Mathematical Tools

Mathematical Tools

Definition

Mathematics is the language of physics. It becomes very easy to describe, understand and apply the physical principles, if we have a good knowledge of mathematics. The techniques of mathematics such as algebra, trigonometry, calculus, graph and logarithm can be used to make predictions from the basic equation.In this introductory chapter we will learn some fundamental mathematics.

Quadratic Equation

An algebraic equation of second order (highest power of the variable is equal to 2) is called a quadratic equation. The equation $ax^2 + bx + c = 0$....(i) is the general form of quadratic equation The general solution of above equation is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If values of x are x_1 and x_2 then $x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$

and $x_2 = \frac{-b - \sqrt{b^2 - 4\,ac}}{2\,a}$.

Here x_1 and x_2 are called roots of equation (i). We can easily see that sum of roots = $x_1 + x_2 = -\frac{b}{a}$

and product of roots = $x_1 x_2 = \frac{c}{a}$

Concept Reminder

The name quadratic comes from "quad" meaning square, because the variable is squared (like x^2)

Ex. Solve the equation $10x^2 - 27x + 5 = 0$

Sol. By comparing the given equation with standard equation $ax^2 + bx + C$, we get

$a = 10$, $b = -27$, and $c = 5$

Using $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{-(-27) \pm \sqrt{(-27)^2 - 4 \times 10 \times 5}}{2 \times 10} = \frac{27 \pm 23}{20}$$

KEY POINTS

- Quadratic equation
- Binomial theorem

$\therefore \quad x_1 = \frac{27+23}{20} = \frac{5}{2}$ and $x_2 = \frac{27-23}{20} = \frac{1}{5}$

$\therefore$ Roots of the equation are $\frac{5}{2}$ and $\frac{1}{5}$.

Ex. Find roots of equation $2x^2 - x - 3 = 0$.

Sol. Compare this equation with standard quadratic equation $ax^2 + bx + c = 0$,

we have, a = 2, b = –1, c = –3.

Now from

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a};\; x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-3)}}{2(2)}$$

$$x = \frac{1 \pm \sqrt{1+24}}{4} = \frac{1 \pm 5}{4}$$

$$\Rightarrow \quad x = \frac{6}{4} \text{ or } x = \frac{-4}{4}$$

$$\Rightarrow \quad x = \frac{3}{2} \text{ or } x = -1$$

Ex. A balloon is at a height of 81 m and is ascending upwards with a velocity of 12m/s. A body of 2 kg weight is dropped from it. If g =10 m/s², when body will reach the surface of the earth? Hint apply $h = ut + \frac{1}{2}gt^2$.

Sol. As the balloon is going up we will take initial velocity of falling body = –12m/s, h = 81m, g = +10m/s²

By applying $h = ut + \frac{1}{2}gt^2;\; 81 = -12t + \frac{1}{2}(10)t^2$

$$\Rightarrow \quad 5t^2 - 12t - 81 = 0$$

$$\Rightarrow \quad t = \frac{12 \pm \sqrt{144 + 1620}}{10} = \frac{12 \pm \sqrt{1764}}{10} \approx 5.4 \text{ sec}.$$

Binomial Theorem

If n is any number positive, negative or fraction and x is any real number, such that x < 1 i.e. x lies between – 1 and + 1 then according to binomial theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ...$$

Here 2 ! (Factorial 2) = 2 × 1, 3! (Factorial 3) = 3 × 2 × 1 and 4! (Factorial 4) = 4 × 3 × 2 × 1

If |x| << 1 then only the first two terms are significant. It is so because the values of second and the higher order terms being very small, can be neglected. So, the expression can be written as

$(1 + x)^n = 1 + nx$

$(1 + x)^{-n} = 1 - nx$

$(1 - x)^n = 1 - nx$

$(1 - x)^{-n} = 1 + nx$

Ex. Evaluate $(1001)^{1/3}$ upto six places of decimal.

Sol. $(1001)^{1/3} = (1000 + 1)^{1/3} = 10(1 + 0.001)^{1/3}$

By comparing the given equation with standard equation

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots\ldots$$

Here x = 0.001 and n = 1/3

$\therefore$ $10(1+0.001)^{1/3}$

$$= 10\left[1 + \frac{1}{3}(0.001) + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\times(0.001)^2}{2!} + \ldots\right]$$

$$= 10\left[1 + 0.00033 - \frac{1}{9}(0.000001) + \ldots\right]$$

= 10[1.0003301] = 10.003301 (Approx.)

Rack your Brain

Find the value of $(999)^{\frac{1}{3}}$ up to 5 places of decimal.

Ex. The value of acceleration due to gravity (g) at a height h above the surface of earth is given by $g' = \frac{gR^2}{(R+h)^2}$. If h<<R then find approximate value of g'.

Sol. $g' = g\left(\frac{R}{R+h}\right)^2 = g\left(\frac{1}{1+h/R}\right)^2 = \left(1+\frac{h}{R}\right)^{-2}$

$$= g\left[1 + (-2)\frac{h}{R} + \frac{(-2)(-3)}{2!}\left(\frac{h}{R}\right)^2 + \ldots\ldots\right]$$

$$g' = g\left(1 - \frac{2h}{R}\right)$$

(If h<<R then by neglecting higher power of $\frac{h}{R}$.)

Ex. The mass m of a body moving with a velocity v is given by $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$

where m_0 = rest mass of body = 10 kg and c = speed of light = 3×10^8 m/s. Find the value of m, at v = 3×10^7 m/s.

Sol. $m = m_0\left(1-\frac{v^2}{c^2}\right)^{-1/2} = 10\left[1-\left(\frac{3\times10^7}{3\times10^8}\right)^2\right]^{-1/2}$

$$= 10\left[1-\frac{1}{100}\right]^{-1/2}$$

$$\approx 10\left[1-\left(-\frac{1}{2}\right)\left(\frac{1}{100}\right)\right] = 10+\frac{10}{200} \approx 10.05 \text{ kg}$$

Arithmetic Progression

It is a sequence of numbers which are arranged in increasing order and having a constant difference between them.

Example : 1, 3, 5, 7, 9, 11, 13, ...

or 2, 4, 6, 8, 10, 12, ...

In general arithmetic progression can be written as a_0, a_1, a_2, a_3, a_4, a_5 ....

(i) n^{th} term of arithmetic progression $a_n = a_0 + (n-1)d$

a_0 = First term, n = Number of terms,

d = Common difference = $(a_1 - a_0)$

or $(a_2 - a_1)$ or $(a_3 - a_2)$

(ii) Sum of arithmetic progression

$$S_n = \frac{n}{2}\left[2a_0 + (n-1)d\right] = \frac{n}{2}\left[a_0 + a_n\right]$$

Concept Reminder

- n^{th} term of arithmetic progression is

$$a_n = a_0 + (n-1)d$$

- Sum of arithmetic progression is

$$S_n = \frac{n}{2}\left[a_0 + a_n\right]$$

Ex. Find the sum of series 7 + 10 + 13 + 16 + 19 + 22 + 25

Sol. $S_n = \frac{n}{2}\left[a_0 + a_n\right] = \frac{7}{2}\left[7+25\right] = 112$

[As $n = 7; a_0 = 7; a_n = a_6 = 25$]

Geometric Progression

It is a sequence of numbers in which every term is obtained by multiplying the previous term by a constant quantity. This constant quantity is called the common ratio.

Example: 4, 8, 16, 32, 64, 128 ......

or 5, 10, 20, 40, 80, ...

In general geometric progression can be written as a, ar, ar^2, ar^3, ar^4, ...

Here a = first term, r = common ratio

(i) Sum of 'n' terms of G.P.

$$\text{if } r < 1,\ S_n = \frac{a(1-r^n)}{1-r}$$

$$\text{if } r > 1,\ S_n = \frac{a(r^n-1)}{r-1}$$

(ii) Sum of infinite terms of G.P.

$$\text{if } r < 1,\ S_\infty = \frac{a}{1-r}$$

$$\text{if } r > 1,\ S_\infty = \frac{a}{r-1}$$

Ex. Find the sum of series

$$Q = 2q + \frac{q}{3} + \frac{q}{9} + \frac{q}{27} + \ldots\ldots$$

Sol. Above equation can be written as

$$Q = q + \left[q + \frac{q}{3} + \frac{q}{9} + \frac{q}{27} + \ldots\ldots\right]$$

By using the formula of sum of infinite terms of

$$\text{G.P. } Q = q + \left[\frac{q}{1-\frac{1}{3}}\right] = q + \frac{3}{2}q = \frac{5}{2}q$$

Some common formulae of algebra

(i) $(a + b)^2 = a^2 + b^2 + 2ab$

(ii) $(a - b)^2 = a^2 + b^2 - 2ab$

Definitions

The constant amount we multiply by each term to get next term in Geometric progression is known as Common ratio.

Rack your Brain

Find the sum of following infinite G.P.

$$\frac{1}{3}, \frac{-2}{9}, \frac{4}{27}, \frac{-8}{81} \ldots\ldots$$

KEY POINTS

- Arithmetic progression
- Geometric progression
- Algebra
- Componendo and dividendo

(iii) $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$
(iv) $(a + b)(a - b) = a^2 - b^2$
(v) $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$
(vi) $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$
(vii) $(a + b)^2 - (a - b)^2 = 4ab$
(viii) $(a + b)^2 + (a - b)^2 = 2(a^2 + b^2)$
(ix) $a^3 - b^3 = (a - b)(a^2 + b^2 + ab)$
(x) $a^3 + b^3 = (a+b)(a^2+b^2 - ab)$

Componendo and dividendo: If $\frac{a}{b} = \frac{c}{d}$ then

$$\frac{a+b}{a-b} = \frac{c+d}{c-d}$$

Trigonometry

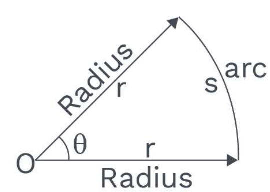
The word trigonometry is derived from two Greek words 'trigonon' and 'metron'. The word 'trigonon' means a triangle and the word 'metron' means a measure. Hence the word trigonometry means the study of properties of triangles. This involves the measurement of angles and lengths.

Concept Reminder

Trigonometry involves the measurement of angles and lengths.

Angle

It is measure of change in direction.



$$\text{Angle}(\theta) = \frac{\text{Arc(s)}}{\text{Radius(r)}}$$

Angles measured in anticlockwise and clockwise direction are usually taken positive and negative respectively.

Rack your Brain

Which of the following are the acute angles.

$30°, -120°, -73°, 150°,$

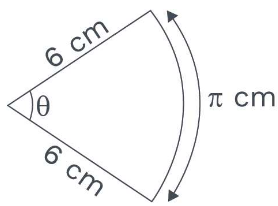
$\frac{\pi}{3}, \frac{-\pi}{6}, \frac{2\pi}{3}.$

Ex. The circular wire of radius 7 cm is cut and bent again into an arc of a circle of radius 12 cm. Find angle subtended by an arc at the centre of the circle.

Sol. Given the diameter of circular wire = 14 cm. Therefore length of wire $= 14\pi\,\text{cm}$.

