Gravitation

Gravitation

Introduction

Newton observed that any object when released near the earth surface is accelerated towards the earth. As acceleration is caused by an unbalanced force, there must be a force pulling objects towards the earth. If someone throws a projectile with some initial velocity, then instead of that particle moving off into place (space) in a straight line, then particle is continuously acted on by a force pulling it back to the earth. Let we throw the projectile with greater speed, then the path of projectile would be different than earlier, and its range is also increased. If the projection speed is further increased until at some initial speed, the particle would not hit the earth surface at all but would go right around it in an earth orbit. But at any point along its path length the projectile motion would still have a force acting on it is pulling it towards the earth surface. Newton was led to the result that the equal force that causes the apple to fall down to the earth also causes the moon to be pulled by the earth. Therefore, the moon moves in its orbit about the earth because it is pulled toward the earth. But if there is a force present between the moon and the earth, why not a force act between the sun and the other planet or why not a force between the sun and the earth? Newton proposed that the equal force, named gravitational force which acts on particle near the earth surface also acts on all the heavenly particle. He result that there was a force of gravitation between each and every mass in the universe.

Newton's Law of Gravitation

It says that everybody in the universe attracts every another object with a force which is proportional to product of their masses and is inversely proportional to square of the distance between them.

KEY POINTS

- Gravitation
- Newton's Law of gravitation

Concept Reminder

Issac Newton was struck on head by a falling apple. This simple incident stirred Newton to think about falling bodies and eventually led him to discovery of the law of gravitation.

Definitions

Newton's Law of Gravitation

It states that everybody in the universe attracts every other body with a force which is directly proportional to the product of their masses and is inversely proportional to the square of the distance between them.

$F \propto m_1 m_2$ and $F \propto \frac{1}{r^2}$

So, $F \propto \frac{m_1 m_2}{r^2}$

$\therefore \quad F = \frac{Gm_1 m_2}{r^2}$

[G = Universal gravitational constant = 6.67 × 10^{-11} Nm^2/kg^2]

Note: This formula is only applicable for point masses.

Vector form of Newton's law of Gravitation:

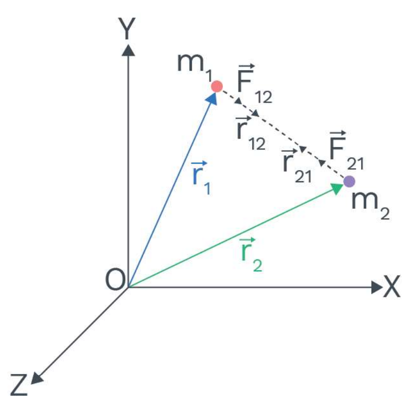
Let $\vec{r}_{12}$ = Displacement vector from m_1 to m_2

$\vec{r}_{21}$ = Displacement vector from m_2 to m_1

$\vec{F}_{21}$ = Gravitational force exerted on m_2 by m_1

$\vec{F}_{12}$ = Gravitational force exerted on m_1 by m_2

$$\vec{F}_{12} = -\frac{Gm_1 m_2}{r_{21}^2}\hat{r}_{21} = -\frac{Gm_1 m_2}{r_{21}^3}\vec{r}_{21}$$



Negative sign shows that:

(i) The direction of $\vec{F}_{12}$ is opposite to that $\vec{F}_{21}$

Concept Reminder

- $\vec{F}_{12} = \frac{-Gm_1 m_2}{r_{21}^3}\vec{r}_{21}$
- $\vec{F}_{21} = \frac{-Gm_1 m_2}{r_{12}^3}\vec{r}_{12}$

$\Rightarrow \vec{F}_{12} = -\vec{F}_{21}$

KEY POINTS

- Vector form of Newton's law of gravitation
- Gravitational constant

(ii) The gravitational force is attractive in nature

Similarly, $\vec{F}_{21} = -\frac{Gm_1m_2}{r_{12}^2}\hat{r}_{12}$ or $\vec{F}_{21} = -\frac{Gm_1m_2}{r_{12}^3}\vec{r}_{12}$

$\Rightarrow \vec{F}_{12} = -\vec{F}_{21}$

Gravitational constant "G"

- Gravitational constant is a scalar quantity.
- **Unit:-** SI: $N-m^2/kg^2$
  Dimensions:- $[M^{-1}L^3T^{-2}]$
- Its value is equal throughout the universe, G independent on the nature and size of the bodies, **it also independent upon nature of the medium between the bodies.**
- Its value was first found out by the scientist **"Henry Cavendish"** with the experiment of "Torsion Balance.
- Value of G is small therefore gravitational force is weaker than electrostatic and nuclear forces.

