of a stiff body with respect to other point in the stiff body is always same.

For example:

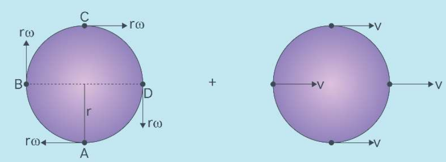
Now for ω_{DA} :

$$\omega_{DA} = \frac{\sqrt{2}\,v}{\sqrt{2}\,r} = \frac{v}{r}$$

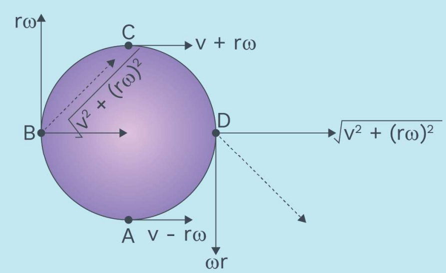


(v = rω)

Concept Reminder

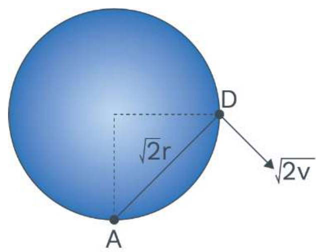


Then combine the result of above both

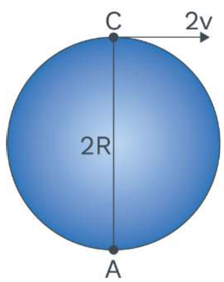


Concept Reminder

In combined rotational and translational motion angular velocity of any point of a rigid body with respect to other point in the rigid body is always same.

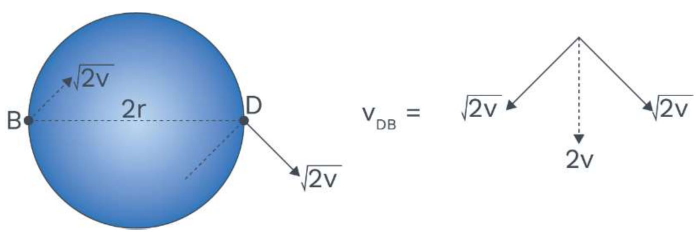


Now for ω_{CA} :



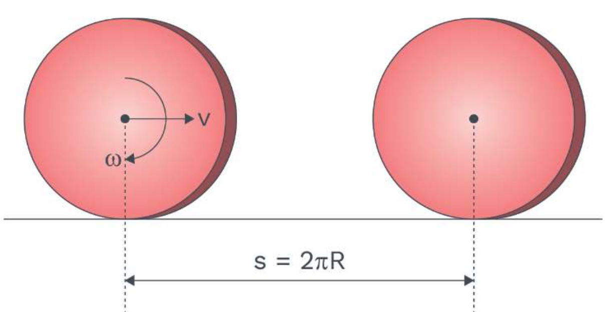
$$\omega_{CA} = \frac{2v}{2r} = \frac{v}{r}$$

Now for ω_{DB} :



$$\Rightarrow \quad \omega_{DB} = \frac{2v}{2r} = \frac{v}{r}$$

- Distance travelled by the COM of the rigid body in one full rotation is $2\pi R$.



Rack your Brain

A balls rolls without slipping. The radius of gyration of the ball about an axis passing through its centre of mass is K. If radius of ball is R, then find fraction of total energy associated with its rotational motion.

Concept Reminder

In one complete rotation, distance moved by COM of rigid body is:
(1) $s = 2\pi R$ (for pure rolling)
(2) $s > 2\pi R$ (for forward slipping)
(3) $s < 2\pi R$ (for backward slipping)

This can be shown as under:

In one rotation angular displacement $\theta = 2\pi = \omega t$

$$s = v \cdot T = (\omega R)\left(\frac{2\pi}{\omega}\right) = 2\pi R$$

In forward slipping

$s > 2\pi R$ (as $v > \omega R$)

and in backward slipping

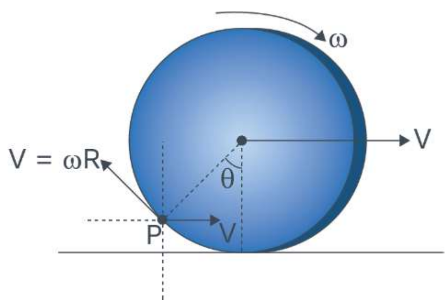
$s < 2\pi r$ (as $v < \omega R$)

- The speed of any point on the circumference of the body at any instant time t is $2R\omega \sin\frac{\omega t}{2}$.

Proof:

$v_{xp} = v - v\cos\theta = v[1 - \cos\theta]$

$v_{yp} = v\sin\theta$

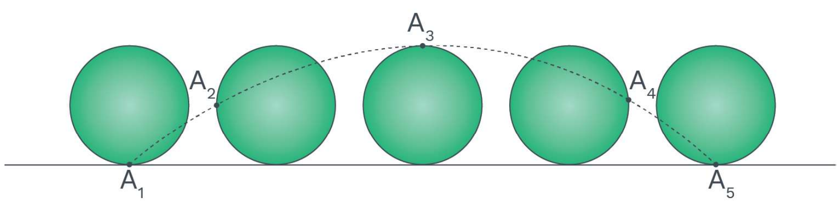


$$|\vec{v}_p| = \sqrt{v^2\sin^2\omega + v^2(1-\cos\theta)^2}$$

$$v = \sqrt{2v^2 - 2v^2\cos\theta} = \sqrt{2}\,v(1-\cos\theta)^{1/2}$$

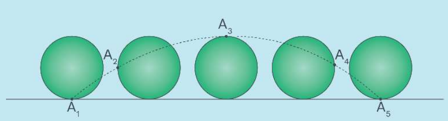
$$= 2v\sin\left(\frac{\theta}{2}\right) = 2v\sin\left(\frac{\omega t}{2}\right) = 2R\omega\sin\left(\frac{\omega t}{2}\right)$$

- The path length of a point on circumference is a cycloid and the distance travelled by this point in one full rotation is 8R.



Concept Reminder

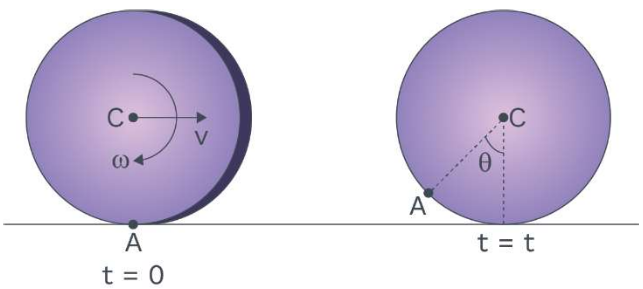
The path of a point on circumference is a cycloid and the distance moved by this point in one full rotation is 8R.



In the agove figure, the dotted line is a cycloid and the distance A_1, A_2, ..., A_5 is 8R. This can be confirmed as under.

According to point '3', speed of point A at any instant is,

$$v_A = 2R\omega \sin\left(\frac{\omega t}{2}\right)$$



Distance moved by 'A' in time dt is,

$$ds = v_A dt = 2R\omega \sin\left(\frac{\omega t}{2}\right)dt$$

Therefore, the total distance moved in '1' full rotation is,

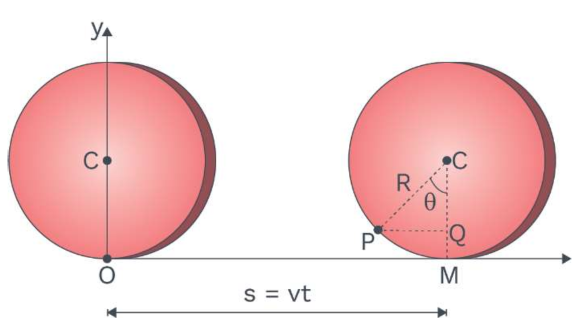
$$s = \int_0^{T=2\pi/\omega} ds$$

or $$s = \int_0^{T=2\pi/\omega} 2R\omega \sin\left(\frac{\omega t}{2}\right)dt$$

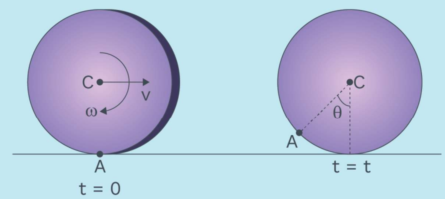
On integration we get,

s = 8R

- x and y coordinates of the lowest point at any time t.



Concept Reminder



$$v_A = 2R\omega \sin\left(\frac{\omega t}{2}\right)$$

At time t the lowest point will rotate an angle $\theta = \omega t$ with respect to the centre of the disc 'C'. The centre C will move a distance s = vt.

In the figure,

$PQ = R \sin\theta = R \sin\omega t$

$CQ = R \cos\theta = R \cos\omega t$

Coordinates of point P at time t are,

$x = OM - PQ = vt - R \sin\omega t$

and $y = CM - CQ = R - R \cos\omega t$

$\therefore \quad (x, y) = (vt - R \sin\omega t, R - R \cos\omega t)$

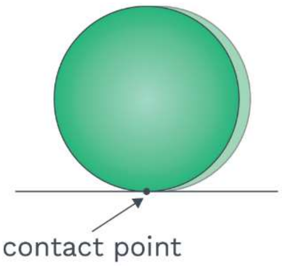
ROLLING

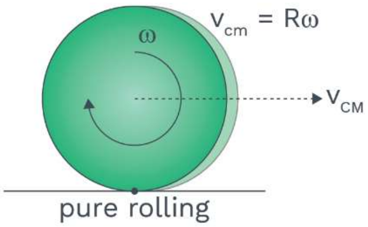
Rolling Motion: When a spherical body performs translatory motion as well as rotatory motion then it is known as rolling. The velocity of centre of mass represents linear motion while angular velocity represents rotatory motion.

PURE ROLLING:

If the velocity of point of contact with respect to the surface is zero then it is known as pure rolling. If a body is performing rolling then the velocity of any point of the body with respect to the surface is given by

$$\vec{v} = \vec{v}_{CM} + \vec{\omega} \times \vec{R}$$



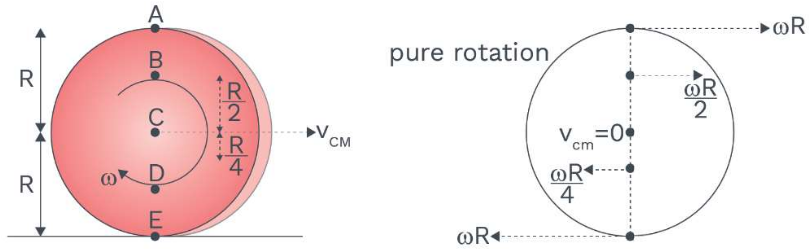


Definitions

When a spherical body performs translatory motion as well as rotatory motion then it is known as rolling.

Concept Reminder

The velocity of centre of mass represents linear motion while angular velocity represents rotatory motion.



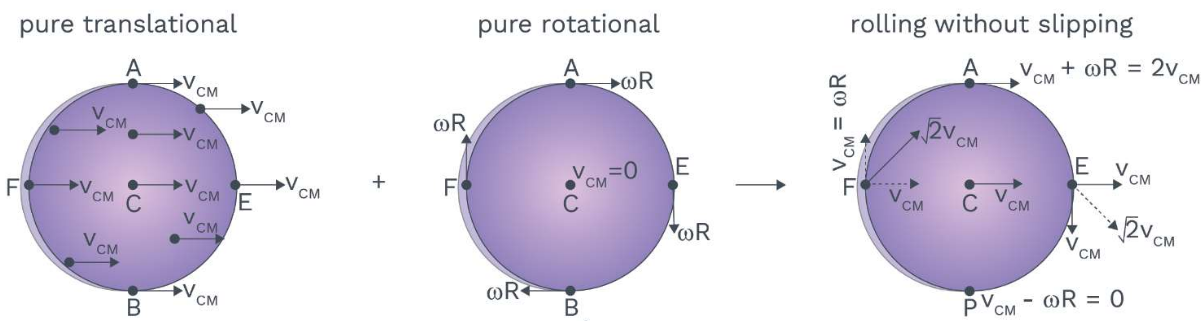
$$v_A = v_{CM} + \omega R = v_{CM} + v_{CM} = 2v_{CM}$$

$$v_B = v_{CM} + \frac{\omega R}{2} = v_{CM} + \frac{v_{CM}}{2} = \frac{3}{2}v_{CM}$$

$$v_C = v_{CM} + 0 = v_{CM}$$

$$v_D = v_{CM} - \frac{\omega R}{4} = v_{CM} + \frac{v_{CM}}{4} = \frac{3}{4}v_{CM}$$

$$v_E = v_{CM} + \omega R = v_{CM} - v_{CM} = 0$$

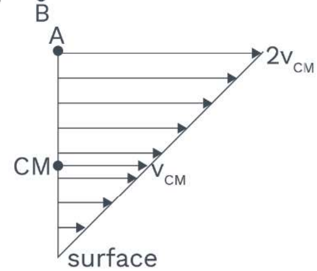


$v_A = 2v_{CM}$

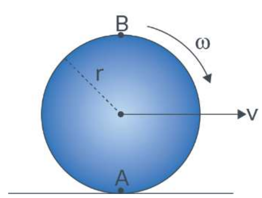
$v_E = \sqrt{2}v_{CM}$

$v_F = \sqrt{2}v_{CM}$

$v_P = 0$



Ex. A wheel of radius 'r' rolls (rolling without slipping) on a level road as shown in figure. Find out the velocity of point 'A' and 'B'.

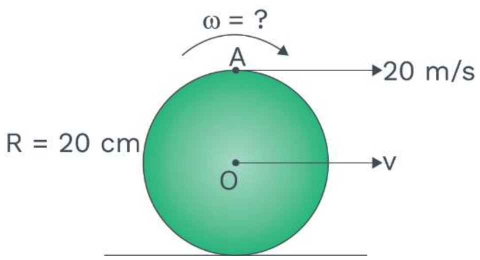


Sol. Contact surface is at rest for pure rolling velocity of point A is zero,

So, $v = \omega r$

Velocity of point B = $v + \omega r = 2v$

Ex. Pure rolling then



Sol. $v_A = 2v_{CM}$

$\Rightarrow$ $20 = 2v_{CM}$

$v_{CM} = 10$ m/s

$v_{CM} = \omega R$

$$\Rightarrow \quad \omega = \frac{10 \times 100}{20} = 50 \text{ rad / s}$$

Angular momentum of a stiff body in combined rotation and translation:

Assume 'O' be a fixed point in an inertial frame of reference. Angular momentum of body about point 'O' is.

$$\vec{L} = \vec{L}_{CM} + M(\vec{r}_0 \times \vec{v}_0)$$

The first term '$\vec{L}_{CM}$' represents the angular momentum of the body as seen from the centre of mass frame. The second term '$M(\vec{r}_0 \times \vec{v}_0)$' equals the angular momentum of centre of mass about point 'O'.