Hence, required angle $= \dfrac{7\pi}{6} \times \dfrac{180°}{\pi} = 210°$

Ex. A circular arc of length π cm. Find angle subtended by it at the centre in radian and degree.



Sol. $\theta = \dfrac{s}{r} = \dfrac{\pi\ \text{cm}}{6\,\text{cm}} = \dfrac{\pi}{6}\text{rad} = 30°$

System of measurement of an angle

[A] Sexagesimal system

In this system, angle is measured in degrees.

In this system, 1 right angle = 90°, 1° = 60' (arc minutes), 1' = 60'' (arc seconds)

[B] Circular System

In this system, angle is measured in radian.

if arc = radius then $\theta = 1$ rad

Relation between degrees and radian

2 rad = 360°

π rad = 180° $\Rightarrow$ 1 rad $= \dfrac{180°}{\pi} = 57.3°$

To convert from degree to radian multiply by $\dfrac{\pi}{180°}$

To convert from radian to degree multiply by $\dfrac{180°}{\pi}$

Concept Reminder

$1° = \dfrac{\pi}{180}\text{rad}$

Mathematical Tools

Ex. The degree measure corresponding to the given radian $\left[\frac{2\pi}{15}\right]^c$

Sol. We have, π radians = 180°

$$\therefore \quad 1^c = \left[\frac{180}{\pi}\right]^0;$$

$$\therefore \quad \left[\frac{2\pi}{15}\right]^c = \left[\frac{2\pi}{15} \times \frac{180}{\pi}\right]^0 = 24^\circ$$

KEY POINTS

- Angle
- Radian
- Arc minute
- Arc seconds

Ex. The angles of a quadrilateral are in A.P. and the greatest angle is 120°, the angles in radians are

Sol. Let the angles in degrees be

$\alpha - 3\delta, \alpha - \delta, \alpha + \delta, \alpha + 3\delta$

Sum of the angles $= 4\alpha = 360^\circ$

$\therefore \quad \alpha = 90^\circ$

Also greatest angle $= \alpha + 3\delta = 120^\circ$,

Hence, $3\delta = 120^\circ - \alpha = 120^\circ - 90^\circ$

$\therefore \quad \delta = 10$

Hence the angles are 90°–30°, 90° – 10°, 90°+10° and 90° + 30°

That is, the angles in degrees are 60°, 80°, 100° and 120°

$\therefore$ In terms of radians the angles are

$$60 \times \frac{\pi}{180}, 80 \times \frac{\pi}{180}, 100 \times \frac{\pi}{180} \text{ and } 120 \times \frac{\pi}{180}$$

$$\frac{\pi}{3}, \frac{4\pi}{9}, \frac{5\pi}{9} \text{ and } \frac{2\pi}{3}.$$

Ex. The minute hand of a clock is 10 cm long. How far does the tip of the hand move in 20 minutes

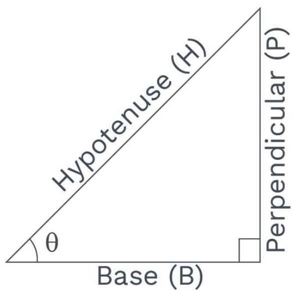
Sol. We know that the tip of the minute hand makes one complete round in one hour i.e. 60 minutes since the length of the hand is 10 cm. the distance moved by its tip in 60 minutes $= 2\pi \times 10\text{cm} = 20\pi$ cm

Hence the distance in 20 minutes

$$= \frac{20\pi}{60} \times 20\,\text{cm} = \frac{20\pi}{3}\,\text{cm}.$$

Trigonometric Ratios (T-ratios)

Following ratios of the sides of a right-angled triangle are known as trigonometrical ratios.

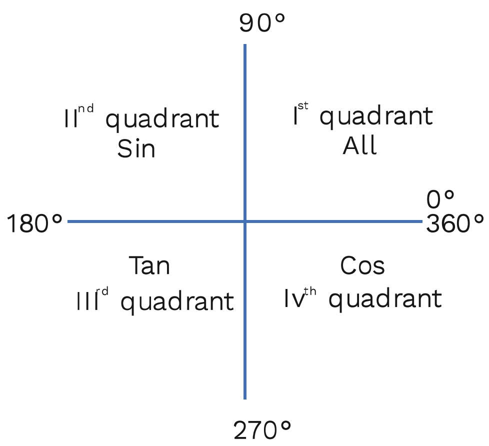


$$\sin\theta = \frac{P}{H}\ ;\ \cos\theta = \frac{B}{H}\ ;\ \tan\theta = \frac{P}{B}$$

Commonly Used Values of Trigonometric Functions

Anlge (θ)	0°	30°	37°	45°	53°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{1}{\sqrt{2}}$	$\frac{4}{5}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{4}{5}$	$\frac{1}{\sqrt{2}}$	$\frac{3}{5}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	$\frac{3}{4}$	1	$\frac{4}{3}$	$\sqrt{3}$	∞

Four Quadrants and ASTC Rule



Rack your Brain

Given that $\sin\theta = \frac{5}{12}$ then find value of $\cos\theta$ and $\tan\theta$.

Concept Reminder

Only in 1^{st} quadrant, all T-ratios are positive.

In first quadrant, all trigonometric ratios are positive.
In second quadrant, only $\sin\theta$ and $\text{cosec}\,\theta$ are positive.

In third quadrant,only $\tan\theta$ and $\cot\theta$ are positive.

In fourth quadrant, only $\cos\theta$ and $\sec\theta$ are positive.

Trigonometric Identities

As $P^2 + B^2 = H^2$

Divide by H^2,

$$\left(\frac{P}{H}\right)^2 + \left(\frac{B}{H}\right)^2 = 1$$

$$\Rightarrow \quad \sin^2\theta + \cos^2\theta = 1$$

Divide by B^2,

$$\left(\frac{P}{B}\right)^2 + 1 = \left(\frac{H}{B}\right)^2$$

$$\Rightarrow \quad 1 + \tan^2\theta = \sec^2\theta$$

Divide by P^2,

$$1 + \left(\frac{B}{P}\right)^2 = \left(\frac{H}{P}\right)^2$$

$$\Rightarrow \quad 1 + \cot^2\theta = \text{cosec}^2\theta$$

Addition/Subtraction Formulae for Trigonometrical Ratios

- sin (A+B)=sinA cosB + cosA sinB
- sin(A–B)=sinA cosB–cosA sinB
- cos(A+B)=cosA cosB –sinA sinB
- cos(A–B)=cosA cosB + sinA sinB

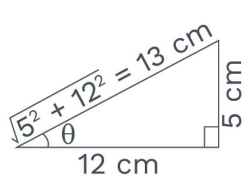
Ex. The two shorter sides of right-angled triangle are 5 cm and 12 cm. Let θ denote the angle opposite to the 5 cm side. Find $\sin\theta$, $\cos\theta$ and tan θ.

Sol.

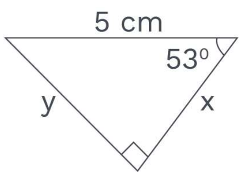
$$\sin\theta = \frac{P}{H} = \frac{5\,\text{cm}}{13\,\text{cm}} = \frac{5}{13}$$

$$\cos\theta = \frac{B}{H} = \frac{12\,\text{cm}}{13\,\text{cm}} = \frac{12}{13}$$

$$\tan\theta = \frac{P}{B} = \frac{5\,\text{cm}}{12\,\text{cm}} = \frac{5}{12}$$



Ex. Find x, y and perimeter of the triangle



Sol. $\frac{y}{5} = \sin 53^\circ = \frac{4}{5} \Rightarrow y = 4\,\text{cm}$

and $\frac{x}{5} = \cos 53^\circ = \frac{3}{5} \Rightarrow x = 3\,\text{cm}$

Perimeter of the triangle

$= x + y + 5 = 3 + 4 + 5 = 12$ cm

Ex. Find the value of :

(i) sin30° + cos60° (ii) sin 0° – cos 0°

(iii) tan 45°–tan 37° (iv) sin 390°

(v) cos 405° (iv) tan 420°

(viii) sin 150° (viii) cos 120°

(ix) tan 135° (x) sin (330°)

(xi) cos 300° (xii) sin(–30°)

(xiii) cos(–60°) (xiv) tan(–45°)

(xvi) sin(–150°)

Sol. (i) $\sin 30^\circ + \cos 60^\circ = \frac{1}{2} + \frac{1}{2} = 1$

(ii) $\sin 0^\circ - \cos 0^\circ = 0 - 1 = -1$

(iii) $\tan 45^\circ - \tan 37^\circ = 1 - \frac{3}{4} = \frac{1}{4}$

(iv) $\sin 390^\circ = \sin(360^\circ + 30^\circ) = \sin 30^\circ = \frac{1}{2}$

(v) $\cos 405^\circ = \cos(360^\circ + 45^\circ) = \cos 45^\circ = \frac{1}{\sqrt{2}}$

(vi) $\tan 420^\circ = \tan(360^\circ + 60^\circ) = \tan 60^\circ = \frac{1}{2}$

(vii) $\sin 150^\circ = \sin(90^\circ + 60^\circ) = \cos 60^\circ = \frac{1}{2}$

or $\sin 150^\circ = \sin(180^\circ - 30^\circ) = \sin 30^\circ = \frac{1}{2}$

Rack your Brain

Find cos75° and tan75°.

(viii) $\cos 120° = \cos(180° - 60°) = -\cos 60° = -\frac{1}{2}$

(ix) $\tan 135° = \tan(180° - 45°) = -\tan 45° = -1$

(x) $\sin 330° = \sin(360° - 30°) = -\sin 30° = -\frac{1}{2}$

(xi) $\cos 300° = \cos(360° - 60°) = \cos 60° = \frac{1}{2}$

(xii) $\sin(-30°) = -\sin 30° = -\frac{1}{2}$

(xiii) $\cos(-60°) = +\cos 60° = \frac{1}{2}$

(xiv) $\tan(-45°) = -\tan 45° = -1$

(xv) $\sin(-150°) = -\sin(150°) = -\sin(180° - 30°)$

$= -\sin 30° = -\frac{1}{2}$

Ex. Find sin 75°

Sol. $\sin 75° = \sin(45° + 30°) = \sin 45° \cos 30° + \sin 30°.\cos 45°$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{2} \times \frac{1}{\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

Small Angle Approximation

If θ is small (say < 5°) then $\sin\theta \approx \theta$, $\cos\theta \approx 1$ and $\tan\theta \approx \theta$

Note: here θ must be in radian.

Ex. Find the approximate values of (i) sin 1° (ii) tan 2° (iii) cos 1°.