Concept Reminder

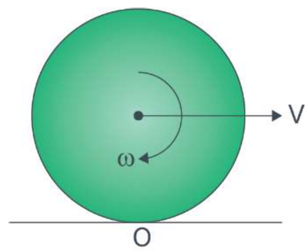
Angular momentum of a rigid body in combined rotation and translation:

Angular momentum of the body about O is

$$\vec{L} = \vec{L}_{CM} + M(\vec{r}_0 \times \vec{v}_0)$$

Ex. A circular disc of mass 'm' and radii 'R' is set into the motion on a horizontal floor with a linear speed 'v' in the forward direction and an angular speed $\omega = \frac{v}{R}$ in clockwise direction as shown in diagram. Find out the magnitude of the total angular momentum of the disc about bottommost point O of the disc.

Sol. $\vec{L} = \vec{L}_{CM} + m(\vec{r}_0 \times \vec{v}_0)$ ....(i)

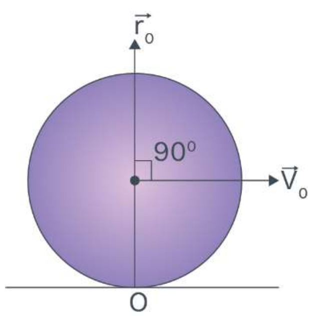


Here, $\vec{L}_{CM} = I\omega$

$$= \left(\frac{1}{2}mR^2\right)\left(\frac{v}{R}\right) = \frac{1}{2}mvR$$

and $m(\vec{r}_0 \times \vec{v}_0) = mRv$ (perpendicular to paper inwards)

Since, both the terms of right hand side of Eq. (i) are in the same direction.



$$\therefore \quad |\vec{L}| = \frac{1}{2}mvR + mvR$$

or $$|\vec{L}| = \frac{3}{2}mvR$$

Concept Reminder

If a body of mass M is rolling on a plane such that velocity of its centre of mass is V and its angular speed is ω, its kinetic energy is given by

$$KE = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

Kinetic Energy of a Rolling Body:

If a body of mass M is rolling on a plane such that velocity of its COM is 'v' and its angular speed is ω, its kinetic energy is given by

$$KE = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

'I' is moment of inertia of body about the axis passing through COM. In case of rolling without slipping.

$$KE = \frac{1}{2}M\omega^2R^2 + \frac{1}{2}I\omega^2 \quad [\because v = \omega R]$$

$$= \frac{1}{2}[MR^2 + I]\omega^2 = \frac{1}{2}I_c\omega^2$$

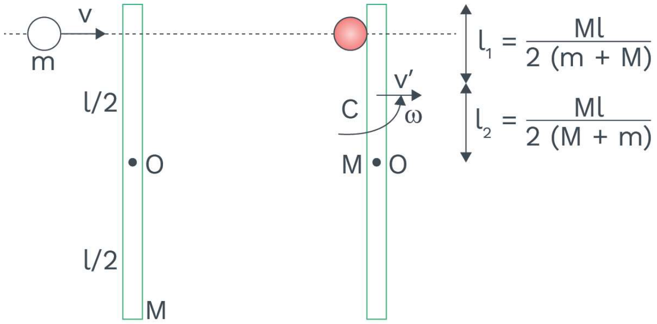
I_c is moment of inertia of the body about axis passing through point of contact.

Ex. A uniform rod of length l lies on the smooth horizontal table. A particle moving on table has a mass 'm' and a speed 'v' before the collision and it sticks to rod after the collision. The rod has a mass 'M' then find out.

(a) The moment of inertia of system about the vertical axis passing through the COM 'C' after the collision.

(b) The velocity of the COM 'C' and the angular velocity of the system about the COM after the collision.

Sol. Figure shows situation of the system just before and just after the collision.
Initially the COM of the rod is at point 'O'. After collision when particle sticks to the rod. COM is shifted from point 'O' to 'C' as shown in figure. Now the system is rotated about the axis passing through 'C'.



Before collision After collision

Now from the linear momentum conservation

$$mv = (M+m)v' \quad \Rightarrow v' = \frac{mv}{M+m}$$

(a) Let us assume that the moment of inertia of the system about 'C' is '1'. Then

$$I = I_{(rod)C} + I_{(part)C}$$

$$I = I_0 + M\ell_2^2 + m\ell_1^2$$

$$I = \frac{M\ell^2}{12} + \frac{Mm^2\ell^2}{4(m+M)^2} + \frac{mM^2\ell^2}{4(m+M)^2}$$

$$\Rightarrow \quad I = \frac{M(M+4m)}{12(m+M)}\ell^2$$

(b) From Angular momentum conservation about A

$L_i = L_f$

$0 + 0 = I\omega - (m + M)\, v'l_1$

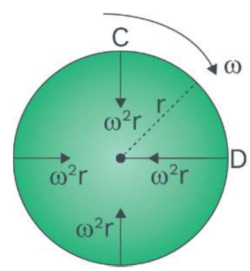
$\Rightarrow \quad I\omega = (m + M)\, v'l_1$

Put the value of I, v', & l_1 we get

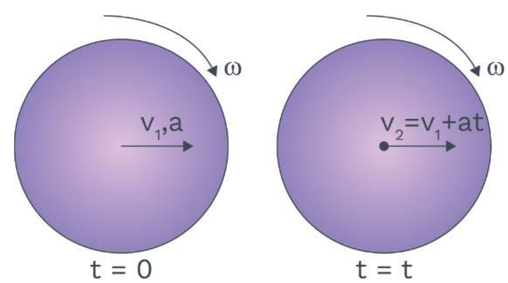
$$\omega = \frac{6mv}{(M+4m)\ell}$$

Acceleration of a point on the circumference of body in R + T motion:

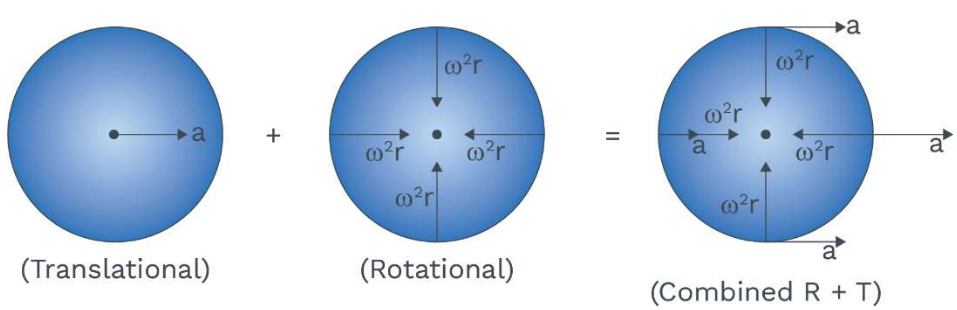
(a) Both ω & v are constant:



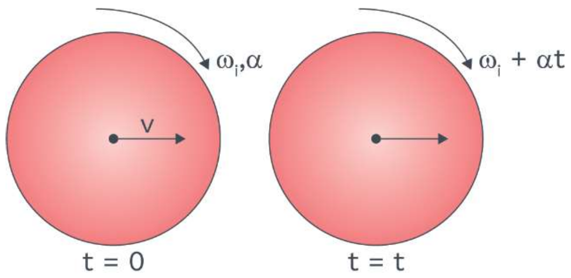
(b) When ω is constant and 'v' is variable:



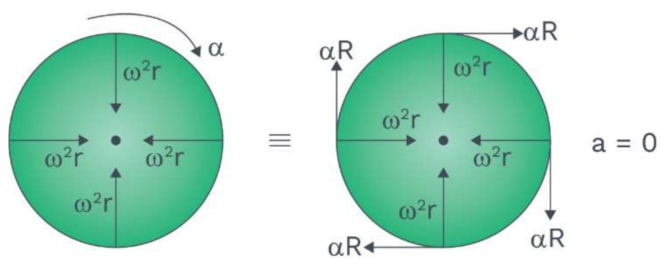
So, the acceleration of different point on body is given by following figure.



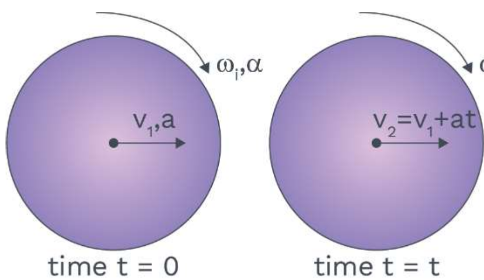
(c) When ω is variable and 'v' is constant:



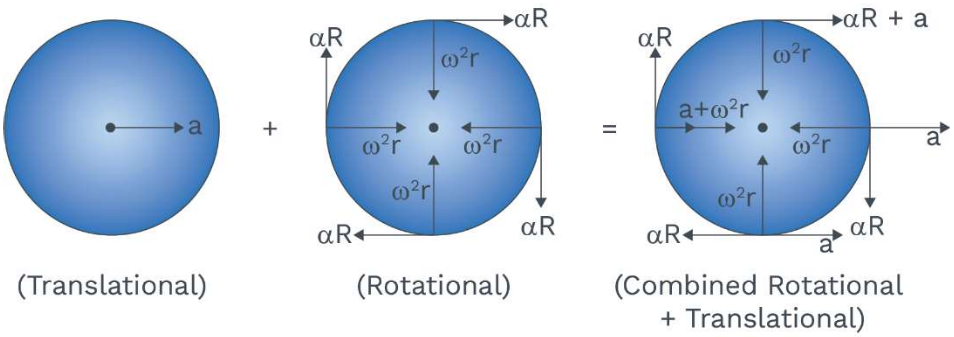
So, acceleration of different point on body is given by the following way



(d) When both ω & 'v' are variable:

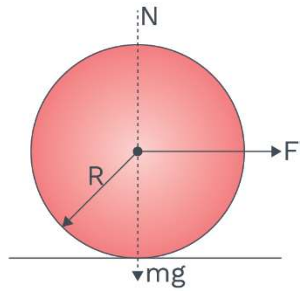


Now, net acceleration of different points on rigid body is given by following way.



Ex. A force 'F' acts at centre of a thin spherical shell of mass 'm' and radius 'R'. Find the acceleration of shell if the surface is smooth.

Sol. ∵ Force F, mg & N passes through centre so $\tau_{net} = 0$, i.e., body is in rotational equilibrium



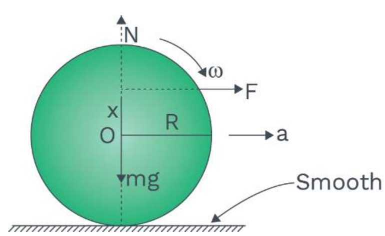
But $\vec{F}_{net} = F$ so body moves with constant acceleration $a = \frac{F}{m}$.

Ex. In a previous problem if force 'F' applied at a distance 'x' above the centre then find out linear and angular acceleration.

Sol. This force F translate the body linearly as well as rotate it. So, Net torque about O it $\tau_0 = Fx$

From rotational motion $\tau_0 = I\alpha$

$$\alpha = \frac{\tau_0}{I} = \frac{Fx}{\frac{2MR^2}{3}} \Rightarrow \alpha = \frac{3Fx}{2MR^2}$$

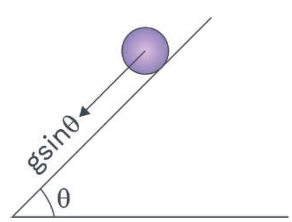


From linear motion of sphere

$$F = ma \Rightarrow a = \frac{F}{m}$$

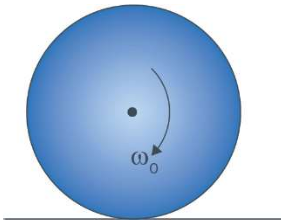
Ex. A rigid body of mass 'm' and radius 'r' starts coming down an inclined plane of inclination θ. Then find out acceleration of centre of mass if friction is absent.

Sol. Since friction is absent so body is moving down the incline without rolling so acceleration of centre of mass is g sin θ



Rotational Motion

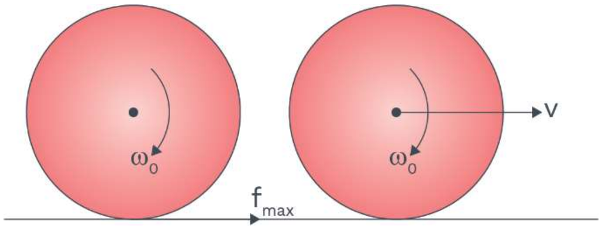
Ex. A solid sphere of radius 'r' is gently placed on the rough horizontal ground with an initial angular speed ω_0 and no linear velocity. If the coefficient of friction is μ, find the linear velocity 'v' and angular velocity ω at the end of slipping.



Sol. 'm' be the mass of sphere.

Since, this is a case of backward slipping, so friction force is in forward direction. Limiting friction will act in this case.

Net torque on sphere about the bottommost point is zero. Therefore, the angular momentum of the sphere will remain conserved about bottommost point.



$$L_i = L_f$$

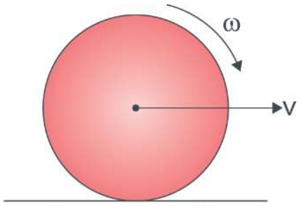
$$\therefore \quad I\omega_0 = I\omega + mrv$$

$$\text{or} \quad \frac{2}{5}mr^2\omega_0 = \frac{2}{5}mr^2\omega + mr(\omega r)$$

$$\therefore \quad \omega = \frac{2}{7}\omega_0 \text{ and } v = r\omega = \frac{2}{7}r\omega_0$$

THE NATURE OF FRICTION IN FOLLOWING CASES ASSUME BODY IS PERFECTLY RIGID

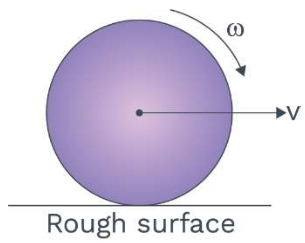
(i) $v = \omega R$



Smooth surface

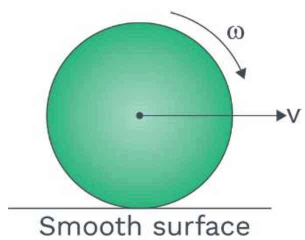
No friction and pure rolling,

(ii) $v = \omega R$



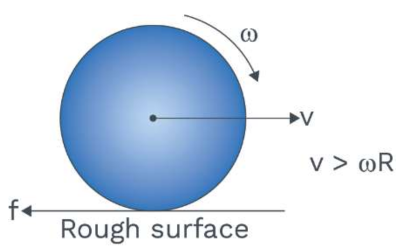
No friction and pure rolling (if body is not perfectly rigid, then there is a small friction acting in this case which is known as rolling friction)

(iii) $v > \omega R$ or $v < \omega R$



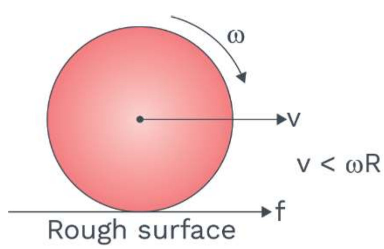
No friction force but not pure rolling.

(iv) $v > \omega R$



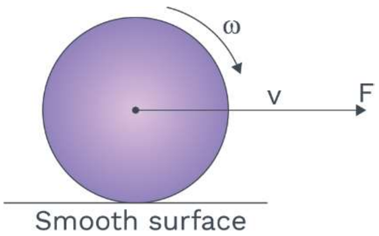
There is relative motion at the point of contact so Kinetic Friction, f = mN will act in backward direction. This kinetic friction decrease v and increase w, so after some time $v = \omega R$ and pure rolling will resume like in case (ii).

(v) $v < \omega R$



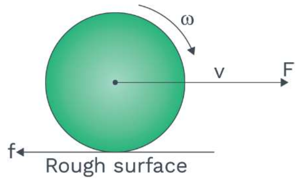
There is relative motion at the point of contact so Kinetic Friction, f = mN will act in forward direction. This kinetic friction increase v and decrease w, so after some time v = ωR and pure rolling will resume like in case (ii).

(vi) v = ωR (initial)



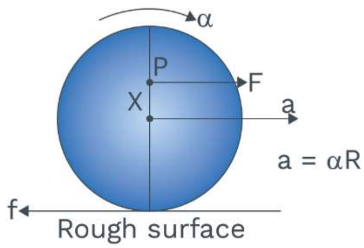
No friction and no pure rolling.

(vii) v = ωR (initial)



Static friction whose value can lie between zero and mN will act in backward direction. If the coefficient of friction is appropriately high, then f_s compensates for increasing 'v' due to 'F' by increasing ω and body may continue in pure rolling with increases v as well as ω.

Ex. A rigid body of mass 'm' and radius 'r' rolls without slipping on the rough surface. A force is acting on the rigid body 'x' distance from centre as shown in figure. Find value of 'x' so that static friction is zero.



Sol. Torque about centre of mass

$Fx = I_{cm}\alpha$ ...(i)

$$F = ma \qquad ...(ii)$$

From equation (i) and (ii),

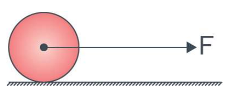
$$max = I_{cm}a \qquad (a = \alpha R)$$

$$x = \frac{I_{cm}}{mR}$$

Conclusion:

This $x\left(\frac{I_{cm}}{MR}\right)$ is the value for which friction will be zero. It means if a force is applied at this distance, then even on smooth surface, pure rolling is possible.

Ex. A horizontal force F acts on a sphere of mass M at its centre as shown. Coefficient of friction between ground and the sphere is μ. What is maximum value of F, for which there is no slipping?



Sol. For linear motion $F - f = Ma$

and for rotational motion $\tau = I\alpha$

$$\Rightarrow \quad f \cdot R = \frac{2}{5}MR^2 \cdot \frac{a}{R}$$

$$\Rightarrow \quad f = \frac{2}{5}Ma \text{ or } Ma = \frac{5}{2}f$$

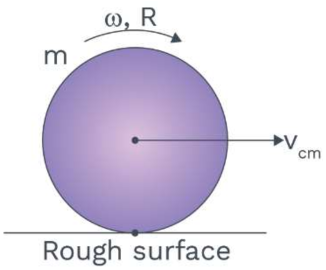
$$\therefore \quad F - f = \frac{5}{2}f \text{ or } f = \frac{2F}{7} \quad (\because f \le \mu Mg)$$

$$\therefore \quad \frac{2F}{7} \le \mu Mg \quad \text{so, } F \le \frac{7}{2}\mu Mg$$

Note:

- For pure rolling if any type of friction is required then the friction force will be static friction. It can be zero, backward direction or forward direction depending on the value of 'x'. If 'F' below the point 'P' then friction force will act in backward direction or above the point P friction face will act in forward direction.
- On smooth horizontal surface pure rolling can be possible.
- If friction is present, then it will be of static nature.
- Work done by static friction in pure rolling is zero on fixed surface.
- In pure rolling contact point is momentarily at rest but Its acceleration is not zero.

KINETIC ENERGY IN PURE ROLLING



$v_{cm} = \omega R$

Total (KE) = Translational (KE) + Pure Rotational (KE)

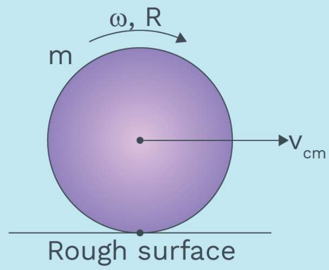
$$= \frac{1}{2}mv_{cm^2} + \frac{1}{2}I_{cm}\omega^2$$

$$= \frac{1}{2}mv_{cm^2} + \frac{1}{2}mK^2 \cdot \frac{v_{cm}\omega^2}{R^2}$$

$$\text{Total KE} = \frac{1}{2}mv_{cm^2}\left[1+\frac{K^2}{R^2}\right]$$

$$\frac{(KE)_{Rot}}{(KE)_{total}} = \frac{\frac{K^2}{R^2}}{\left(1+\frac{K^2}{R^2}\right)}$$

Concept Reminder

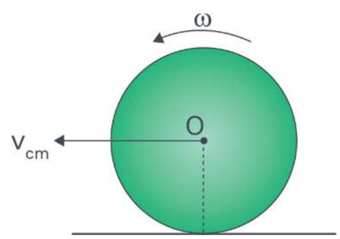


$$\frac{(KE)_{Rot}}{(KE)_{total}} = \frac{\frac{K^2}{R^2}}{\left(1+\frac{K^2}{R^2}\right)}$$

BODY	$\frac{K^2}{R^2}$	$\frac{E_{trans}}{E_{rotation}} = \frac{1}{\left(\frac{K^2}{R^2}\right)}$	$\frac{E_{trans}}{E_{total}} = \frac{1}{\left(1+\frac{K^2}{R^2}\right)}$	$\frac{E_{rotation}}{E_{total}} = \frac{K^2/R^2}{\left(1+\frac{K^2}{R^2}\right)}$
Ring	1	1	$\frac{1}{2}$	$\frac{1}{2}$
Disc	$\frac{1}{2}$	2	$\frac{2}{3}$	$\frac{1}{3}$
Solid sphere	$\frac{2}{5}$	$\frac{5}{2}$	$\frac{5}{7}$	$\frac{2}{7}$
Spherical shell	$\frac{2}{3}$	$\frac{3}{2}$	$\frac{3}{5}$	$\frac{2}{5}$
Solid cylinder	$\frac{1}{2}$	2	$\frac{2}{3}$	$\frac{1}{3}$
Hollow cylinder	1	1	$\frac{1}{2}$	$\frac{1}{2}$

ANGULAR MOMENTUM IN COMBINED ROTATIONAL & TRANSLATION MOTION

$$\vec{L}_P = \vec{L}_{cm} + (\vec{r} \times \vec{p})$$



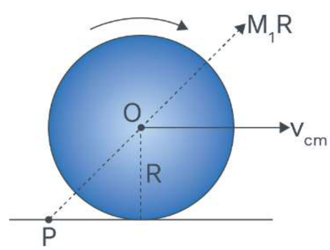
Where $\vec{r}$ = position vector of COM w.r.t. P

$L_P = I_{cm}\,\omega + mv_{cm}\,R$

If pure rolling, $\omega = \frac{v_{cm}}{R}$

$$\boxed{L_P = I_{cm}\frac{v_{cm}}{R} + mv_{cm}R}$$

Ex. Solid sphere starts pure rolling, find L about point P.



Sol. $\vec{L}_p = \left(-\frac{2}{5}mR^2 \times \frac{v_{cm}}{R} - mv_{cm}R\right)\hat{K}$

$\vec{L}_p = -\left(\frac{7}{5}mv_{cm}R\right)\hat{K}$

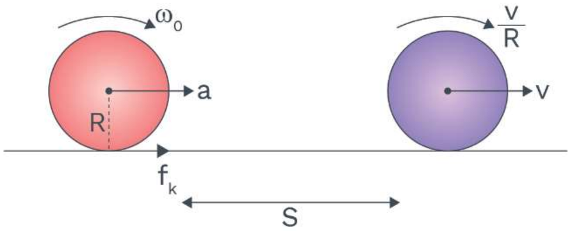
Note: Angular momentum about any axis passing through contact points is conserved in pure rolling.

Ex. Find KE in the above example.

Sol. $KE = \frac{1}{2}mv_{cm}^2\left(1 + \frac{K^2}{R^2}\right)$

$= \frac{1}{2}mv_{cm}^2\left(1 + \frac{2}{5}\right) = \frac{1}{2}mv_{cm}^2\left(\frac{7}{5}\right) = 0.7\ mv_{cm}^2$

Ex. A cylinder is given angular velocity ω and kept on the horizontal rough surface the initial velocity is zero. Find out out distance travelled by the cylinder before it performs pure rolling?



Sol. $\mu MgR = \frac{MR^2\alpha}{2}$

$\alpha = \frac{2\mu g}{R}$ ...(i)

Initial velocity u = 0

$v^2 = u^2 + 2as$

$v^2 = 2as$ ...(ii)

$f_K = Ma$

$\mu Mg = Ma$

$a = \mu g$ ...(iii)

$\omega = \omega_0 - at$

From equation (i),

$\omega = \omega_0 - \frac{2\mu g}{R}t$, $v = u + at$

From equation (iii),

$v = \mu gt$

$\omega = \omega_0 - \frac{2v}{R}$

$\omega = \omega_0 - 2\omega$

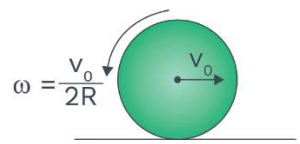
$\Rightarrow \omega = \frac{\omega_0}{3}$

From equation (ii),

$\left(\frac{\omega_0 R}{3}\right)^2 = (2as) = 2\mu gs$

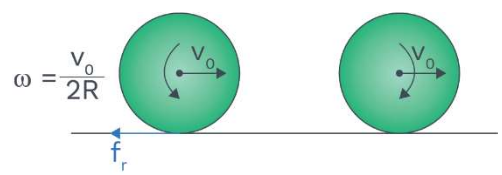
$s = \left(\frac{\omega_0^2 R^2}{18\mu g}\right)$

Ex. A solid sphere of radius 'R' is set into motion on the rough horizontal surface with a linear speed v_0 in the forward direction and an angular velocity $\omega_0 = \frac{v_0}{2R}$ in counter clockwise direction as shown in figure. If coefficient of friction is m, then find-



(a) The linear speed of sphere when it starts pure rolling.
(b) The time after which sphere starts pure rolling,
(c) The work done by friction over a long time.

Sol. (a) Since the net torque about the axis fixed to the ground and passing through the point of contact is zero, using the conservation of angular momentum about this axis,



$$-\frac{2}{5}mR^2\omega_0 + mv_0R = \frac{2}{5}mR^2\omega + mvR$$

$$-\frac{2}{5}mv_0R + mv_0R \Rightarrow \frac{3}{5}mv_0R = \frac{7}{5}mvR$$

$$\therefore \quad v = \frac{3v_0}{7}$$

(b) Now, $v = v_0 - \mu gt$

$$\Rightarrow \quad \frac{3v_0}{7} = v_0 - \mu gt$$

$$\therefore \quad t = \frac{3v_0}{7\mu g}$$

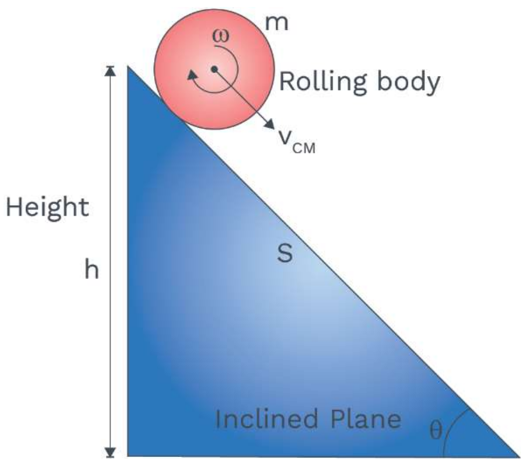
(c) Work done by friction,

$\Delta W_{fr} = K_f - K_i$

$$= \frac{7}{10}mv^2 - \frac{11}{20}mv_0^2 = \frac{-59}{140}mv_0^2$$

Rolling Motion on an inclined plane:

Applying Conservation of energy



$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}mK^2\left(\frac{v^2}{R^2}\right)$$

$$mgh = \frac{1}{2}mv^2\left(1+\frac{K^2}{R^2}\right) \quad ...(i)$$

$$h = s\sin\theta \quad ...(ii)$$

From equation (i) and (ii),

$$v_{rolling} = \sqrt{\frac{2gh}{1+\frac{K^2}{R^2}}} = \sqrt{\frac{2gs\sin\theta}{1+\frac{K^2}{R^2}}}$$

When body slide, $\frac{K^2}{R^2} = 0$

Velocity when body slides

$$v_{sliding} = \sqrt{2gh} = \sqrt{2gs\sin\theta}$$

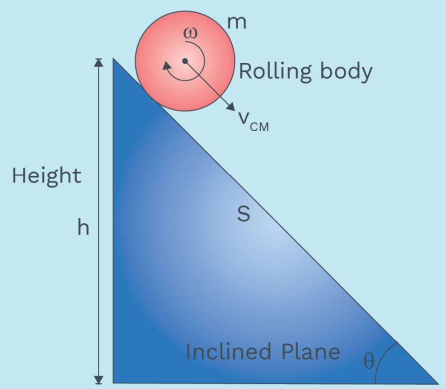
$v_s > v_R$

Acceleration of body

$$v^2 = u^2 + 2as \quad (u = 0)$$

$$v^2 = 2as$$

Concept Reminder



$$v_{rolling} = \sqrt{\frac{2gh}{1+\frac{K^2}{R^2}}} = \sqrt{\frac{2gs\sin\theta}{1+\frac{K^2}{R^2}}}$$