Sol. (i) $\sin 1° = \sin\left(1° \times \frac{\pi}{180°}\right) = \sin\frac{\pi}{180} \approx \frac{\pi}{180}$

(ii) $\tan 2° = \tan\left(2° \times \frac{\pi}{180°}\right) = \tan\frac{\pi}{90} \approx \frac{\pi}{90}$

(iii) $\cos 1° = \cos\left(1° \times \frac{\pi}{180°}\right) = \cos\frac{\pi}{180} = 1$

Maximum and Minimum Values of Some useful Trigonometric Functions

- $-1 \le \sin\theta \le 1$
- $-1 \le \cos\theta \le 1$
- $-\sqrt{a^2+b^2} \le a\cos\theta + b\sin\theta \le \sqrt{a^2+b^2}$

Ex. Find maximum and minimum values of y:

(i) y = 2 sinx

(ii) y = 4−cosx

(iii) y = 3sinx + 4 cosx

Sol. (i) y_{max}=2(1)=2 and y_{min}=2(−1) = −2

(ii) y_{max} = 4 − (−1) = 4 + 1 = 5 and

y_{min} = 4−(1) = 3

(iii) $y_{max} = \sqrt{3^2 + 4^2} = 5$ and $y_{min} = -\sqrt{3^2 + 4^2} = -5$

Rack your Brain

Find minimum value of $y = \dfrac{2}{3\sin\theta + 4\cos\theta}$

Ex. A ball is projected with speed u at an Angel θ to the horizontal. The range R of the projectile is given by

$$R = \frac{u^2 \sin 2\theta}{g}$$

for which value of θ will the range be maximum for a given speed of projection?

(Here g = constant)

Sol. As sin $2\theta \le 1$ so range will be maximum if sin 2θ = 1. Therefore

$2\theta = \dfrac{\pi}{2} \Rightarrow \theta = \dfrac{\pi}{4}$ rad.

Ex. The position of a particle moving along x-axis varies with time t according to equation $x = \sqrt{3}\ \sin\omega t - \cos\omega t$ where ω is constants. Find the region in which the particle is confined.

Sol. ∴ $x = \sqrt{3}\sin\omega t - \cos\omega t$

∴ $x_{max} = \sqrt{\left(\sqrt{3}\right)^2 + \left(-1\right)^2} = 2$

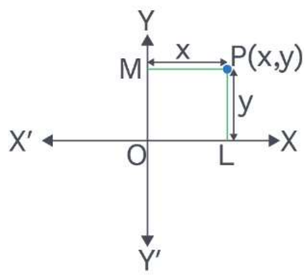
and $x_{min} = -\sqrt{\left(\sqrt{3}\right)^2 + \left(-1\right)^2} = -2$

Thus, the particle is confined in the region $-2 \le x \le 2$.

Cartesian Coordinate System:

To specify the position of a point in space, we use right-handed rectangular axes coordinate system. This system consists of (i) origin (ii) axis or axes. If point is known to be on a given line or in a particular direction, only one coordinate is necessary to specify its position, if it is in a plane, two coordinates are required, if it is in space three coordinates are needed.

Let us consider two intersecting lines XOX' and YOY', which are perpendicular to each other. Let P be any point in the plane of lines. Draw the rectangle OLPM with its adjacent sides OL and OM along the lines XOX', YOY' respectively. The position of the point P can be fixed in the plane provided the locations as well as the magnitudes of OL, OM are known.



Concept Reminder

In cartesian coordinate abscissa means x-coordinate and ordinate means y-coordinate.

Axis of x: The line XOX' is called axis of x.
Axis of y: The line YOY' is called axis of y.
Co-ordinate axes: x axis and y axis together are called axis of co-ordinates or axis of reference.
Origin: The point 'O' is called the origin of co-ordinates or the origin.
Let OL = x and OM = y which are respectively called the abscissa (or x-coordinate) and the ordinate (or y-coordinate).
The co-ordinate of P are (x, y).

Ex. For point (2, 14) find abscissa and ordinates. Also find distance from y and x-axis.

Sol. Abscissa = x-coordinate = 2 = distance from y-axis.
Ordinate = y-coordinate = 14 = distance from x-axis.

Distance Formula

The distance between two points (x_1, y_1) and (x_2, y_2) is given by $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Ex. Find value of a if distance between the points (–9 cm, a cm) and (3 cm, 3cm) is 13 cm.

Sol. By using distance formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\Rightarrow \quad 13 = \sqrt{[3 - (-9)]^2 + [3 - a]^2}$$

$$13^2 = 12^2 + (3-a)^2$$
$$\Rightarrow (3-a)^2 = 13^2 - 12^2 = 5^2$$
$$\Rightarrow (3-a) = \pm 5$$
$$\Rightarrow a = -2 \text{ cm or } 8 \text{ cm}$$

Ex. A dog wants to catch a cat. The dog follows the path whose equation is $y-x = 0$ while the cat follows the path whose equation is $x^2 + y^2 = 8$. The coordinates of possible points of catching the cat are:

Sol. Let catching point be (x_1, y_1) then, $y_1-x_1=0$ and $x_1^2 + y_1^2 = 8$

Therefore, $2x_1^2 = 8$

$\Rightarrow x_1^2 = 4$

$\Rightarrow x_1 = \pm 2$

So, possible points are (2, 2) and (−2, −2).

Rack your Brain

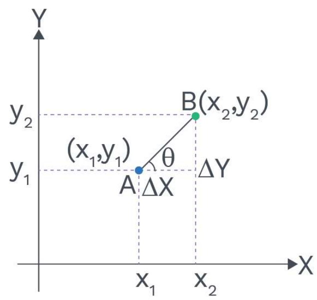
Find the slope of lines which are parallel and perpendicular to x-axis.

Slope of a Line

The slope of a line joining two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is denoted by m and is given by

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \tan\theta$$

[If both axes have identical scales]



Here θ is the angle made by line with positive x-axis. Slope of a line is a quantitative measure of inclination.

Different Forms of Equations of Line:

Slope form

Equation of a line through the origin and having slope m is $y = mx$.

One point form or Point slope form

Equation of a line through the point (x_1, y_1) and having slope m is

$y - y_1 = m(x - x_1)$.

Slope intercept form

The equation of a line with slope m and the x-intercept d is y = m (x - d)

Intercept form

Equation of a straight line cutting off intercepts a and b on x–axis and y–axis respectively is $\frac{x}{a}+\frac{y}{b}=1$.

Two -point form

Equation of the line through the points $A(x_1, y_1)$ and

$B(x_2, y_2)$ is $(y-y_1)=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$.

Concept Reminder

- Cartesian coordinate
- Slope of line
- Intercept

Ex. Find equation to the straight line cutting off an intercept 2 from the negative direction of the axis of y and inclined at 30° to the positive direction of x.

Sol. Let the equation of the straight line is y = mx + c.

Here m = tan 30° = $\frac{1}{\sqrt{3}}$ and c = – 2

Hence, the required equation is

$$y=\frac{1}{\sqrt{3}}x-2 \Rightarrow \sqrt{3}y-x+2\sqrt{3}=0.$$

Ex. Find equation of a straight line passing through (–3, 2) and cutting an intercept equal in magnitude but opposite in sign from the axes.

Sol. Let the equation be

$$\Rightarrow \frac{x}{-3}+\frac{y}{2}=1$$

$$\Rightarrow 2x-3y=-6$$

GRAPHS :

Straight line

General Equation y = mx + c

Where,

m = slope of the line

m = $\tan\theta$.

θ = Slope angle

Different values of m :

m —
- ⊕ : $0° < \theta < 90°$ (For acute slope angle)
- ⊖ : $0° < \theta < 180°$ (For obtuse slope angle)
- Zero : $\theta = 0°$ (For line || to x axis)
- ∞ : $\theta = 90°$ (For line || to y axis)

Rack your Brain

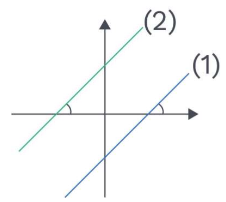
Find slope and y-intercept of $2y = 3x - 5$.

Different value of c :

c —
- ⊕ when line intersects ⊕ y-axis
- ⊖ when line intersects ⊖ y-axis
- Zero When line passes through origin
- ∞ When line is || to y axis

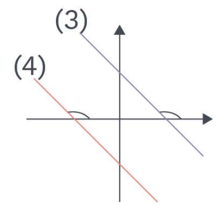
Ex.

For line 1	For line 2
m = ⊕	m = ⊕
c = ⊖	c = ⊕
eg: $y = 2x - 3$	eg : $y = 2x + 3$



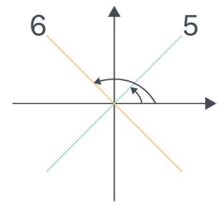
Ex.

For line 3	For line 4
$m = \ominus$	$m = \ominus$
$c = \oplus$	$c = \ominus$
$y = -x + 5$	$y = x - 5$

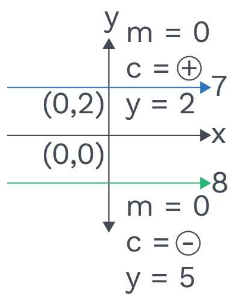


Ex.

For line 5	For line 6
$m = \oplus$	$m = \ominus$
$c = 0$	$c = 0$
$y = 2x$	$y = -2x$



Ex.



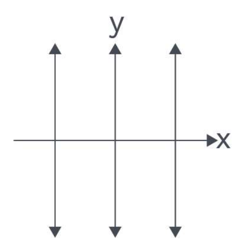
Ex.

Note : All the lines | | to x-axis have equation $y = \text{constant}$

All the lines | | to y axis have equation $x = \text{constant}$

Equation of x axis $\to y = 0$

y axis $\to x = 0$



Ex. Find slope, slope angle and y-intercept.

(i) $y = \frac{1}{\sqrt{3}}x + 2$

$y = mx + c$

$m = \frac{1}{\sqrt{3}} \Rightarrow \tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = 30^\circ$

$c = 2$

(ii) $y = -\sqrt{3}x - 5$

$$y = mx + c$$

$$m = -\sqrt{3} \Rightarrow \tan\theta = -\sqrt{3} \Rightarrow \theta = 120°$$

$$c = -5$$

(iii) $3x + 4y = 5$

$$y = \frac{5 - 3x}{4}$$

$$m = \frac{-3}{4} \Rightarrow \tan\theta = \frac{-3}{4} \Rightarrow \theta = 143°$$

$$c = \frac{5}{4}$$

Ex. Find slope and y-intercept:

$$lx + my + n = 0$$

$$y = \frac{-lx - n}{m}$$

Slope = $\frac{-l}{m}$

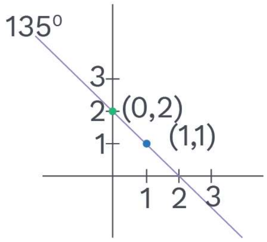
y-intercept = $\frac{-n}{m}$

Ex. $x + y = 2$

$y = 2 - x$

at $x = 0$

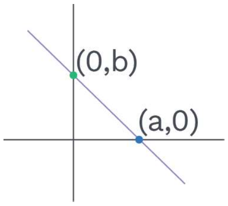
and $m = (-1)$



Concept Reminder

For a particle moving in a plane, relation between x-coordinate and y-coordinate gives the information of trajectory of that particle.