Rack your Brain

A solid cylinder of mass M and radius R rolls without slipping down an inclined plane of length L and height h. What is the speed of its centre of mass when cylinder reaches its bottom.

$$\frac{2\,gs\sin\theta}{1+\frac{K^2}{R^2}} = 2\,as$$

Acceleration when body rolls

$$\boxed{a_R = \frac{g\sin\theta}{1+\frac{K^2}{R^2}}}$$

When body slides, $\frac{K^2}{R^2} = 0$

Acceleration when slides

a_S = g sin θ   ($a_S > a_R$)

Time taken by the rolling body to reach the bottom

$$s = \frac{1}{2}at^2$$

$$\Rightarrow \quad t = \sqrt{\frac{2s}{a}}, \quad s = \frac{h}{\sin\theta}$$

Time of descend when body rolls

$$t_R = \sqrt{\frac{2s}{g\sin\theta}\left(1+\frac{K^2}{R^2}\right)} = \sqrt{\frac{2h}{g\ \sin^2\theta}\left(1+\frac{K^2}{R^2}\right)}$$

$$= \frac{1}{\sin\theta}\sqrt{\frac{2h}{g}\left(1+\frac{K^2}{R^2}\right)}$$

When body slides,

$$\frac{K^2}{R^2} = 0$$

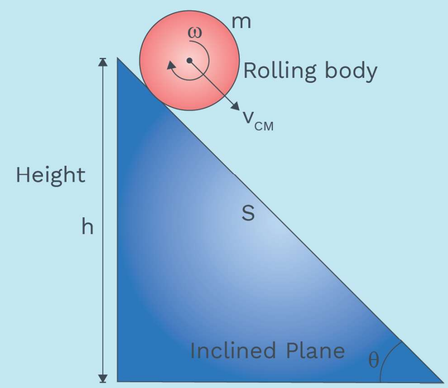
Time of descend when body slides

$$\boxed{t_S = \sqrt{\frac{2s}{g\sin\theta}} = \sqrt{\frac{2h}{g\sin^2\theta}} = \frac{1}{\sin\theta}\sqrt{\frac{2h}{g}}}$$

So, $t_R > t_S$

If different body are rolled down on an inclined plane then body which has

Concept Reminder



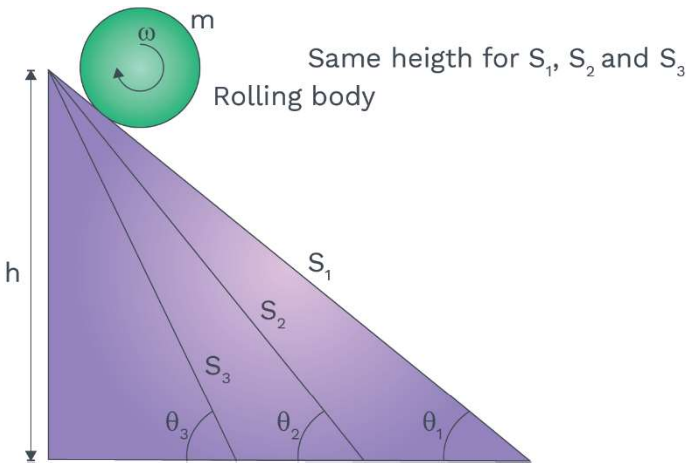
Acceleration when body rolls

$$\boxed{a_R = \frac{g\sin\theta}{1+\frac{K^2}{R^2}}}$$

Concept Reminder

Time of descend when body slides

$$\boxed{t_S = \sqrt{\frac{2s}{gsn\theta}} = \sqrt{\frac{2h}{g\sin^2\theta}} = \frac{1}{\sin\theta}\sqrt{\frac{2h}{g}}}$$

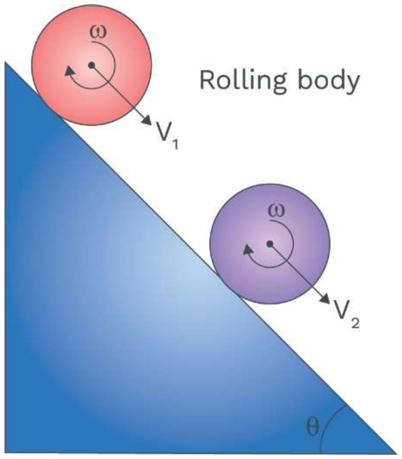


(a) $\frac{K^2}{R^2}$ Least, will reach first

(b) $\frac{K^2}{R^2}$ Maximum, will reach last

(c) $\frac{K^2}{R^2}$ Equal, will reach together

(d) From figure $\theta_3 > \theta_2 > \theta_1$
$a_3 > a_2 > a_1$
$t_3 < t_2 < t_1$
$v_1 = v_2 = v_3$
In next figure if $v_2 > v_1$
Change in kinetic energy due to rolling



Rack your Brain

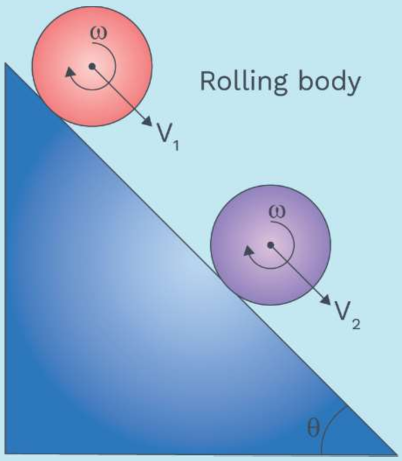
Two rotating bodies A and B of masses m and 2m with moment of inertia I_A and I_B ($I_B > I_A$) have equal kinetic energy of rotation. If L_A and L_B be their angular momenta respectively then:

(1) $L_A = \frac{L_B}{2}$ (2) $L_A = 2L_B$

(3) $L_B > L_A$ (4) $L_A > L_B$

Concept Reminder

Change in kinetic energy due to rolling



$$KE = \frac{1}{2}m\left(1+\frac{K^2}{R^2}\right)(v_2^2 - v_1^2)$$

$$= \frac{1}{2}mv_2^2\left(1+\frac{K^2}{R^2}\right) - \frac{1}{2}mv_1^2\left(1+\frac{K^2}{R^2}\right)$$

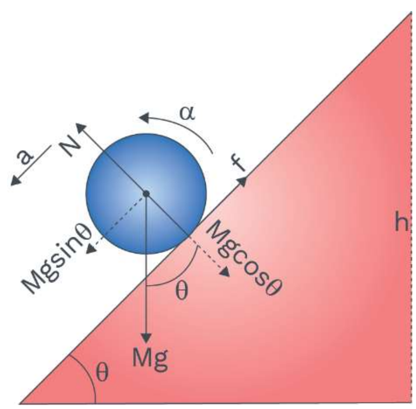
$$= \frac{1}{2}m\left(1+\frac{K^2}{R^2}\right)(v_2^2 - v_1^2)$$

Rolling v/s Sliding:

$$\frac{v_R}{v_S} = \sqrt{\frac{1}{1+\frac{K^2}{r^2}}}, \quad \frac{a_R}{a_S} = \frac{1}{1+\frac{K^2}{r^2}}, \quad \frac{t_R}{t_S} = \sqrt{1+\frac{K^2}{r^2}}$$

$v_R < v_S$, $a_R < a_S$, $t_R > t_S$

ROLL OF FRICTION TO EXECUTE ROLLING MOTION ON AN INCLINED PLANE



A body of mass 'M' and radius 'R' rolling down on an plane inclined at an angle θ with the horizontal. The body rolls without slipping. The COM of the body moves in a straight line. External forces acting on the body are:

Weight Mg of the body vertically downwards through the center of mass of the body.

The normal reaction N of the inclined plane. The frictional force f acting upwards and parallel to the inclined plane.

For linear motion,

$$Mg \sin\theta - f = Ma$$

Concept Reminder

- $\frac{v_R}{v_S} = \sqrt{\frac{1}{1+\frac{K^2}{r^2}}}$
- $\frac{a_R}{a_S} = \frac{1}{1+\frac{K^2}{r^2}}$
- $\frac{t_R}{t_S} = \sqrt{\frac{1+\frac{K^2}{r^2}}{1}}$

For angular motion,

$\tau = fR = I\alpha$

Pure rolling condition,

$a = \alpha R$

$$\Rightarrow \quad Mg\sin\theta - f = M\left(\frac{fR^2}{I}\right) = M\left(\frac{fR^2}{MK^2}\right)$$

$$\Rightarrow \quad f = \frac{Mg\sin\theta}{\left(1+\frac{R^2}{K^2}\right)}$$

In critical position

$$\text{but } f = \mu Mg\cos\theta \quad \Rightarrow \frac{Mg\sin\theta}{1+\frac{R^2}{K^2}} = \mu Mg\cos\theta$$

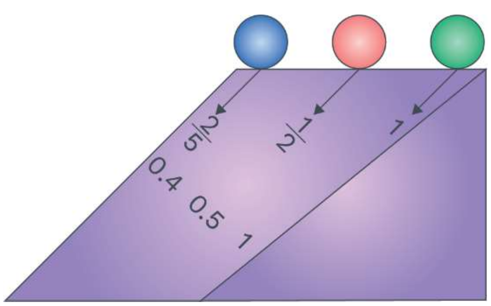
$$\Rightarrow \quad \mu = \frac{\tan\theta}{1+\frac{R^2}{K^2}}$$ for pure rolling motion on inclined plane

$$\mu_{min} = \frac{\tan\theta}{1+\frac{R^2}{K^2}}$$

Concept Reminder

Don't write the value of friction $f = \mu N$ unless limiting condition is applied since friction force is static in nature in pure rolling motion.

Ex. Increasing order of following-



(i) $a_{cm} = ?$ (ii) $v_{cm} = ?$

(iii) Time = ? (iv) $a_A : a_B : a_C = ?$

Sol. (i) C < B < A

(ii) C < B < A

(iii) A < B < C

(iv) $\frac{5}{7} : \frac{2}{3} : \frac{1}{2}$

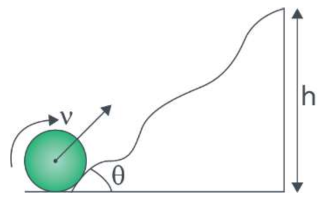
30 : 28 : 21

Concept Reminder

For pure rolling motion on inclined plane

$$\mu_{min} = \frac{\tan\theta}{1+\frac{R^2}{K^2}}$$

Ex. Disc (M, R) rolls without slipping then maximum height attained?



Sol. $v^2 = \dfrac{2g(\ell \sin\theta)}{1+\dfrac{1}{2}}$

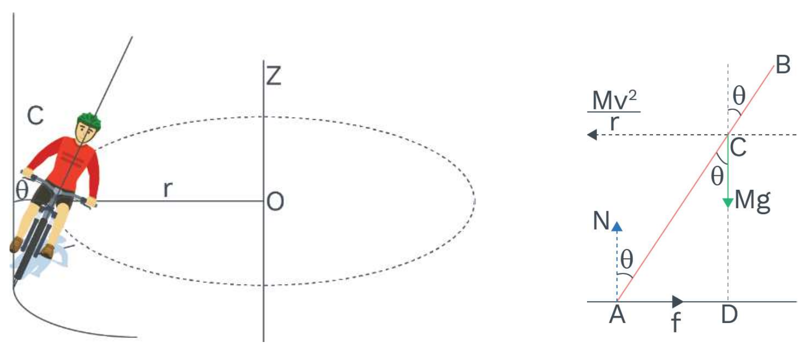
$$v^2 = \frac{2gh}{\frac{3}{2}}$$

$$\frac{3v^2}{2} = 2g(h)$$

$$h = \frac{3v^2}{4g}$$

BENDING ON CYCLIST ON HORIZONTAL TURN

Suppose a cyclist is moving at a speed v on a circular horizontal road of radius r. Consider cycle and the rider together as the system. The COM 'C' (figure a) of the system is going in a circle with the centre at 'O' and radius 'r'.



As seen from A the frame rotating with the same angular velocity as the system is in equilibrium

So, $F_{net} = 0$ and $\tau_{net} = 0$

For rotational equilibrium

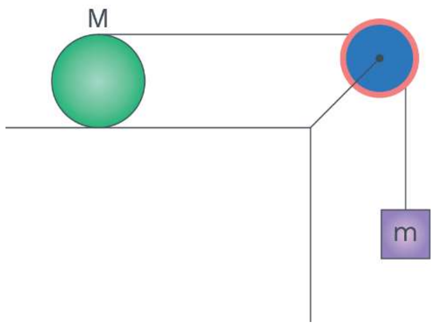
$$\tau_A = 0 \Rightarrow Mg(AD) = \frac{Mv^2}{r}(CD)$$

$$\Rightarrow \frac{AD}{CD} = \frac{v^2}{rg} \Rightarrow \tan\theta = \frac{v^2}{rg}$$

Thus the cyclist bends at an angle $\tan^{-1}\left(\frac{v^2}{rg}\right)$ with the vertical.

Note: In case of overturning inner wheels leaves the surface first.

Ex. Consider the arrangement shown in figure. The string is wrapped around the uniform cylinder which rolls without slipping. The other end of string is passed over a massless, frictionless pulley to a falling weight. Calculate the acceleration of the falling mass 'm' in terms of only the mass of the cylinder 'M', the mass 'm' and 'g'.



Sol. Let 'T' be the tension in the string and 'f' the force of (static) friction, between the cylinder and the surface.

a_1 = acceleration of COM of cylinder towards right.

a_2 = downward acceleration of block m

α = angular acceleration of cylinder (clockwise)

Equations of motion are:

For block, $mg - T = ma_2$ ...(i)

For cylinder, $T + f = Ma_1$ ...(ii)

$$\alpha = \frac{(T - f)R}{\frac{1}{2}MR^2} \quad ...(iii)$$

The string attaches the mass 'm' to the highest point of the cylinder, hence

$$v_m = v_{COM} + R\omega$$

Differentiating, we get

$$a_2 = a_1 + R\alpha \quad ...(iv)$$

We also have (for rolling without slipping)

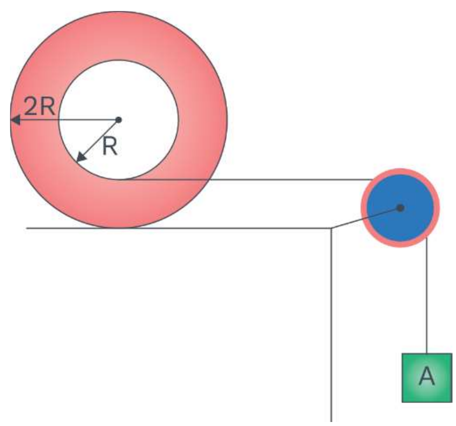
$$a_1 = R\alpha \quad ...(v)$$

By solving the equations, we get

$$a_2 = \frac{8mg}{3M + 8m}$$

Note: Work done by the friction in pure rolling on the stationary ground is zero because the point of application of force is at rest. Therefore, mechanical energy can be conserved if all other dissipative forces are ignored.

Ex. A thin massless thread is wound on the reel of mass 3kg and the moment of inertia 0.6 kg-m^2. The hub radius is R = 10 cm & peripheral radius is 2R = 20 cm. The reel is placed on the rough table and friction is enough to prevent the slipping. Find acceleration of the centre of reel and of hanging the mass of 1 kg.



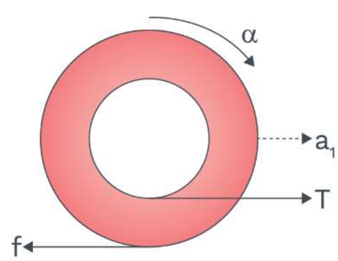
Sol. Let, a_1 = acceleration ofCOM of reel

a_2 = acceleration of the block of 1 kg

α = angular acceleration of the reel (clockwise)

T = tension in the string

and f = force of the friction



F.B.D. of reel is as shown below: (only horizontal forces are shown).

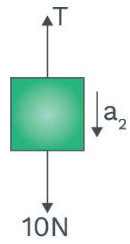
Equations of motion are:

$$T - f = 3a_1 \quad \text{...(i)}$$

$$\alpha = \frac{\tau}{I} = \frac{f(2R) - T \cdot R}{I}$$

$$= \frac{0.2f - 0.1T}{0.6} = \frac{f}{3} - \frac{T}{6} \quad \text{...(ii)}$$

Free body diagram of mass is,



Equation of motion is,

$10 - T = a_2$ ...(iii)

For no slipping condition,

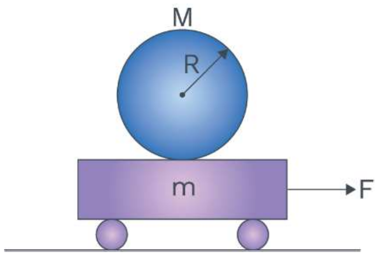
$a_1 = 2R\alpha$ or $a_1 = 0.2a$ ...(iv)

and $a_2 = a_1 - R\alpha$ or $a_2 = a_1 - 0.1a$ ...(v)

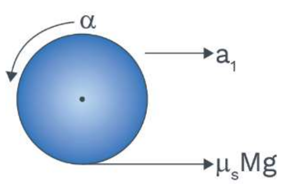
Solving the above five equations, we get

$a_1 = 0.27\ m/s^2$ and $a_2 = 0.135\ m/s^2$

Ex. Find the maximum horizontal force F that may be applied to the plank of mass 'm' for which the solid sphere doesn't slip as it begins to roll on the plank. The sphere has a mass 'M' and radius 'R'. The coefficient of the static and kinetic friction between sphere and plank are μ_s and μ_k respectively.



Sol. The F.B.D. of the sphere and the plank are as shown below:

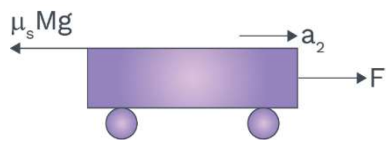


Writing equations of motion:

For sphere: Linear acceleration

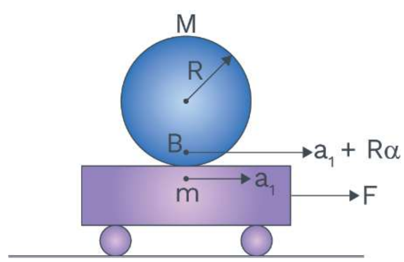
$$a_1 = \frac{\mu_s Mg}{M} = \mu_s g \quad ...(i)$$

Angular acceleration



$$\alpha = \frac{(\mu_s Mg)R}{\frac{2}{5}MR^2} = \frac{5}{2}\frac{\mu_s g}{R} \qquad \text{...(ii)}$$

For plank: Linear acceleration



$$a_2 = \frac{F - \mu_s Mg}{m} \qquad \text{...(iii)}$$

For no slipping acceleration of point B and A is same,

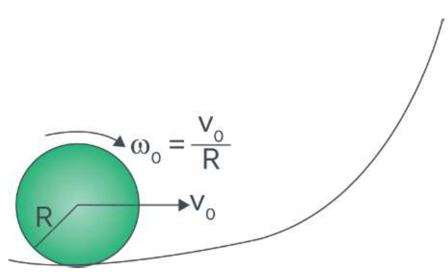
So, $a_2 = a_1 + R\alpha$

Solving the above four equation, we get

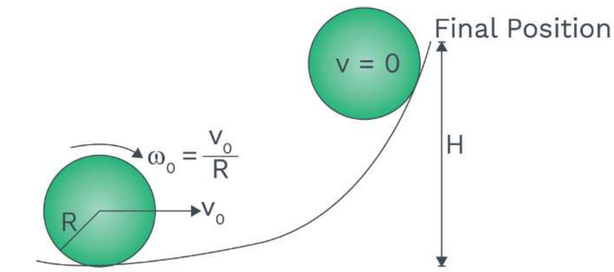
$$F = \mu_s g\left(M + \frac{7}{2}m\right)$$

Thus, maximum value of F can be $\mu_s g\left(M + \frac{7}{2}m\right)$.

Ex. Find the maximum height attained by solid sphere on a friction less track as shown.



Sol. Let us suppose that sphere attain a maximum height 'H' on the track.



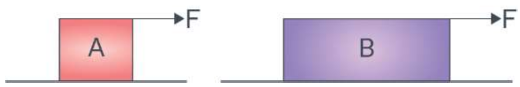
As the sphere move upward the speed is decreased because of gravity but there is no force to change the ω_0 (friction less track). So from energy conservation

$$\frac{1}{2}mv_0^2 + \frac{1}{2}I\omega_0^2 = mgH_{max} + \frac{1}{2}I\omega_0^2$$

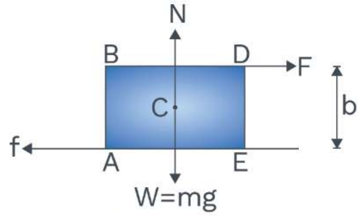
$$H_{max} = \frac{v_0^2}{2g}$$

TOPPLING:

You might have seen in your daily life that if a force 'F' is applied to a block 'A' of smaller width it is more likely to topple down, before sliding while if the same force 'F' is applied to an another block 'B' of broader base, chances of the sliding are more compared to its toppling. Have you ever thought why it happens so to understand it in a better way let us take an example.

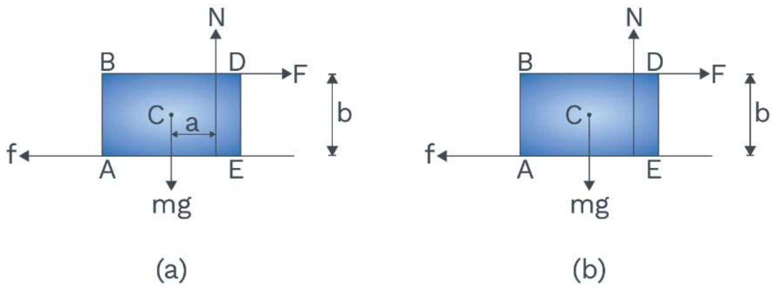


Suppose a force 'F' is applied at a height 'b' above the base AE of the block. Further, suppose the friction 'f' is sufficient to prevent sliding. In this case, if the normal reaction 'N' also passes through 'C', then despite the fact that the block is in translational equilibrium (F = f and N = mg), an unbalanced torque (because of the couple of forces 'F' and 'f') is there. This torque has a tendency to topple block about point 'E'. To cancel effect of this unbalanced torque normal reaction 'N' is shifted towards right a distance 'a' such that, the net anticlockwise torque is equal to the net clockwise torque, or



Fb = (mg) a

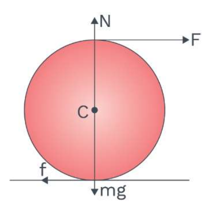
or $a = \frac{Fb}{mg}$



Now, as 'F' or 'b' (or both) are increased, distance 'a' also increases. But it cannot go beyond the right edge of block. So, in the extreme case (beyond which block will topple down), the normal reaction passes through 'E' as shown in figure.

If 'F' or 'b' are further increased, block will topple down. This is why the block having broader base has less chances of toppling in comparison to the block of smaller base. Because the block of bigger base has more margin for normal reaction to shift.

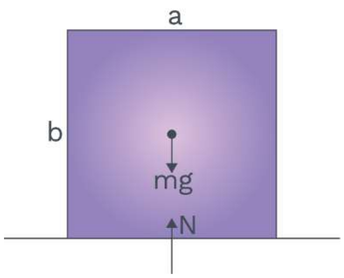
Why the rolling is so easy on the ground:



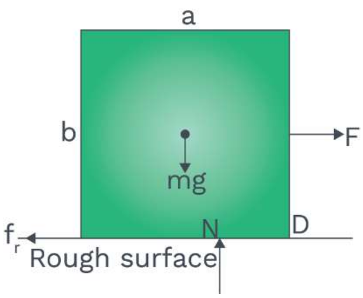
Because in this case normal reaction has zero margin to shift. so even if body is in translational equilibrium (F = f, N = mg) an unbalanced torque is left behind and body starts rolling clockwise. As soon as body starts rolling the force of friction is so adjusted (both in the magnitude and direction) that either pure rolling starts (if friction is sufficient enough) or body starts sliding. Let us take some examples related to toppling.

In many situations an external force is applied to the body to cause it to slide along the surface. In certain cases, the body many tip over before sliding ensues. This is known as toppling.

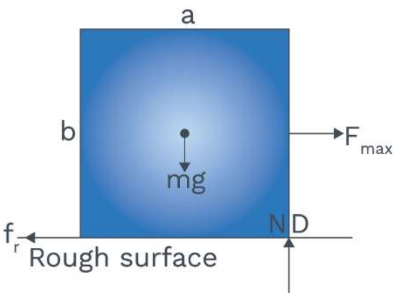
- There is a no horizontal force so pressure at the bottom is uniform and normal is collinear with mg.



- If a force is applied at C.O.M., pressure is not uniform normal shifts right so that the torque of 'N' can counter balance torque of friction.



- If F is continuously increased N keeps shifting towards right until it reaches the right most point D.
  Here we have assumed that surface is sufficiently rough so that there is no sliding on increase F to F_{max}.
  If force is increased any further, then torque of N can not counter balance torque of friction, f_r and body will topple.
  The value of force now is the max value for which toppling will not occur F_{max}.

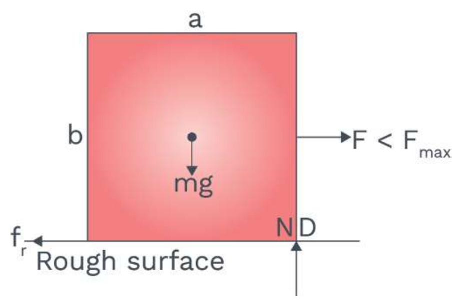


$$F_{max} = f_r$$
$$N = mg$$
$$f_r \cdot \frac{b}{2} = N \cdot \frac{a}{2}$$
$$\Rightarrow \quad f_r = \frac{Na}{b} = mg\frac{a}{b} \Rightarrow F_{max} = mg\frac{a}{b}$$

- If surface is not sufficiently rough and body slides before 'F' is increased to F_{max} = mg a/b then body will slide before the toppling. Once body starts sliding friction becomes constant and hence no toppling. This is the case if

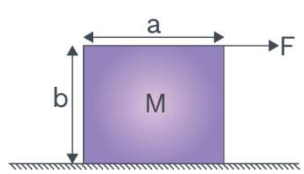
$$F_{max} > f_{limit} \quad \Rightarrow mg\frac{a}{b} > \mu mg$$

$$\mu < \frac{a}{b}$$

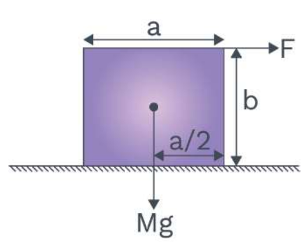


Condition for toppling $\mu \geq \frac{a}{b}$ in this case body will topple if $F > mg\frac{a}{b}$ but if $\mu < \frac{a}{b}$, body will not topple any value of F applied a COM.

Ex. Find the minimum value of F for the block to topple about an edge.



Sol. When the block is about to topple the normal reaction N shifts to the edge through O. FBD during toppling



Taking torque about O,

$$F(b) = Mg\left(\frac{a}{2}\right) \quad \Rightarrow F_{min} = \frac{Mga}{2b}$$

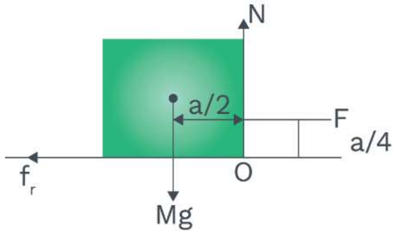
Ex. A uniform cube of side 'a' and mass 'm' rests on the rough horizontal table. The horizontal force 'F' is applied normal to one of faces at a point directly below centre of the face, at a height of a/4 above the base.

(i) What is the minimum value of 'F' for which the cube begins to tip about an edge?

(ii) Find the minimum value of μ_s so that toppling occurs.

(iii) If $\mu_s = \mu_{min}$, find minimum force for topping.

Sol. (i) In the limiting case normal reaction will pass through 'O'. The cube will tip about 'O' if torque of 'F' about 'O' exceeds the torque of mg.



Hence, $F\left(\frac{a}{4}\right) > mg\left(\frac{a}{2}\right)$

or $F > 2$ mg

Therefore, minimum value of 'F' is 2 mg

(ii) In this case since it is not acting at the COM, toppling can occur even after the body started sliding because increasing the torque of 'F' about COM.

Hence $\mu_{min} = 0$.