Intercept form of straight line :



$$\frac{x}{a} + \frac{y}{b} = 1$$

eg. $2x + 3y = 5$

intercepts $y = \frac{5}{3}, x = \frac{5}{2}$

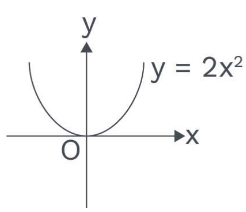
$$2x + 3y = 5$$

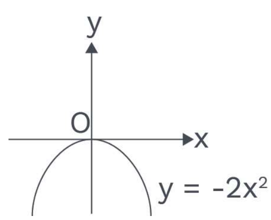
$$3y = 5 - 2x$$

$$y = \frac{-2x}{3} + \frac{5}{3}$$

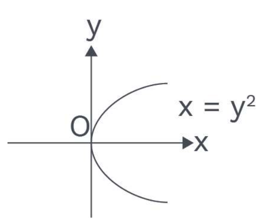
Parabola
standard Equation

(1) $\boxed{y = kx^2}$

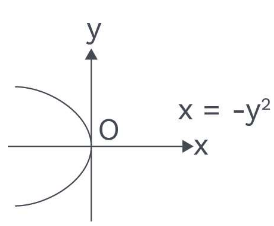




(2) $x = ky^2$



General equation



Circle

A circle is defined as the locus of a point which moves in a plane such that its distance from a fixed point in that plane always remains the same i.e. constant.

The fixed point is called the centre of the circle and the fixed distance is called the radius of the circle.

Concept Reminder

The distance between centre of a circle and at any point on circumference remains constant. And it is known as radius of circle.

General equation of a circle

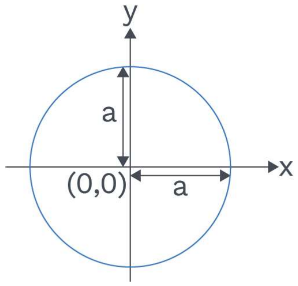
The general equation of a circle is $x^2 + y^2 + 2gx + 2fy + c = 0$, where g, f, c are constant.

Centre of the circle is (–g, –f).

i.e., $\left(-\frac{1}{2}\text{coefficient of x}, -\frac{1}{2}\text{ coefficient of y}\right)$

Radius of the circle is $\sqrt{g^2 + f^2 - c}$.

Circle : $x^2 + y^2 = a^2$



Central form of equation of a circle

The equation of a circle having centre (h, k) and radius r is $(x - h)^2 + (y - k)^2 = r^2$

If the centre is origin, then the equation of the circle is

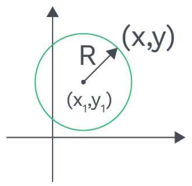
$x^2 + y^2 = r^2$

Circle not centred at origin

$$(x - x_1)^2 + (y - y_1)^2 = R^2$$

Centre = (x_1, y_1)

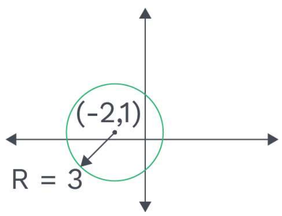
Radius = R



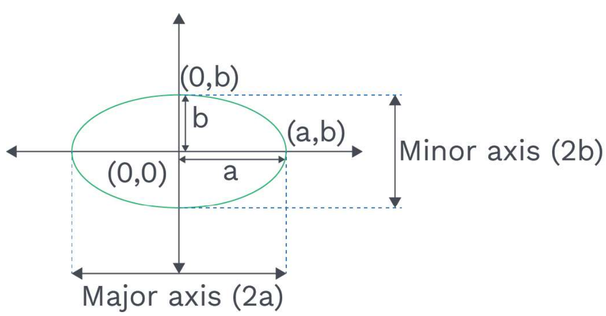
Rack your Brain

Draw the circle of equation $(y + 2)^2 + (x + 1)^2 = 4$.

eg : $(x + 2)^2 + (y - 1)^2 = 9$



ELLIPSE :



KEY POINTS

- Equation of parabola
- Equation of circle
- Equation of ellipse

a = semi major axis
b = semi minor axis

Standard Equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

a – semi horizontal axis
b – semi vertical axis

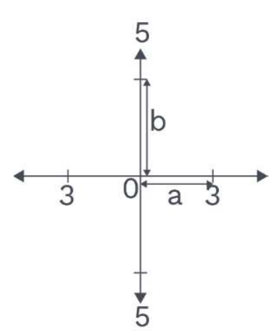
eg. $\frac{x^2}{16} + \frac{y^2}{9} = 1.$

Ex. Given $\frac{x^2}{9} + \frac{y^2}{25} = 1,$

Find semi-major and semi-minor length.

Sol. $5 \times 2 = 10$

$3 \times 2 = 6$



Ex. Find centre coordinate and radius of

$$\frac{x^2}{3} + \frac{y^2}{3} = 1$$

$$3 \times \left[\frac{x^2}{3} + \frac{y^2}{3} = 1\right]$$

Centre – (0, 0)

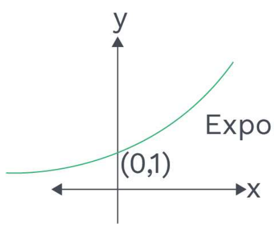
Radius – $(\sqrt{3})$

Exponential Graphs

eg $\Rightarrow y = e^x$

eg $\Rightarrow y = e^{Kx}$

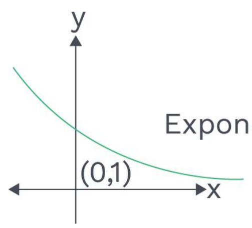
Kx	y
–2	$1/e^2$
–1	1/e
0	1
1	e
2	e^2



$\left.\begin{array}{l} x \to -\infty \\ \Rightarrow y \to 0 \end{array}\right\}$ $\left.\begin{array}{l} x \to \infty \\ \Rightarrow y \to \infty \end{array}\right\}$

eg : $y = e^{-Kx}$

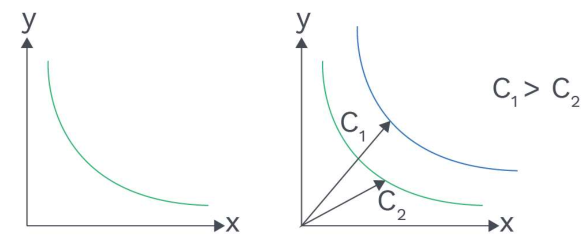
Kx	y
–2	e^2
–1	e
0	1
1	1/e
2	$1/e^2$



Rectangular Hyperbola

xy = constant

$$x \propto \frac{1}{y}$$



eg. $y = \frac{2}{x}$ eg. $y = \frac{5}{x}$

Rack your Brain

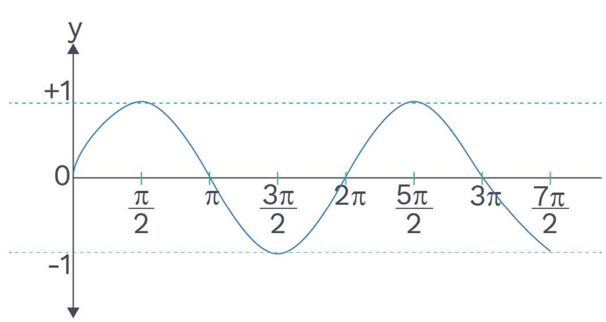
Draw the graph for following equations.

1. $2y + x = 3$
2. $x^2 + (y - 1)^2 = 9$
3. $\frac{x^2}{4} + \frac{y^2}{16} = 1$
4. $y = \frac{-5}{x}$

Trigonometric Graphs

(1) $y = \sin x$

X	Y
0	0
$\pi / 2$	1
π	0
$3\pi / 2$	–1
2π	0



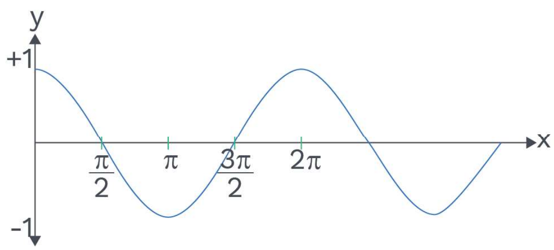
Time period of $\sin x = 2\pi$

(2) y = cos x

X	Y
0	1
$\pi / 2$	0
π	–1
$3\pi / 2$	0
2π	1

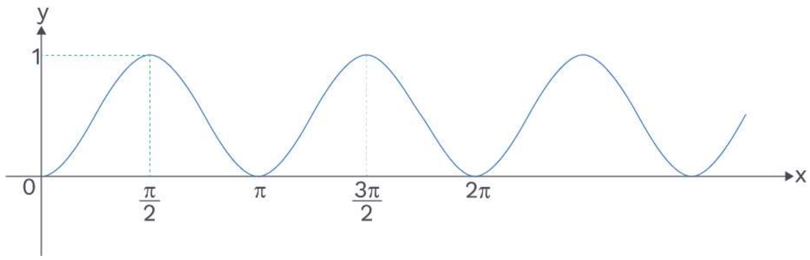
Concept Reminder

- Range of $\sin\theta$ and $\cos\theta$ is from –1 to +1
  i.e., $-1 \leq \sin\theta \leq 1$
  and $-1 \leq \cos\theta \leq 1$



Time period = 2π

(3) y = |sin x|



Ex. The lines 2x – 3y=5 and 3x – 4y = 7 are the diameters of a circle of area 154 square units. Find equation of the circle.

Sol. Centre of circle = Point of intersection of diameters,

On solving equations, 2x – 3y = 5 and 3x - 4y = 7, we get, (x, y) = (1, –1)

$\therefore$ Centre of circle = (1, -1). Now area of circle = 154

$\Rightarrow \pi r^2 = 154$

$\Rightarrow r = 7$

Hence, the equation of required circle is

$(x - 1)^2 + (y+1)^2 = (7)^2$

$x^2 + y^2 - 2x + 2y = 47.$

Function

Physics involves study of natural phenomena and describes them in terms of several physical quantities. A mathematical formulation of interdependence of these physical quantities is necessary for a concise and precise description of the phenomena. These mathematical formulae are expressed in form of equations and are known as function.

Concept Reminder

- Physics is a study of nature and natural phenomena.

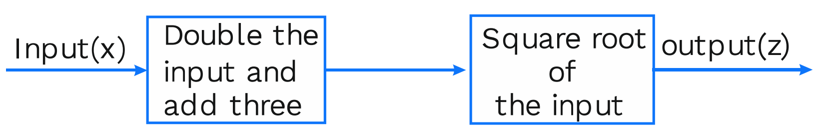
Thus, a function describing a physical process expresses an unknown physical quantity in terms of one or more known physical quantities. We call the unknown physical quantity as dependent variable and the known physical quantities as independent variables. For the sake of simplicity, we consider a function that involves a dependant variable y and only one independent variable x. It is denoted y=f(x) and is read as y equals to f of x. Here f(x) is the value of y for a given x. Following are some examples of functions.

$y=2x+1$, $y=2x^2+3x+1$,

$y=\sin x$, $y = \ln(2x - 1)$

Knowledge of the dependant variable for different values of the independent variable, and how it changes when the independent variable varies in an interval is collectively known as behaviour of the function.

Ex. In the given figure, each box represents a function machine. A function machine illustrates what it does with the input.



Find output.

Sol. $z = \sqrt{2x + 3}$

Graph of a Function

Graph is diagrammatic representation of a function and allows us to visualize it. To plot a graph the dependant variable (here y) is usually taken on the ordinate and the independent variable (here x) on the abscissa. Graph being an alternative way to represent a function does not require elaborate calculations and explicitly shows behaviour of the function in a concerned interval.