(iii) Now body is sliding before toppling, torque equation cannot be applied across it. It can now be applied about COM.

$$F \times \frac{a}{4} = N \times \frac{a}{2} \quad \text{...(i)}$$

$$N = mg \quad \text{...(ii)}$$

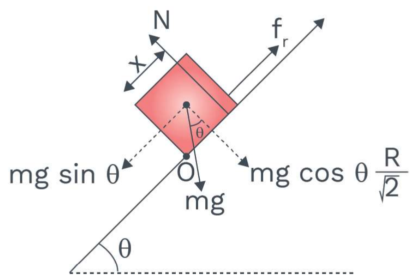
From equation (i) and (ii)

$$F = 2mg$$

Ex. A cubical block of mass 'm' and edge 'a' slides down a rough inclined plane of inclination θ with uniform velocity. Calculate the torque of normal force acting on block, about its centre.

Sol. To avoid toppling

Net torque about 'O' has to be zero



$$\Rightarrow \quad mg\sin\theta\left(\frac{a}{2}\right) - mg\cos\theta\left(\frac{a}{2}\right) + mg\cos\theta(x) = 0$$

$$\Rightarrow \quad x = \frac{a}{2}(1 - \tan\theta)$$

So torque of Normal reaction about centre

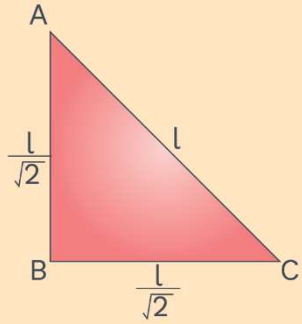
$$= mg\cos\theta\left(x - \frac{a}{2}\right)$$

$$= mg\cos\theta\left(\frac{a}{2}\tan\theta\right)$$

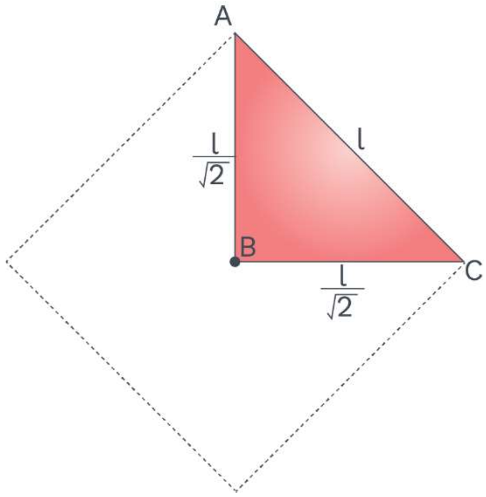
$$\tau = \frac{1}{2}mg\,a\sin\theta$$

EXAMPLES

Q1 **A uniform triangular plate of mass M whose vertices are ABC has lengths $\ell, \frac{\ell}{\sqrt{2}}$ and $\frac{\ell}{\sqrt{2}}$ as shown in figure. Find the moment of inertia of this plate about an axis passing through point B and perpendicular to the plane of the plate.**



Sol:



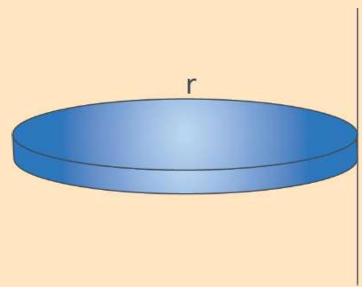
Assuming a square plate ACDE of mass 4M having centre B.

$$I_B = \frac{(4M)\ell^2}{6}$$

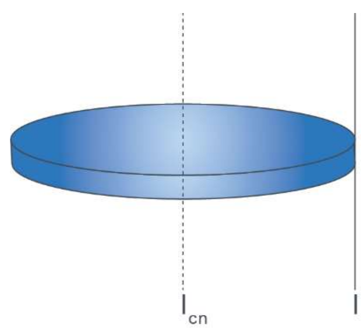
Moment of inertia of plate ABC

$$I_B = \frac{1}{4}\left(\frac{4M\ell^2}{6}\right) = \frac{M\ell^2}{6}.$$

Q2 **Calculate the radius of gyration of a uniform circular disk of radius r and thickness t about a line perpendicular to the plane of this disk and tangent to the disk as shown in figure.**



Sol:



$$I = I_{cm} + md^2$$

$$I = \frac{mr^2}{2} + mr^2 = \frac{3}{2}mr^2 = mK^2$$

$$\left(K = \sqrt{\frac{3}{2}}r\right).$$

Q3 **Two forces $\vec{F}_1 = 2\hat{i} - 5\hat{j} - 6\hat{k}$ and $\vec{F}_2 = -\hat{i} + 2\hat{j} - \hat{k}$ are acting on a body at the points (1, 1, 0) and (0, 1, 2) respectively. Find torque acting on the body about point (– 1, 0, 1).**

Sol:

$\vec{F}_1 = 2\hat{i} - 5\hat{j} - 6\hat{k}$ at point (1, 1, 0)

$\vec{F}_2 = -\hat{i} + 2\hat{j} - \hat{k}$ at point (0, 1, 2)

$$\vec{r}_1 = \left(1\hat{i} + \hat{j} + 0\hat{k}\right) - \left(-\hat{i} + 0\hat{j} + \hat{k}\right)$$

$$\vec{r}_1 = \left(2\hat{i} + \hat{j} - \hat{k}\right)$$

$$\vec{\tau}_1 = \vec{r}_1 \times \vec{F}_1 = \left(2\hat{i} + \hat{j} - \hat{k}\right) \times \left(2\hat{i} - 5\hat{j} - 6\hat{k}\right)$$

$$\vec{\tau}_2 = \vec{r}_2 \times \vec{F}_2 = \left(\hat{i} + \hat{j} + \hat{k}\right) \times \left(-\hat{i} + 2\hat{j} - \hat{k}\right)$$

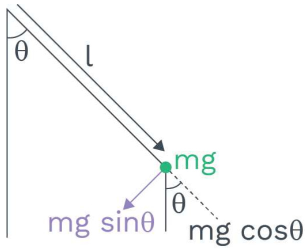
Total $\vec{\tau}_T = \vec{\tau}_1 + \vec{\tau}_2$

$$\vec{\tau}_T = \left(2\hat{i} + \hat{j} - \hat{k}\right) \times \left(2\hat{i} - 5\hat{j} - 6\hat{k}\right) + \left(\hat{i} + \hat{j} + \hat{k}\right) \times \left(-\hat{i} + 2\hat{j} - \hat{k}\right)$$

$$\vec{\tau}_T = \left(-14\hat{i} + 10\hat{j} - 9\hat{k}\right).$$

Q4 **A simple pendulum having bob of mass m and length ℓ is pulled aside to make an angle θ with the vertical. Find the magnitude of the torque of the weight of the bob about the point of suspension. At which position its torque is zero ? At which θ it is maximum ?**

Sol:



Torque of mg about point of suspension is

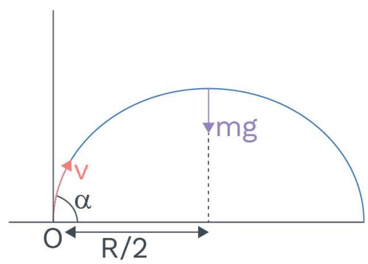
$\tau = (mg \sin\theta)(\ell)$

(When bob is at the lowest position $\tau = 0$)

Torque is maximum when string is horizontal that is $\theta = 90°$.

Q5 **A particle having mass m is projected with a speed v at an angle α with horizontal ground. Find the torque of the weight of the particle about the point of projection when the particle**
(a) is at the highest point
(b) reaches the ground

Sol: **(a)**

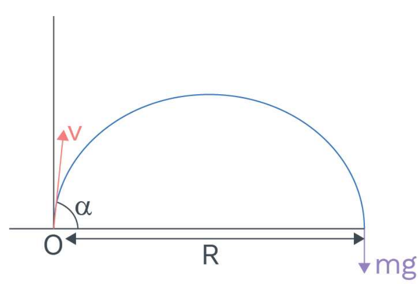


$$\tau_0 = mgR/2 = mg\left(\frac{v^2 \sin 2\alpha}{2g}\right)$$

$$\tau_0 = mg\frac{v^2 \sin 2\alpha}{2g} = \left(\frac{mv^2 \sin 2\alpha}{2}\right)$$

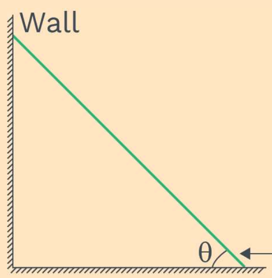
$$\tau_0 = \left(mv^2 \sin\alpha \cos\alpha\right).$$

(b)



τ_1 = mgR
= $(2mv^2 \sin\alpha \cos\alpha)$.

Q6 **Assuming frictionless contacts, determine the magnitude of external horizontal force P applied at the lower end for equilibrium of the rod as shown in figure. The rod is uniform and its mass is ' m'.**

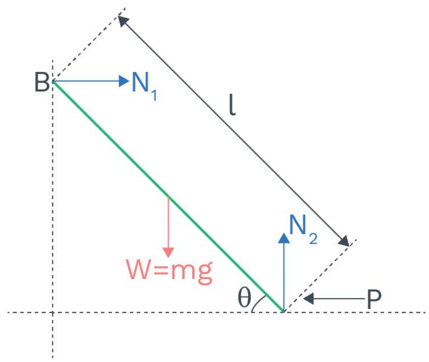


Sol: The F.B.D. of rod is as shown in the diagram

For the rod to be in the translational equilibrium

$N_1 = P$ ...(i)

$N_2 = W = mg$ ...(ii)



For rod to be in the rotational equilibrium, net torque on rod about any axis is zero.

∴ Net torque on rod about 'B' is zero

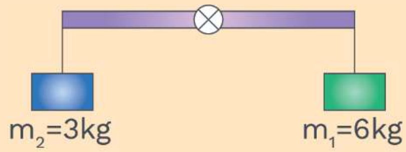
i.e.,

$$mg\frac{\ell}{2}\cos\theta - N_2\ell\cos\theta + P\ell\sin\theta = 0 \qquad ...(iii)$$

From equation (ii) and (iii) solving we get

$$P = \frac{mg}{2}\cot\theta.$$

Q7 **A rod of negligible mass of length ℓ = 2 m is pivoted at its centre and '2' masses of m_1 = 6 kg and m_2 = 3 kg are hung from its ends as shown in figure.**



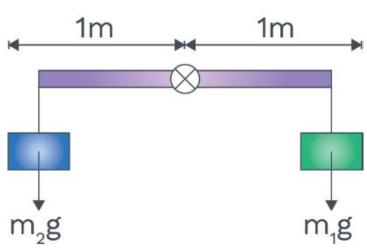
(a) What is initial angular acceleration of rod if it is horizontal initially.

(b) If rod is uniform and has a mass of m_3 = 3 kg.

(i) Calculate the initial angular acceleration of rod.

(ii) Find the amount of tension in the supports to the blocks of mass 3 kg and 6 kg (g = 10 m/s²).

Sol: (a)



Torque about hinge

$$(m_1g - m_2g)\left(\frac{\ell}{2}\right) = I\alpha$$

$$\alpha = \frac{(m_1 - m_2)g(\ell/2)}{m_1\left(\frac{\ell}{2}\right)^2 + m_2\left(\frac{\ell}{2}\right)^2}$$

$$\alpha = \frac{2(m_1 - m_2)g}{(m_1 + m_2)\ell}$$

$$\alpha = \frac{2(6-3)10}{2(6+3)} = \frac{10}{3} \text{ rad/sec}^2.$$

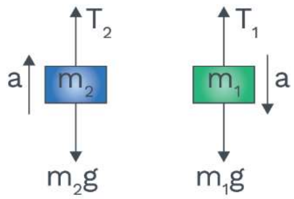
(b) If mass of the rod is 3 kg then torque about hinge

$$\left(m_1g - m_2g\right)\frac{\ell}{2} = I'\alpha'$$

$$\alpha' = \frac{\left(m_1 - m_2\right)g\left(\frac{\ell}{2}\right)}{\left[m_1\left(\frac{\ell}{2}\right)^2 + m_2\left(\frac{\ell}{2}\right)^2 + \frac{m_3\ell^2}{12}\right]}$$

$$\alpha' = \frac{2\left(m_1 - m_2\right)g}{\ell\left[m_1 + m_2 + \frac{m_3}{3}\right]}$$

$$= \frac{2\left(6-3\right)10}{2\left[6+3+\frac{3}{3}\right]} = 3 \text{ rad/s}^2$$



For m_1 block

$$m_1g - T_1 = m_1a$$

$$T_1 = \left(m_1g - \frac{m_1\ell\alpha}{2}\right)$$

$$T_1 = 60 - \frac{6\times2\times3}{2} = 42 \text{ N}$$

For m_2 block

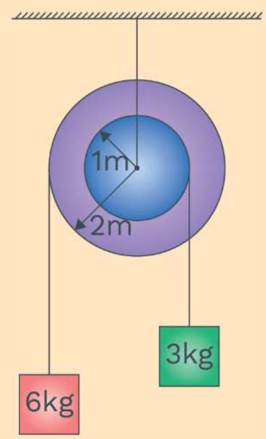
$$T_2 - m_2g = m_2a$$

$$T_2 = m_2g + m_2\frac{\ell\alpha}{2} = 30 + \frac{3\times2\times3}{2}$$

T_2 = 39 N.

Q8

The moment of inertia of a pulley system as shown is 3 kg – m². The radius of bigger and smaller pulleys are '2 m' and '1 m' respectively. As the system is released from the rest, find angular acceleration of the pulley system. (Assume that there is no slipping between the string and pulley and string is light) [Take g = 10 m/s²]



Sol: Let α be the angular acceleration of pulley system.

For 6 kg block

$6g - T_1 = 6(2\alpha)$ ...(i)

For 3 kg block

$T_2 - 3g = 3\alpha$ ...(ii)

For pulley system

$\Rightarrow 2T_1 - T_2 = I\alpha = 3\alpha$ ...(iii)

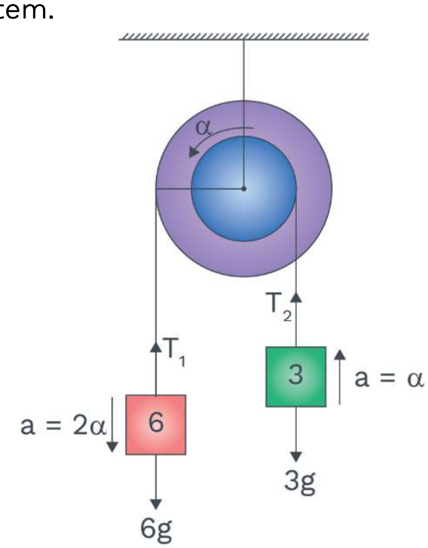
From the equation (i) and (ii) putting values of T_1 and T_2.