KEY POINTS

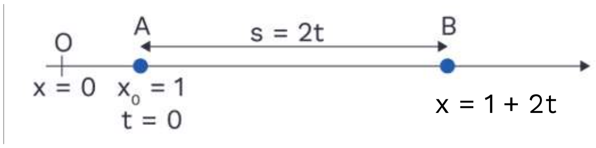
- Function
- Dependent Variable
- Independent Variable

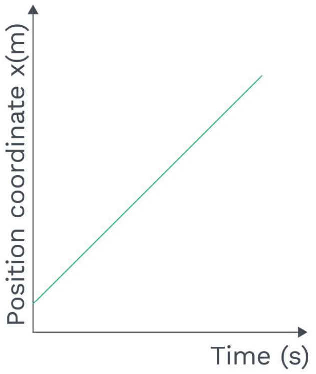
Ex. Consider a body moving with constant speed of 2 m/s in a straight line. When you start your stopwatch, you observe the body 1 m away from a fixed point on the line. Suggest suitable physical quantities, write a function and draw its graph describing motion of the body.

Sol. Distance x of the body from the given fixed point and time t measured by the stopwatch are the suitable variables. If we consider the fixed point as the origin, distance x is known as the position coordinate of the body.

In the following figure it is shown that the body is on point A at the instant t = 0 and after a time t it reaches another point B covering a distance, which equals to product of speed and time interval. Thus, distance which is covered by the body in time t is given by the following equation.

$s = 2t$



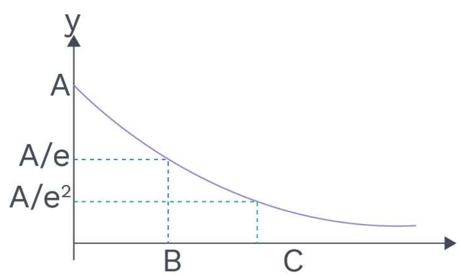


With the help of the above figure, position coordinate x of the body at any time t is given by the following equation, which is the required function describing motion of the body.

$$x=2t+1$$

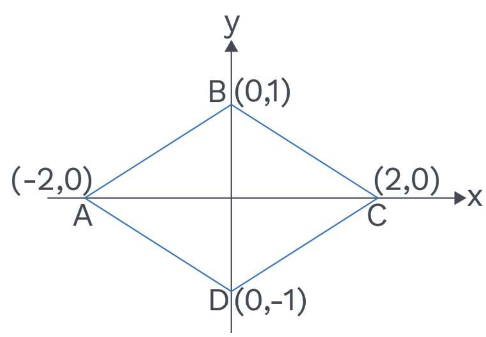
Graph of this equation is also shown in the adjoining figure.

Ex. In the given figure, a function $y = 15e^{-x}$ is shown. What is the numerical value of expression A/(B+C)?



Sol. From graph using $y = 15e^{-x}$ we have A = 15; B =1 ; C =2 . Therefore [A/(B+C) = 15/3 = 5]

Ex. A parallelogram ABCD is shown in figure.



	Column I		**Column II**
(A)	Equation of side AB	(P)	$2y + x = 2$
(B)	Equation of side BC	(Q)	$2y - x = 2$
(C)	Equation of side CD	(R)	$2y + x = -2$
(D)	Equation of side DA	(S)	$2y - x = -2$
		(T)	$y + 2x = 2$

Sol. (A)-Q, (B)-P, (C)-S, (D)-R

For side AB: $m = \dfrac{1-0}{0-(-2)} = \dfrac{1}{2}$, $c = 1 \Rightarrow y = \dfrac{1}{2}x + 1$

For side BC: $m = \frac{2-0}{0-1} = -2, \quad c = 1 \Rightarrow y = -2x + 1$

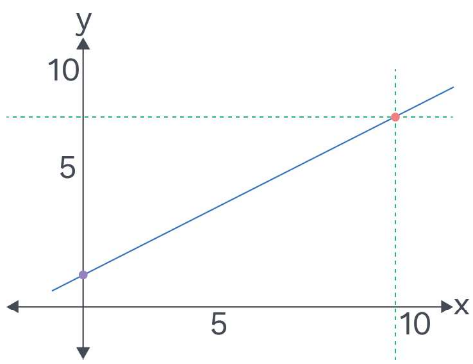
For side CD: $m = \frac{0-(-1)}{2-0} = \frac{1}{2}, \ c = -1 \Rightarrow y = \frac{1}{2}x - 1$

For side DA: $m = \frac{-1-0}{0-(-2)} = -\frac{1}{2}, \ c = -1 \Rightarrow y = -\frac{1}{2}x - 1$

Ex. A variable y increases from $y_1 = 2$ to $y_2 = 8$ linearly with another variable x in the interval $x_1 = 0$ to $x_2 = 10$. Express y as function of x and draw its graph.

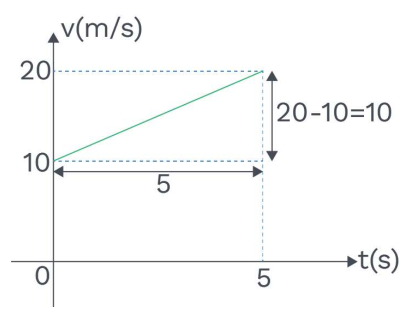
Sol. Linear variation is represented by a linear equation of the form y=mx+c. To represent the function on graph we have to join two points whose coordinates are (x_1, y_1) and (x_2, y_2) i.e. (0, 2) and (10, 8).

Slope of the line is $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8-2}{10-0} = \frac{3}{5}$



From the graph, intercept isc=2. Now the required equation is $y = \frac{3}{5}x + 2$.

Ex. A car changes its velocity linearly from 10 m/s to 20 m/s in 5 seconds. Plot v-t graph and write velocity as a function of time.



Sol. Slope $= \frac{20-10}{5-0} = 2$

y-intercept = 10

$\Rightarrow \quad v = 2t + 10$

Ex. The position of a particle moving in XY-plane varies with time t as $x = t, y = 3t - 5$.

(i) What is the path traced by the particle?

(ii) When does the particle cross-x-axis?

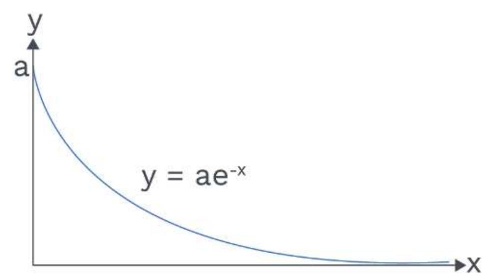
Sol. (i) $x = t, y = 3t - 5$ By eliminating t from above two equations $y = 3x - 5$

This is the equation of a straight line.

(ii) The particle crosses x-axis when y = 0.

So, $0 = 3t - 5 \Rightarrow t = \frac{5}{3}$

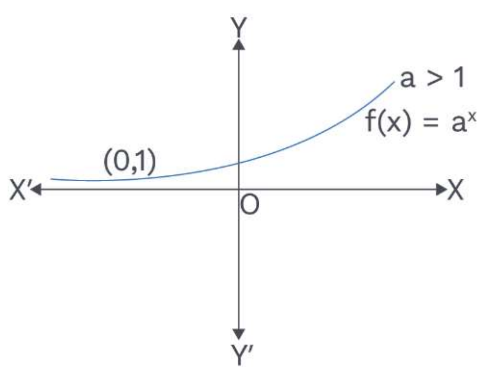
Exponential function and its graph.



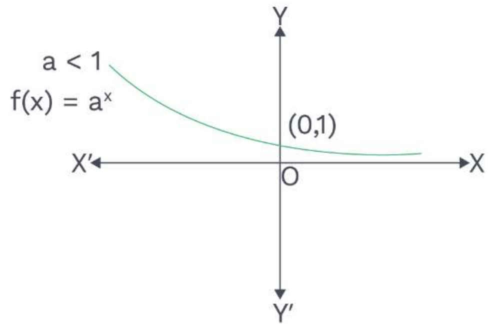
Behaviour of several physical phenomena is described by exponential function to the base e. Here e is known as Euler's Number.

$e = 2.718218$

Most commonly used exponential function has the form $y = ae^{-x}$. In the above figure graph of this function is shown.

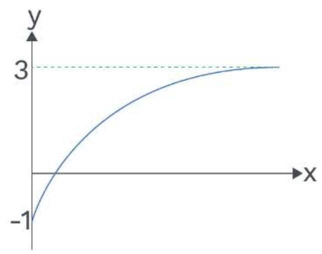


graph of $f(x) = a^x$, when $a > 1$



graph of $f(x) = a^x$, when $a < 1$

Ex. In the given figure is shown a variable y varying exponentially on another variable x. Study the graph carefully. Which of the following equations best suits the shown graph?



(1) $y = 1 - 4e^{-x}$ (2) $y = 2 - e^{-x}$

(3) $y = 4 - e^{-x}$ (4) None of these

Sol. Shift the curve $(-4e^{-x})$ in positive y-direction by 3 units.

At $x = 0$, $y = -1$.

$y = 1 - 4e^{-x}$

Logarithmic function

"The Logarithm of a given number to a given base is the index of the power to which the base must be raised in order to equal the given number."

If $a > 0$ and $\neq 1$, then logarithm of a positive number N is defined as the index x of that power of 'a' which equals N.

i.e., $\log_a N = x$ if $a^x = N$

$a > 0$, $a \neq 1$ and $N > 0$

Definitions

- The logarithm of a given number to a given base is the index of the power to which the base must be raised in order to equal the given number.

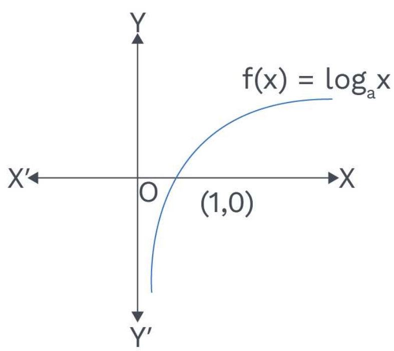
The integral part of a logarithm is called the characteristic and the fractional part is called mantissa.

$$\log_{10} N = \underset{\displaystyle \text{Characters tics}}{\underset{\downarrow}{\text{integer}}} + \underset{\displaystyle \text{Mantissa}}{\underset{\downarrow}{\text{fraction (+ve)}}}$$

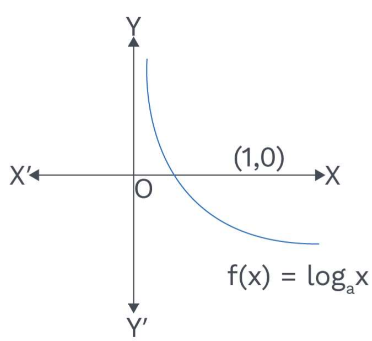
The function defined by is called logarithmic function.

Its domain is $(0, \infty)$ and range is R. a is called the base of the logarithmic function.

When base is 'e' then the logarithmic function is called natural or Napierian logarithmic function and when base is 10, then it is called Common logarithmic function.



graph of $f(x) = \log_a x$, when $a > 1$



graph of $f(x) = \log_a x$, when $a > 1$

Properties of log function are

(1) $\log_a a = 1$, $\log_a 1 = 0$

(2) $\log_a b . \log_b a = 1 = \log_a a = \log_b b$

$$\Rightarrow \quad \log_a b = \frac{1}{\log_b a}$$

(3) $\log_c a = \log_b a . \log_c b$ or $\log_c a = \dfrac{\log_b a}{\log_b c}$

(4) $\log_a (mn) = \log_a m + \log_a n$

(5) $\log_a \left(\dfrac{m}{n}\right) = \log_a m - \log_a n$

(6) $\log_a m^n = n \log_a m$

(7) $a^{\log_a m} = m$

(8) $\log_a \left(\dfrac{1}{n}\right) = -\log_a n$

(9) $\log_{a^\beta} n = \dfrac{1}{\beta} \log_a n$

(10) $\log_{a^\beta} n^\alpha = \dfrac{\alpha}{\beta} \log_a n, \ (\beta \neq 0)$

(11) $a^{\log_c b} = b^{\log_c a}, \quad (a, b, c > 0 \text{ and } c \neq 1)$

Rack your Brain

If $\log_e(2) = 0.693$ and $\log_e(3) = 1.098$. Then find value of $\log(1.5)$ and $\log(24)$.

Ex. If $\log_e \left(\dfrac{a+b}{2}\right) = \dfrac{1}{2}(\log_e a + \log_e b)$, then relation between a and b will be

Sol. $\log_e \left(\dfrac{a+b}{2}\right) = \dfrac{1}{2}(\log_e a + \log_e b)$

$$= \frac{1}{2}\log_e (ab) = \log_e \sqrt{ab}.$$

Differential Calculus

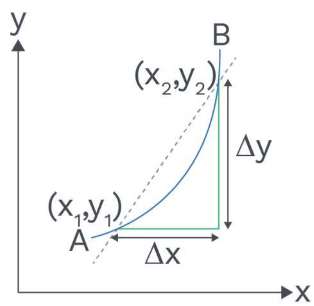
The purpose of differential calculus to study the nature (i.e., increase or decrease) and the amount of variation in a quantity when another quantity (on which first quantity depends) varies independently.

Concept Reminder

- One of most important application of differentiation is that it is used to find instantaneous slope of curve.

Average rate of change

Let a function y = f(x) be plotted as shown in figure. Average rate of change in y w.r.t. x in interval $[x_1, x_2]$ is

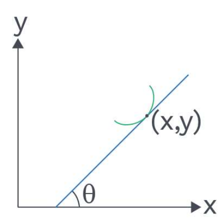


Average rate of change

$$= \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \text{slope of chord AB.}$$

Instantaneous rate of change

It is defined as the rate of change in y with x at a particular value of x. It is measured graphically by the slope of the tangent drawn to the y-x graph at the point (x,y) and algebraically by the first derivative of function y =f(x).



$$\text{Instantaneous rate of change} = \frac{dy}{dx}$$

$$= \text{slope of tangent} = \tan\theta$$

First Derivatives of Commonly used Functions

- $y = \text{constant} \Rightarrow \frac{dy}{dx} = 0$
- $y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}$
- $y = e^x \Rightarrow \frac{dy}{dx} = e^x$
- $y = \ln x \Rightarrow \frac{dy}{dx} = \frac{1}{x}$
- $y = \sin x \Rightarrow \frac{dy}{dx} = \cos x$

KEY POINTS

- Logarithm
- Differential calculus
- Rules of differentiation

- $y = \cos x \Rightarrow \frac{dy}{dx} = -\sin x$
- $y = \tan x \Rightarrow \frac{dy}{dx} = \sec^2 x$
- $y = \cot x \Rightarrow \frac{dy}{dx} = -\text{cosec}^2 x$

- **Method of Differentiation or Rules of Differentiation**

  Function multiplied by a constant i.e.,

  $$y = kf(x) \Rightarrow \frac{dy}{dx} = kf'(x)$$

Ex. Find derivatives of the following functions:

(i) $y = x^3$

(ii) $y = \frac{4}{x}$

(iii) $y = 3e^x$

(iv) $y = 6 \ln x$

(v) $y = 5 \sin x$

Sol. (i) $y = x^3 \Rightarrow \frac{dy}{dx} = 3x^{3-1} = 3x^2$

(ii) $y = \frac{4}{x} = 4x^{-1} \Rightarrow \frac{dy}{dx} = 4\left[(-1)x^{-1-1}\right] = -\frac{4}{x^2}$

(iii) $y = 3e^x \Rightarrow \frac{dy}{dx} = 3e^x$

(iv) $y = 6 \ln x \Rightarrow \frac{dy}{dx} = 6\left(\frac{1}{x}\right) = \frac{6}{x}$

(v) $y = 5 \sin x \Rightarrow \frac{dy}{dx} = 5(\cos x) = 5 \cos x$

Concept Reminder

- $f'(x)$ is used to write $\frac{df}{dx}$ in short form.

- **Sum or Subtraction of Two functions**

  i.e., $y = f(x) \pm g(x) \Rightarrow \frac{dy}{dx} = f'(x) \pm g'(x)$

Ex. Find differentiation of y w.r.t x.

(i) $y = x^2 - 6x$

(ii) $y = x^5 + 2e^x$

(iii) $y = 4 \ln x + \cos x$

Sol. (i) $\frac{dy}{dx} = 2x^{2-1} - 6\left(x^{1-1}\right) = 2x - 6$

(ii) $\frac{dy}{dx} = 5x^{5-1} + 2e^x = 5x^4 + 2e^x$

(iii) $\frac{dy}{dx} = 4\left(\frac{1}{x}\right) + \left(-\sin x\right) = \frac{4}{x} - \sin x$

- **Product of two functions : Product rule**

$y = f(x) \cdot g(x) \Rightarrow \frac{dy}{dx} = f(x).g'(x) + f'(x).g(x)$

Ex. Find first derivative of y w.r.t. x.

(i) $y = x^2 \sin x$

(ii) $y = 4(e^x)\cos x$

Sol. (i) $\frac{dy}{dx} = x^2(\cos x) + (2x)(\sin x) = x^2 \cos x + 2x \sin x$

(ii) $\frac{dy}{dx} = 4\left[(e^x)(\cos x) + (e^x)(-\sin x)\right]$

$= 4e^x\left[\cos x - \sin x\right]$

Rack your Brain

If position of particle varies with time as $x = 4t^2 - 2t$ where x is in metre and t is in second. Then find velocity $\left(v = \frac{dx}{dt}\right)$ at time t = 2s.

- **Division of two functions : Quotient rule**

$$y = \frac{f(x)}{g(x)} \Rightarrow \frac{dy}{dx} = \frac{f'(x)g(x) - f(x)g'(x)}{\left[g(x)\right]^2}$$

Ex. Find differentiation of y w.r.t. x.

(i) $y = \frac{\sin x}{x}$ (ii) $y = \frac{4x^3}{e^x}$

Sol. (i) Here f(x) = sinx, g(x) = x

So f'(x) = cosx, g'(x) = 1

Therefore, $\frac{dy}{dx} = \frac{(\cos x)(x) - (\sin x)(1)}{x^2} = \frac{x\cos x - \sin x}{x^2}$

(ii) Here $f(x) = 4x^3$, $g(x) = e^x$

So, $f'(x) = 12x^2, g'(x) = e^x$

Therefore, $\frac{dy}{dx} = \frac{12x^2(e^x) - 4x^3(e^x)}{(e^x)^2} = \frac{12x^2 - 4x^3}{e^x}$

- **Function of Functions : Chain rule**

Let f be a function of x, which in turn is a function of t. The first derivative of w.r.t. t is equal to the product of $\frac{df}{dx}$ and $\frac{dx}{dt}$.

Therefore $\frac{df}{dt} = \frac{df}{dx} \times \frac{dx}{dt}$

Ex. Find first derivative of y w.r.t. x.

(i) $y = e^{-x}$ (ii) $y = 4 \sin 3x$

Sol. (i) $y = e^{-x} = e^z$ where $z = -x$

So, $\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx} = (e^z)(-1) = -e^z = -e^{-x}$

(ii) $y = 4 \sin 3x = 4 \sin z$ where $z = 3x$

So, $\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx} = 4(\cos z)(3) = 12 \cos 3x$

Ex. The position of a particle moving along x-axis varies with time t as $x = 4t - t^2 + 1$. Find the time interval(s) during which the particle is moving along positive x-direction.

Sol. If the particle moves along positive x-direction, its x-coordinate must increase with time t.

x-coordinate will increase with time t if $\frac{dx}{dt} > 0$.

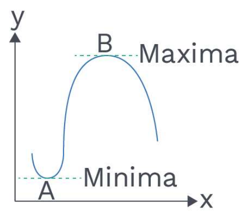
$\frac{dx}{dt} > 4 - 2t$

$\frac{dx}{dt} > 0 \Rightarrow 4 - 2t > 0 \Rightarrow t < 2$

Hence, the particle moves in positive x-direction during time-interval $0 < t < 2$.

Maximum and Minimum value of a Function

Higher order derivatives are used to find the maximum and minimum values of a function. At the points of maxima and minima, first derivative becomes zero.



At point 'A' (minima)

As we see in figure, in the neighbourhood of A, slope is increases so $\frac{d^2y}{dx^2} > 0$

Condition for minima

$$\frac{dy}{dx} = 0 \text{ and } \frac{d^2y}{dx^2} > 0$$

At point 'B' (maxima)

As we see in figure, in the neighbourhood of B, slope is decreases

So, $\frac{d^2y}{dx^2} < 0$

Condition for maxima

$$\frac{dy}{dx} = 0 \text{ and } \frac{d^2y}{dx^2} < 0$$

Ex. The minimum value of $y = 5x^2 - 2x + 1$ is

Sol. For maximum/minimum value $\frac{dy}{dx} = 0$

$$\Rightarrow 5(2x) - 2(1) + 0 = 0$$

$$\Rightarrow x = \frac{1}{5}$$

Now at $x = \frac{1}{5}$, $\frac{d^2y}{dx^2} = 10$ which is positive so minima at $x = \frac{1}{5}$.

Therefore $y_{min} = 5\left(\frac{1}{5}\right)^2 - 2\left(\frac{1}{5}\right) + 1 = \frac{4}{5}$

KEY POINTS

- Chain rule
- Maxima and minima of Curve.

Ex. The radius 'r' of a circular plate is increasing at the rate of 0.1 cm per second. Calculate the rate at which the area will increase when the radius of plate is $\frac{5}{\pi}$ cm?

Sol. Area of disk, $A = \pi r^2$ (where r = radius of disk)

$$\frac{dA}{dt} = \pi\left(2r\frac{dr}{dt}\right) = 2\pi r\frac{dr}{dt} \text{ so } \frac{dA}{dt} = 2\pi \times \frac{5}{\pi} \times 0.1 = 1 \text{ cm}^2/\text{s}$$

Ex. An object moves along the curve $12y = x^3$. Which coordinate will change at faster rate at x=10?