$\Rightarrow 2\left[6g - 12\alpha\right] - \left[3g + 3\alpha\right] = 3\alpha$

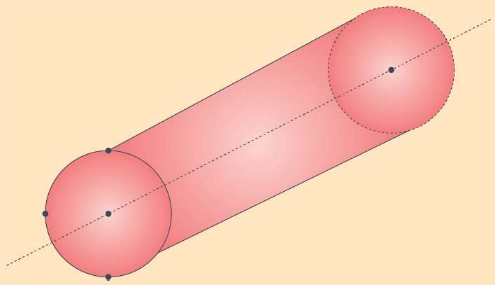
$\Rightarrow 12g - 24\alpha - 3g - 3\alpha = 3\alpha$

$\Rightarrow 30\alpha = 9g$

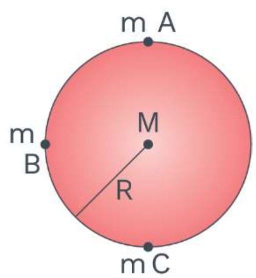
$\Rightarrow \alpha = \frac{90}{30} = 3 \text{ rad/s}^2$.



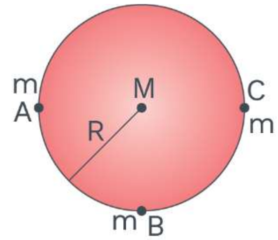
Q9 **A solid cylinder of mass M = 1 kg and radius R = 0.5 m is pivoted at its centre and has three particles of mass m = 0.1 kg mounted at its perimeter in vertical plane as shown. The system is initially at the rest. Find the angular speed of cylinder, when it has swung through 900 in anticlockwise direction.**
[Take g = 10 m/s²]



Sol:



After rotating 90°



Using Energy conservation

$$U_i + K_i = U_f + K_f$$

$$\left(2mgR + mgR + 0 + 0\right) = mgR + O + mgR + \frac{1}{2}I\omega^2$$

$$3mgR = 2mgR + \frac{1}{2}I\omega^2$$

$$\left(mgR = \frac{1}{2}I\omega^2\right) \quad \left[I = \frac{MR^2}{2} + 3mR^2\right]$$

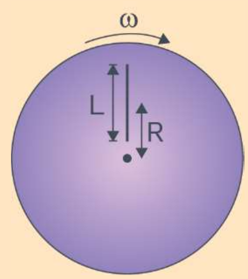
$$2mg = \left(\frac{MR}{2} + 3mR\right)\omega^2$$

$$\omega^2 = \left(\frac{4mg}{MR + 6mR}\right)$$

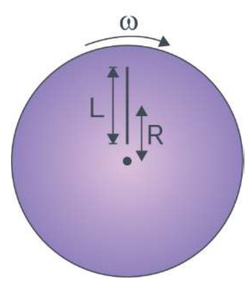
$$\omega = \sqrt{\frac{4mg}{MR + 6mR}} = \sqrt{\frac{4 \times 0.1 \times 10}{1 \times 0.5 + 6 \times 0.1 \times 0.5}}$$

$$\Rightarrow \quad \omega = \sqrt{5} \text{ rad/s.}$$

Q10 **A uniform rod of mass 'm' and length 'L' lies radially on a disc rotating with angular speed ω in a horizontal plane about its axis. The rod does not slip on disc and the centre of the rod is at a distance 'R' from the centre of the disc. Find out the kinetic energy (K.E.) of the rod.**



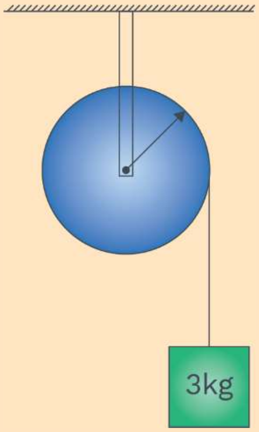
Sol: Moment of inertia of rod w.r.t. the axis through the centre of the disc is : (by parallel axis theorem).



$$I = \frac{mL^2}{12} + mR^2$$

And K.E. of rod w.r.t. disc = $\frac{1}{2}I\omega^2 = \frac{1}{2}m\omega^2\left[R^2 + \frac{L^2}{12}\right]$.

Q11 **The moment of inertia of pulley system as shown is 3 kgm². Its radius is '1 m'. The system is released from the rest. What will be the linear velocity of block, when it has descended through '40 cm'. (Suppose that there is no slipping between the string & pulley and string is light).**
[Take g = 10 m/s²]



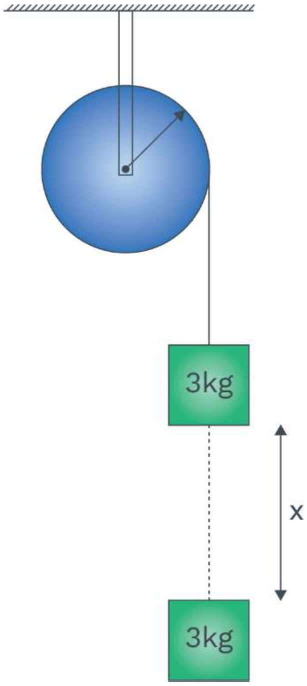
Sol: When the block is descended through 'x', let its velocity be 'v'.
From energy conservation

$$mgx = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2$$

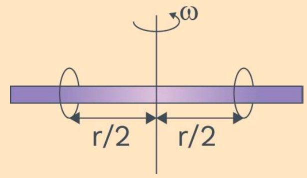
$$mgx = \frac{1}{2}I\left(\frac{v}{r}\right)^2 + \frac{1}{2}mv^2$$

$$\Rightarrow \quad 2mgx = v^2\left[\frac{I}{r^2} + m\right]$$

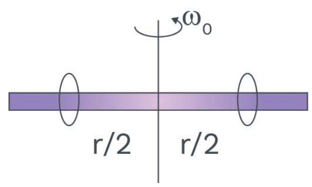
Putting all given values v = 2 m/s.

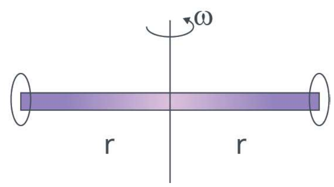


Q12 **Two beads (each of mass m) can move freely in a frictionless wire whose rotational inertia w.r.t. the vertical axis is 'I'. The system is rotated with the angular velocity of ω_0 when the beads are at a distance of r/2 from axis. What is the angular velocity of system when the beads are at a distance 'r' from the axis ?**



Sol:





Initial position   Final position

No external torque so $\vec{L} = \text{constant}$

$L_i = L_f$

$(I_i\omega_0 = I_f\omega)$

$$\left(I + \frac{mr^2}{4} + \frac{mr^2}{4}\right)\omega_0 = \left(I + mr^2 + mr^2\right)\omega$$

$$\left[\omega = \left(\frac{I + \frac{mr^2}{2}}{I + 2mr^2}\right)\omega_0\right].$$

Q13 **A system consists of '2' identical small balls of mass 2 kg each connected to two ends of a '1 m' long light rod. The system is rotating about the fixed axis through centre of the rod and perpendicular to it at the angular speed of 9 rad/s. An impulsive force of average magnitude 10 N acts on one of the masses in direction of its velocity for 0.20 s. Calculate new angular velocity of the system.**

Sol: $\left[\vec{\tau}_{ext} = \frac{d\vec{L}}{dt}\right]$

$d\vec{L} = \left(\vec{\tau}_{ext} dt\right)$

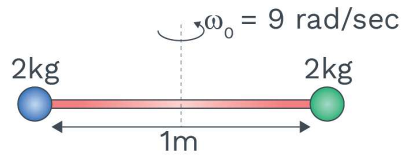
$\left(\vec{r} \times \vec{F}\right) = \frac{d\vec{L}}{dt}, \quad d\vec{L} = \left(\vec{r} \times \vec{F}\right)dt$

$\left(L_f - L_i\right) = \int\left(F\ r\ dt\right), \quad \left(I\omega_f - I\omega_i\right) = \left(r F t\right)$

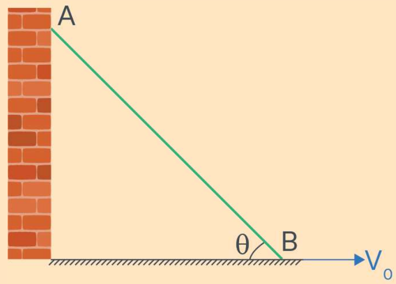
$I\left(\omega - 9\right) = \left(0.5\right)\left(10\right)\left(0.20\right)$

$$\omega - 9 = \frac{0.5 \times 10 \times 0.20}{2\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right)^2} = 1$$

ω = 10 rad/s.



Q14 **The end 'B' of uniform rod 'AB' which makes an angle θ with floor is being pulled with a velocity of v_0 as shown. Taking length of the rod as ℓ, calculate the following at instant when θ = 37°.**



(a) The velocity of end A
(b) The angular velocity of rod
(c) Velocity of CM of the rod.

Sol: (a) $v_a \sin\theta = v_0 \cos\theta$

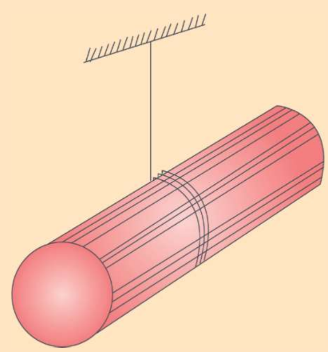
$$v_a = \frac{v_0}{\tan\theta} = \frac{4v_0}{3}.$$

(b) $$\omega = \frac{v_0 \sin\theta + v_A \cos\theta}{\ell} = \frac{3v_0 + 4\left(\frac{4v_0}{3}\right)}{5\ell} = \frac{9v_0 + 16v_0}{15\ell} = \frac{5v_0}{3\ell}.$$

(c) $$v_x = \frac{\ell}{2}\left(\frac{v_{Ax} + v_{Bx}}{\ell}\right) = \frac{v_0}{2}.$$

$$v_y = \frac{1}{2}\left(v_{Ay} + v_{By}\right) = \frac{2v_0}{3}.$$

Q15 **A string is wrapped over a curved surface of the uniform solid cylinder and the free end is fixed with the rigid support. The solid cylinder moves down, unwinding the string. Find the downward acceleration of solid cylinder.**



Sol: For linear motion:

$mg - T = ma$ ...(i)

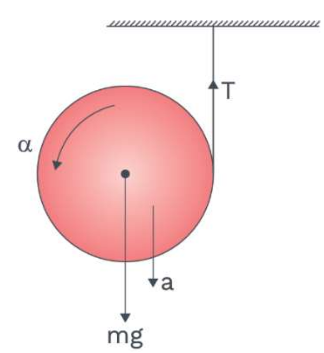
For angular motion:

$$T.R. = \left(\frac{mR^2}{2}\right)\alpha$$

$$T = \frac{mR\alpha}{2}$$ ...(ii)

For no slipping:

$a = R\alpha$ ...(iii)

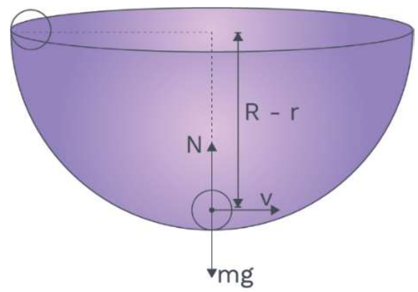


From equation (i), (ii) & (iii)

$$a = \frac{2}{3}g.$$

Q16 **A uniform disk of mass 'm' is released from rest from rim of a fixed hemi-spherical bowl so that it rolls along the surface. If rim of the hemisphere is kept horizontal, find the normal force exerted by the bowl on the disk when it reaches bottom of bowl.**

Sol:



Let R and r be the radii of hemispherical bowl & disc respectively

From energy conservation,

$$mg(R-r) = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

For pure rolling,

$v = r\omega$

$$mg(R-r) = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{1}{2}mr^2\right)\left(\frac{v}{r}\right)^2$$

$$mg(R-r) = \frac{3}{4}mv^2 \qquad \text{...(i)}$$

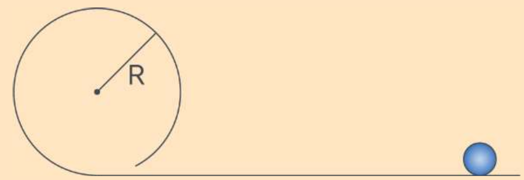
From FBD of bottom:

$$N - mg = \frac{mv^2}{(R-r)} \qquad \text{...(ii)}$$

From equation (i) & (ii),

$$N = \frac{7}{3}\ mg.$$

Q17 **There is a rough track, a portion of which is in form of a cylinder of radius 'R' as shown. Find minimum linear speed of a uniform ring of radius r with which it should be set rolling without sliding on horizontal part so that it can complete round the circle without sliding on the cylindrical part.**



Sol: Let v_1 and v_2 be minimum speed of ring of bottom and top of cylindrical part

At top of path

$$N + mg = \frac{mv_2^2}{(R - r)}$$

for minimum speed N = 0

$$v_2^2 = g(R - r) \qquad \text{...(i)}$$

From energy conservation between bottom and top point of cylindrical part

$$\frac{12}{2}mv_1^2 + \frac{1}{2}I\omega_1^2 = 2mg(R - r) + \frac{1}{2}mv_2^2 + \frac{1}{2}I\omega_2^2$$

For pure rolling $\omega_1 = \frac{v_1}{r}, \omega_2 = \frac{v_2}{r}$

$$\Rightarrow \frac{1}{2}mv_1^2 + \frac{1}{2}(mr^2)\frac{v_1^2}{r^2} = 2mg(R - r) + \frac{1}{2}mv_2^2 + \frac{1}{2}(mr^2)\frac{v_2^2}{r^2}$$

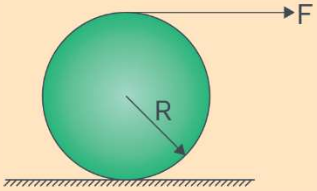
$$\Rightarrow mv_1^2 = 2mg(R - r) + mv_2^2 \qquad \text{...(ii)}$$

From equation (i) & (ii)

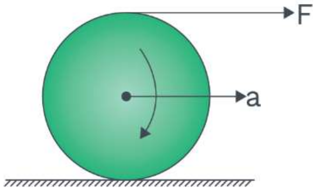
$$\Rightarrow mv_1^2 = 2mg(R - r) + mg(R - r)$$

$$\Rightarrow v_1 = \sqrt{3g(R - r)}.$$

Q18 **A uniform solid sphere of radius 'R' is placed on a smooth horizontal surface. It is pulled by a constant force acting along tangent from the highest point. Calculate the distance travelled by the COM of the solid sphere during the time it makes one full revolution.**



Sol:



For linear motion,

$F = ma$ ...(i)

For angular motion,

$$F.R. = \left(\frac{2}{5}mR^2\right)\alpha$$

$$\alpha = \frac{5F}{2mR} \quad ...(ii)$$

$$\theta = \omega_0 t + \frac{1}{2}at^2$$

$$t^2 = \left[\frac{8\pi mR}{5F}\right]$$

Distance covered by the sphere during one full rotation

$$s = ut + \frac{1}{2}at^2 = 0 + \frac{1}{2}\left(\frac{F}{m}\right)\left(\frac{8\pi mR}{5F}\right)$$

$$s = \frac{4\pi R}{5}.$$

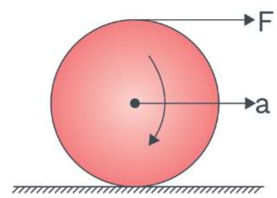
Q19 **A uniform hollow sphere of mass m = 1 kg is placed on the rough horizontal surface for which coefficient of static friction between the surfaces in contact is μ = 2/5. Find maximum constant force which can be applied at highest point in the horizontal direction so that sphere can roll without slipping. (Take g = 10 m/s²)**

Sol: For linear motion

$F + f = ma$ ...(i)

for angular motion

$(F - f)R = \left(\frac{2}{3}mR^2\right)\alpha$ ...(ii)



for pure rolling $a = R\alpha$ ...(iii)

From equation (i), (ii) & (iii)

$$\frac{F+f}{F-f} = \frac{3}{2}$$

$\Rightarrow$ $F = 5f$

$F_{max} = 5f_{max}$

$F_{max} = 5\mu mg = 20$ N.