Sol. $12y = x^3 \Rightarrow 12dy = 3x^2dx$

$$\Rightarrow \frac{dy}{dt} = \left(\frac{x}{2}\right)^2\left(\frac{dx}{dt}\right)$$

Therefore for $\left(\frac{x}{2}\right)^2 > 1$ or x>2, y–coordinate varies at faster rate.

Ex. Water pours out at the rate of 'Q' from a tap, into a cylindrical vessel of radius 'r'. At what rate the height of water level will rises when the height is 'h'.

Sol. $\therefore$ Volume: $V = \pi r^2 h$

$$\therefore \frac{dV}{dt} = \pi r^2\frac{dh}{dt} \text{ But } \frac{dV}{dt} = Q \text{ so } \frac{dh}{dt} = \frac{Q}{\pi r^2}$$

Ex. If surface area of a cube changes at a rate of 5 m^2/s, calculate the rate of change of body diagonal when side length is 1 m.

Sol. Surface area of cube $S=6a^2$

(where a = side of cube)

Body diagonal $l = \sqrt{3}a$. Therefore $S = 2l^2$.

Differentiating it w.r.t. time $\frac{dS}{dt} = 2(2l)\frac{dl}{dt}$

$$\Rightarrow \frac{dl}{dt} = \frac{1}{4\left(\sqrt{3}a\right)}\frac{dS}{dt} = \frac{5}{4\sqrt{3}} \text{ m/s}$$

Integral Calculus:-

It is the reverse process of differentiation. By help of integration we can easily find a function whose derivative is known. Assume a function F(x) whose differentiation with respect to 'x' is equal to f(x) then

$$\int f(x)dx = F(x) + c$$

here 'c' is the constant of and this is called indefinite integration.
Few basic formulae of integration are :

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\int \sin x dx = -\cos x + c$$

$$\int \sec^2 x dx = \tan x + c$$

$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$$

$$\int \frac{dx}{ax+b} = \frac{\ln(ax+b)}{a} + c$$

$$\int \sin(ax+b) dx = \frac{-\cos(ax+b)}{a} + c$$

$$\int \cos(ax+b) dx = \frac{\sin(ax+b)}{a} + c$$

$$\int \sec^2(ax+b) dx = \frac{\tan(ax+b)}{a} + c$$

$$\int e^{ax+b} dx = \frac{e^{ax+b}}{a} + c$$

$$\int \frac{1}{x} dx = \ln x + c$$

$$\int \cos x dx = \sin x + c$$

$$\int e^x dx = e^x + c$$

KEY POINTS

- Indefinite integration
- Definite integration

Ex. Integrate the following w.r.t. x.

(i) $4x^3$ (ii) $x - \frac{1}{x}$

(iii) $\frac{1}{2x+3}$ (iv) $\cos(4x+3)$

(v) $\cos^2 x$

Sol. (i) $\int 4x^3 dx = 4\left(\frac{x^{3+1}}{3+1}\right) + c = \frac{4x^4}{4} + c = x^4 + c$

(ii) $\int\left(x-\frac{1}{x}\right)dx=\int xdx-\int\frac{1}{x}dx=\frac{x^2}{2}-\ln x+c$

(iii) $\int\frac{dx}{2x+3}=\frac{\ln(2x+3)}{2}+c$

(iv) $\int\cos(4x+3)dx=\frac{\sin(4x+3)}{4}+c$

(v) $\int\cos^2 xdx=\int\frac{2\cos^2 x}{2}dx$

$=\int\frac{(1+\cos 2x)}{2}dx=\frac{1}{2}\int dx+\frac{1}{2}\int\cos 2xdx$

$=\frac{x}{2}+\frac{1}{2}\frac{\sin 2x}{2}+c=\frac{x}{2}+\frac{1}{4}\sin 2x+c$

Definite Integration:-

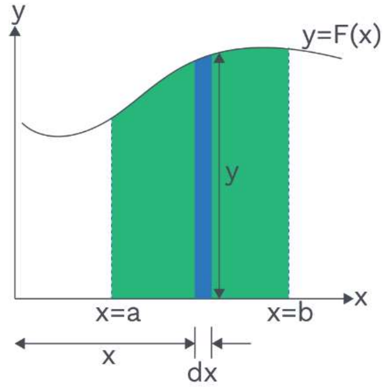
When a function is integrated between lower and an upper limits then it is known a definite integral. Assume a function F(x) whose differentiation with respect to 'x' is equal to f(x), in an interval a≤ x ≤ b then

$$\int_a^b f(x)dx=F(b)-F(a)$$

Concept Reminder

- If lower limit and upper limit are not given for integration then it is known as indefinite integral.

Area under a curve and definite integration



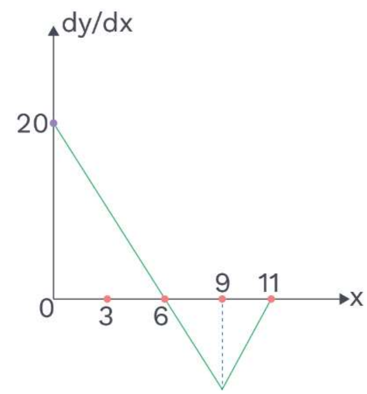
Area of small element = ydx = f(x) dx

If we sum up all areas between x=a and x= b then $\int_a^b f(x)dx=$ shaded area between curve and x-axis.

Ex. The integral $\int_{1}^{5} x^2 dx$ is equal to

Sol. $\int_{1}^{5} x^2 dx = \left[\frac{x^3}{3}\right]_1^5 = \left[\frac{5^3}{3} - \frac{1^3}{3}\right] = \frac{125}{3} - \frac{1}{3} = \frac{124}{3}$

Ex. The curve shown below represent rate of change of a variable 'y' with respect to 'x'. The change in the value of 'y' when 'x' changes from 0 to 11 is:



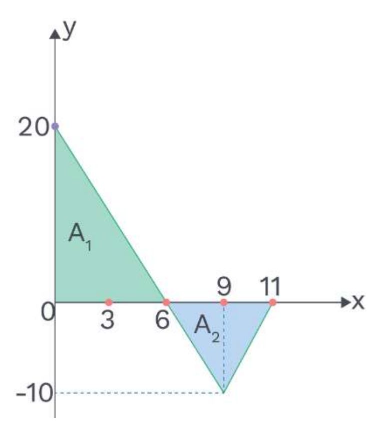
Concept Reminder

- Integration is used to find Area Under Curve & average value of function.

Sol. As $dy = \left(\frac{dy}{dx}\right)dx$

So, $\Delta y = \int dy = \int_{0}^{11} \left(\frac{dy}{dx}\right)dx$

Area under the curve



$$A_1 = \frac{1}{2} \times 6 \times 20 = 60$$

$$A_2 = \frac{1}{2} \times (11-6)(10) = -25$$

$$\Delta y = A_1 + A_2 = 60 - 25 = 35$$

Average value of continuous function in an interval:-

Average value of y = f(x)

for the interval $a \le x \le b$ is given by $y_{av} = \dfrac{\int_a^b y\,dx}{\int_a^b dx} = \dfrac{\int_a^b y\,dx}{b-a}$

Ex. Find the average value of y = 2x +3 in the interval $0 \le x \le 1$.

Sol. $$\int_0^1 (2x+3)dx = \left[2\left(\frac{x^2}{2}\right) + 3x\right]_0^1$$

$$= 1^2 + 3(1) - 0^2 - 3(0) = 1 + 3 = 4$$

Rack your Brain

Find average value of alternating current $I = I_0 \cos \omega t$ in time interval $\left[0, \frac{2\pi}{\omega}\right]$

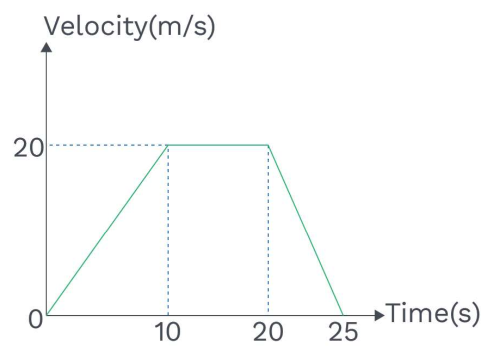
Ex. Find average value of alternating current $I = I_0 \sin \omega t$ in time interval $\left[0, \frac{\pi}{\omega}\right]$.

Sol. $$I_{av} = \frac{\int_0^{\pi/\omega} I\,dt}{\frac{\pi}{\omega} - 0} = \frac{\omega}{\pi}$$

$$\int_\pi^{\pi/\omega} I_0 \sin \omega t\,dt = \frac{\omega}{\pi}\left[\frac{I_0(-\cos \omega t)}{\omega}\right]_0^{\pi/\omega}$$

$$= -\frac{\omega I_0}{\pi\,\omega}\left[\cos \pi - \cos 0\right] = -\frac{I_0}{\pi}\left[-1-1\right] = \frac{2 I_0}{\pi}$$

Ex. The velocity-time graph of a bus moving along a straight road is shown in figure. Find average velocity of the bus in first 25 seconds.



Sol. Average velocity

$$= \frac{\int_0^{25} v\,dt}{25-0} = \frac{\text{Area of } v-t \text{ graph between } t=0 \text{ to } t=25\text{ s}}{25}$$

$$= \frac{1}{25}\left[\frac{1}{2}\times 10\times 20 + 10\times 20 + \frac{1}{2}\times 20\times 5\right] = 14\,\text{m/s}$$

EXAMPLES

Q1 **(i) Convert 45° to radians.**

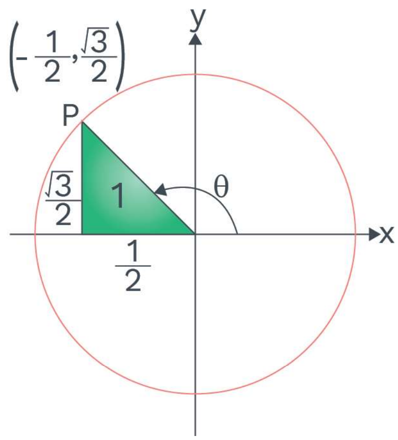
(ii) Convert $\frac{\pi}{6}$ rad to degrees

Sol: (i) $45 \cdot \frac{\pi}{180} = \frac{\pi}{4}\text{rad}$

(ii) $\frac{\pi}{6} \cdot \frac{180}{\pi} = 30°$

Q2 **Find the sin and cos of angle θ shown in the unit circle if coordinate of point 'p' are as shown.**

Sol:



$\cos\theta$ = x-coordinate of P = $-\frac{1}{2}$

$\sin\theta$ = y-coordinate of $P = \frac{\sqrt{3}}{2}$

Q3 **Evaluate cos 135°**

Sol: **Sol.** $\cos 135° = \cos(90° + 45°) = -\sin 45° = -\frac{1}{\sqrt{2}}$

Q4 **Find value of**

(i) $\log_{81} 27$ **(ii) $\log_{10} 100$**

Sol: (i) Let $\log_{81} 27 = x$

$\Rightarrow 27 = 81^x \Rightarrow 3^3 = 3^{4x}$ gives x = ¾

(ii) Let $\log_{10} 100 = x$

$\Rightarrow 100 = 10^x \Rightarrow 10^2 = 10^x$ gives x = 2

Q5 **Find the derivative of $y = (x^2 + 1)(x^3 + 3)$.**

Sol: From the product Rule of diffrentiation with $u = x^2 + 1$ and $v = x^3 + 3$, we find

$$\frac{d}{dx}[(x^2 + 1)(x^3 + 3)] = (x^2 + 1)(3x^2) + (x^3 + 3)(2x)$$

$$= 3x^4 + 3x^2 + 2x^4 + 6x = 5x^4 + 3x^2 + 6x$$

Ex. can be done as well (perhaps better) by multiplying out the original expression for 'y' and differentiating the resulting polynomial. We now check:

$$y = (x^2 + 1)(x^3 + 3) = x^5 + x^3 + 3x^2 + 3$$

$$\frac{dy}{dx} = 5x^4 + 3x^2 + 6x$$

This is in agreement with our 1st calculation.

Q6 **Find the derivative of** $y = \frac{t^2 - 1}{t^2 + 1}$

Sol: We apply the Quotient Rule of diffrentiation with u = $t^2 - 1$ and v = $t^2 + 1$:

$$\frac{dy}{dt} = \frac{(t^2+1).2t - (t^2-1).2t}{(t^2+1)^2} \Rightarrow \frac{d}{dt}\left(\frac{u}{v}\right) = \frac{v(du/dt) - u(dv/dt)}{v^2}$$

$$= \frac{2t^3 + 2t - 2t^3 + 2t}{(t^2+1)^2} = \frac{4t}{(t^2+1)^2}.$$

Q7 **Find derivative for following functions [Q. 7 to 9]**
y = x^2 – sin x:

Sol: $\frac{dy}{dx} = 2x - \frac{d}{dx}(\sin x)$ Difference Rule

$= 2x - \cos x$

Q8 **y = x^2 sinx :**

Sol: $\frac{dy}{dx} = x^2 \frac{d}{dx}(\sin x) + 2x \sin x$ Product rule

$= x^2 \cos x + 2x \sin x$

Q9 $y = \frac{\sin x}{x}$:

Sol:

$$\frac{dy}{dx} = \frac{x.\frac{d}{dx}(\sin x) - \sin x.1}{x^2}$$ Quotient Rule

$$= \frac{x\cos x - \sin x}{x^2}$$

Q10 **Find dy/dx if y = tan x.**

Sol:

$$\frac{d}{dx}(\tan x)=\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)=\frac{\cos x\frac{d}{dx}(\sin x)-\sin x\frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$=\frac{\cos x\cos x-\sin x(-\sin x)}{\cos^2 x}=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}=\frac{1}{\cos^2 x}=\sec^2 x$$

Q11 **The position of a particle is given by the equation**

$$\mathbf{s=f(t)=t^3-6t^2+9t}$$

where 't' is measured in seconds and s in meters.
Find the acceleration at time t. What is the acceleration after 4s?

Sol: The derivative of the position function is defined as velocity function:

$$s=f(t)=t^3-6t^2+9t\Rightarrow v(t)=\frac{ds}{dt}=3t^2-12t+9$$

The derivative of the velocity function is acceleration:

$$a(t)=\frac{d^2s}{dt^2}=\frac{dv}{dt}=6t-12\Rightarrow a(4)=6(4)-12=12\,m/s^2$$

Q12 **Find minimum value of y = 1 + x² – 2x**

Sol: $\frac{dy}{dx}=2x-2$

for minima $\frac{dy}{dx}=0$

2x – 2 = 0

x = 1

$\frac{d^2y}{dx^2}=2\ \Rightarrow\ \frac{d^2y}{dx^2}>0$

at x = 1 there is minima

for minimum value of y

$y_{minimum}=1+1-2=0$

Q13 **Find $\int \frac{1}{\sqrt{x}} dx$**

Sol: $\int \frac{1}{\sqrt{x}} dx = \int x^{-1/2} dx = 2x^{1/2} + C = 2\sqrt{x} + C$

Q14 **Find $\int \cos\frac{x}{2} dx$.**

Sol: $\int \cos\frac{x}{2} dx = \int \cos\frac{1}{2} x dx = \frac{\sin(1/2)x}{1/2} + C = 2\sin\frac{x}{2} + C$

Q15 **Find area under the curve of y = x from x = 0 to x = a.**

Sol: $\int_0^a y dx = \left[\frac{x^2}{2}\right]_0^a = \left[\frac{a^2}{2} - 0\right] = \frac{a^2}{2}$

Mind Map

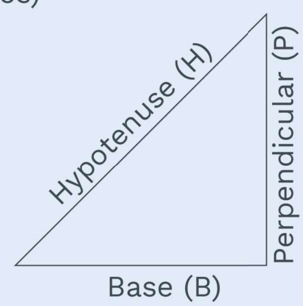
TRIGONOMETRY

Trigonometric Identities

- $\sin^2\theta + \cos^2\theta = 1$
- $1 + \tan^2\theta = \sec^2\theta$
- $1 + \cot^2\theta = \text{cosec}^2\theta$
- $\sin 2\theta = 2\sin\theta.\cos\theta$
- $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

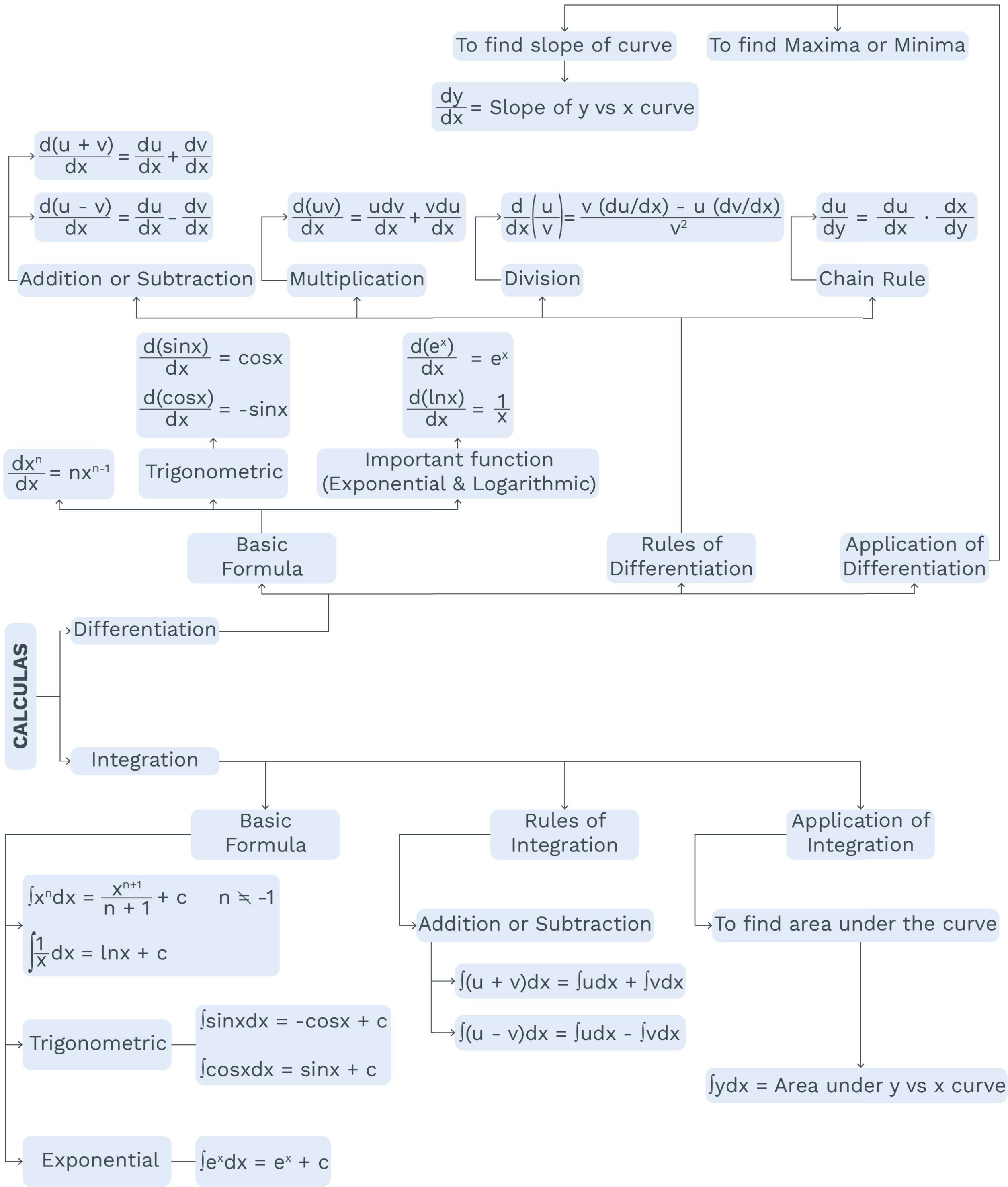
Trigonometric ratios (T-ratios)

- $\sin\theta = \frac{P}{H}$
- $\cos\theta = \frac{B}{H}$
- $\tan\theta = \frac{P}{B}$



Addition/Subtraction Formulae

- $\sin(A + B) = \sin A.\cos B + \cos A.\sin B$
- $\sin(A - B) = \sin A.\cos B - \cos A.\sin B$
- $\cos(A + B) = \cos A.\cos B - \sin A.\sin B$
- $\cos(A - B) = \cos A.\cos B + \sin A.\sin B$


CALCULAS
Differentiation
Integration
Basic Formula
Rules of Differentiation
Application of Differentiation
To find slope of curve
To find Maxima or Minima
$\frac{dy}{dx}$ = Slope of y vs x curve
$\frac{dx^n}{dx} = nx^{n-1}$
Trigonometric
$\frac{d(\sin x)}{dx} = \cos x$
$\frac{d(\cos x)}{dx} = -\sin x$
Important function (Exponential & Logarithmic)
$\frac{d(e^x)}{dx} = e^x$
$\frac{d(\ln x)}{dx} = \frac{1}{x}$
Addition or Subtraction
$\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}$
$\frac{d(u-v)}{dx} = \frac{du}{dx} - \frac{dv}{dx}$
Multiplication
$\frac{d(uv)}{dx} = \frac{udv}{dx} + \frac{vdu}{dx}$
Division
$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\,(du/dx) - u\,(dv/dx)}{v^2}$
Chain Rule
$\frac{du}{dy} = \frac{du}{dx} \cdot \frac{dx}{dy}$
Basic Formula
Rules of Integration
Application of Integration
$\int x^n dx = \frac{x^{n+1}}{n+1} + c \quad n \neq -1$
$\int \frac{1}{x} dx = \ln x + c$
Trigonometric
$\int \sin x dx = -\cos x + c$
$\int \cos x dx = \sin x + c$
Exponential
$\int e^x dx = e^x + c$
Addition or Subtraction
$\int (u+v)dx = \int udx + \int vdx$
$\int (u-v)dx = \int udx - \int vdx$
To find area under the curve
$\int ydx$ = Area under y vs x curve