Q20 **A uniform rod having mass 'm_1' and length 'L' lies on a smooth horizontal surface. A particle of mass 'm_2' moving with the speed 'u' on the horizontal surface strikes free rod perpendicularly at an end and it sticks to the rod.**

(a) Calculate velocity of the COM 'C' of the system constituting "the rod plus the particle".

(b) Calculate the velocity of the particle w.r.t. 'C' before the collision.

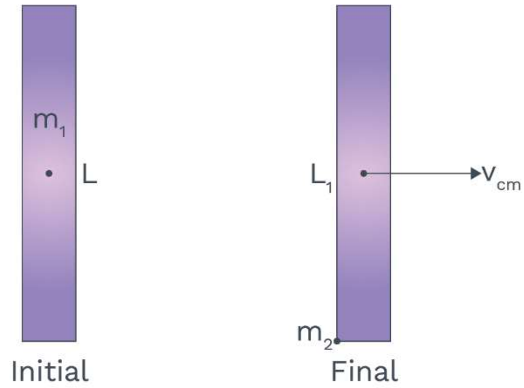
(c) Calculate the velocity of the rod w.r.t. 'C' before the collision

(d) Calculate the angular momentum of particle and of the rod about the com 'C' before the collision.

(e) Calculate the moment of inertia of rod plus particle about vertical axis through centre of mass 'C' after the collision.

(f) Calculate velocity of the com C and the angular velocity of the system about the centre of mass after the collision.

Sol:



(a) $P_i = m_2 v$

$$P_f = (m_1 + m_2) v_{cm}$$

$$m_2 v = (m_1 + m_2) v_{cm}$$

$$v_{cm} = \left(\frac{m_2 v}{m_1 + m_2}\right)$$

(b) $v' = (u - v_{cm})$

$$v' = v - \frac{m_2 u}{m_1 + m_2} = \left(\frac{m_1 u}{m_1 + m_2}\right)$$

(c) $v' = -v_{cm} = \left(\frac{-m_1 u}{m_1 + m_2}\right)$

(d) $X_{cm} = \dfrac{m_1(0) + m_2\left(\frac{L}{2}\right)}{(m_1 + m_2)} = \dfrac{m_2 L}{2(m_1 + m_2)}$

$$L' = \frac{L}{2} - \frac{m_2 L}{2(m_1 + m_2)} \Rightarrow L' = \frac{1}{2}\left(\frac{m_1 L}{m_1 + m_2}\right)$$

momentum of particle

$$\Rightarrow P_i = \left[m_2 (u - V_{cm}) L'\right] = m_2 \frac{m_1 L}{2(m_1 + m_2)}\left(u - \frac{m_2 u}{m_1 + m_2}\right) = \left(\frac{m_2 m_1^2 u L}{2(m_1 + m_2)}\right)$$

Momentum for rod = $m_1 v_{cm} \times L_{cm} = \dfrac{m_1 L}{2} \dfrac{m_2^2 u}{(m_1 + m_2)^2}$

(e) For particle

$$I_1 = m_2L^2 = \frac{m_2 m_1^2 L^2}{4\left(m_1 + m_2\right)^2}$$

$$I_2 = \frac{m_1L^2}{12} + m_1\left(\frac{m_2L}{2\left(m_1 + m_2\right)}\right)^2$$

$$I = I_1 + I_2$$

$$= \frac{m_1\left(m_1 + 4m_2\right)L^2}{12\left(m_1 + m_2\right)}$$

(f) Velocity of centre of mass

$$= -\left(\frac{m_2 v}{m_1 + m_2}\right)$$

Using angular momentum conservation

$$m_2 v \times L_{cm} = I_{cm}\omega$$

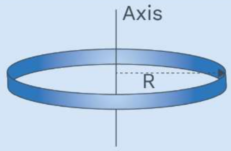
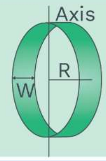
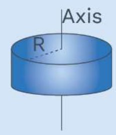
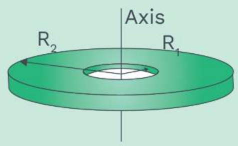
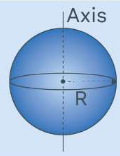
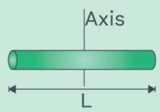
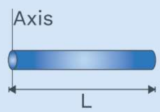
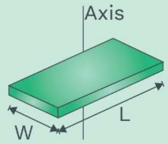
$$= m_2 u \frac{m_1 L}{2\left(m_1 + m_2\right)} = I_{cm}.\omega$$

$$= m_2 u \frac{m_1 L}{2\left(m_1 + m_2\right)} = \frac{m_1\left(m_1 + 4m_2\right)L^2}{12\left(m_1 + m_2\right)} \times \omega$$

$$\Rightarrow \quad \omega = \frac{6m_2 v}{\left(m_1 + 4m_2\right)L}.$$

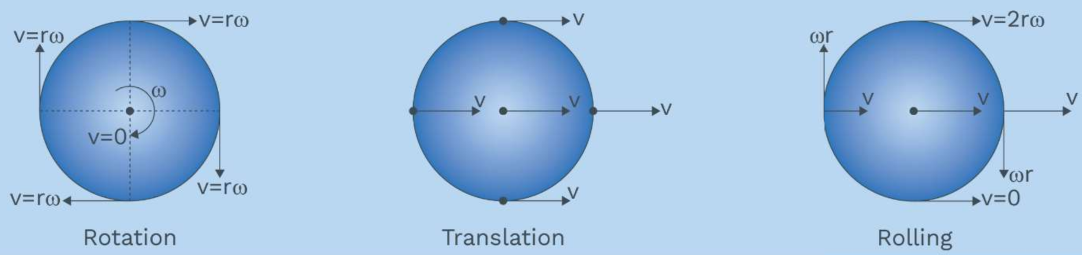
Mind Map

Rotational Motion

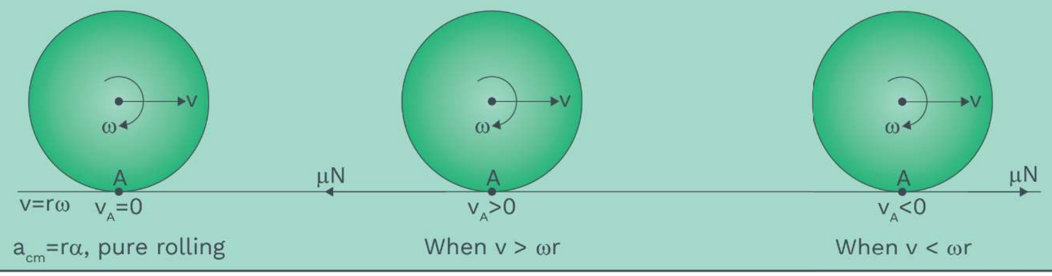
S.No.	Object	Location of Axis	Figure	Moment of Inertia (I)
(a)	Thin hoop, radius R	Through center		MR^2
(b)	Thin hoop, radius R width W	Through central diameter		$\frac{1}{2}MR^2 + \frac{1}{12}MW^2$
(c)	Solid cylinder, radius R	Through center		$\frac{1}{2}MR^2$
(d)	Hollow cylinder, inner radius R_1 outer radius R_2	Through center		$\frac{1}{2}M(R_1^2 + R_2^2)$
(e)	Uniform sphere, radius R	Through center		$\frac{2}{5}MR^2$
(f)	Long uniform rod, length L	Through center		$\frac{1}{12}ML^2$
(g)	Long uniform rod, length L	Through end		$\frac{ML^2}{3}$
(h)	Rectangular thin plate, length L width W	Through center		$\frac{1}{12}M(L^2 + W^2)$

ROLLLING OVER A PLANE SURFACE

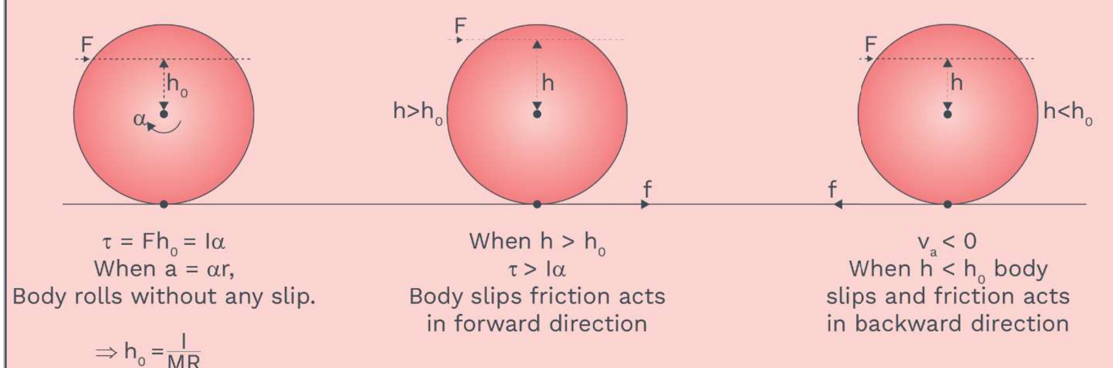
Rolling is a combination of translatory and rotatory motion.

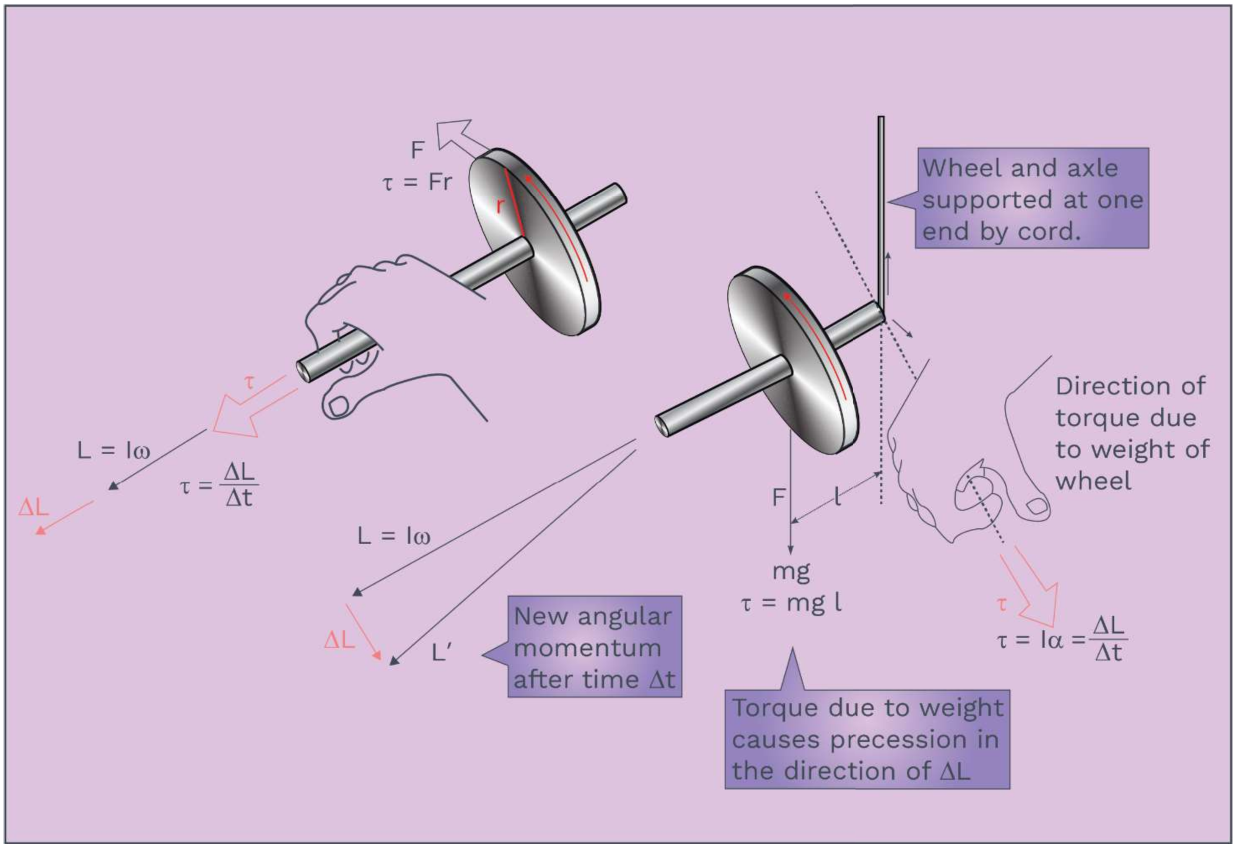


In pure rolling motion, friction is either 0 or static.



When a force is applied over a body, it may roll or slips depending on torque produced by force.




F
τ = Fr
r
Wheel and axle supported at one end by cord.
τ
L = Iω
τ = ΔL/Δt
ΔL
L = Iω
ΔL
L′
New angular momentum after time Δt
F
l
mg
τ = mg l
Direction of torque due to weight of wheel
τ
τ = Iα = ΔL/Δt
Torque due to weight causes precession in the direction of ΔL