Important point about gravitational force:

1. $\therefore \quad \vec{F}_{12} = -\vec{F}_{21}$

$\therefore \quad \vec{F}_{12} + \vec{F}_{21} = 0$ means, $\vec{F}_{Net} = \vec{0}$

Since, the net force on system is zero. Therefore, linear momentum of the system will be conserved but the conservation of linear momentum is not applicable on individual bodies because net force on individual bodies is not zero.

2. Gravitational force is a central force because its direction is along the line joining centre of bodies. Standard form of central force:

$$\boxed{\vec{F} = F(r)\hat{r}}$$

Properties of gravitational force: -

- Gravitational force acts along the line joining the two interacting particles i.e., gravitational force is a central force.
- Gravitational force is independent of the presence of other particles.

Rack your Brain

Two sphere of masses m and M are situated in air and gravitational force between them is F. The space around the masses is now filled with a liquid of specific gravity. Find new gravitational force.

Concept Reminder

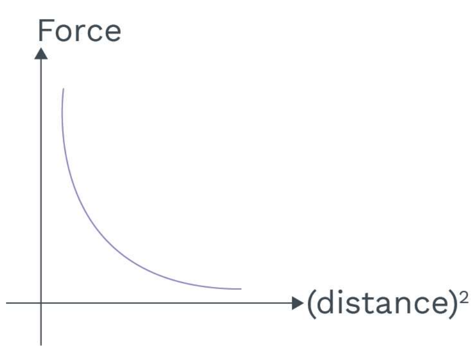
Properties of Gravitational force.

1. Central force
2. Conservative force
3. Always attractive
4. Independent of medium
5. Forms action reaction pair.
6. Follow superposition principle.

- Gravitational force forms an action - reaction pair.
  So, gravitational force obeys Newton III[rd] Law

$$\Rightarrow \overrightarrow{F_{12}} = -\overrightarrow{F_{21}} \Rightarrow \overrightarrow{F_{12}} + \overrightarrow{F_{21}} = 0$$

- F - r^2 graph is a rectangular hyperbola as shown.



- Nature of gravitational force is conservative i.e., the amount of work done by gravitational force in displacing a object from one place to another place is independent of the path traversed.
- The net gravitational force on any particle is the vector sum of all individual gravitational forces unit, by all other particles in the system.

$$\vec{F}_{1,net} = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14} + ... + \vec{F}_{1n}$$

we can express this equation more compactly as

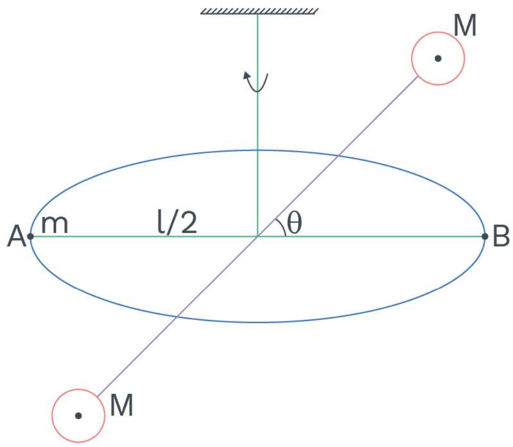
vector sum $\vec{F}_{1,net} = \sum_{i=2}^{n} \vec{F}_{1i}$

Sun exerts gravitational force on earth, but earth does not move towards sun because the gravitational pull of the sun on the earth provides the necessary centripetal force to earth, so the orbit is stable.

Definitions

Superposition principle.
The net gravitational force on any particle is the vector sum of all individual gravitational force on it, by all other particles in the system.
$\vec{F}_{1,net} = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14} + ... + \vec{F}_{1n}$

Universal Gravitational Constant:



Definitions

From Cavendish experiment

$$G = \frac{K\theta r^2}{Mml}$$

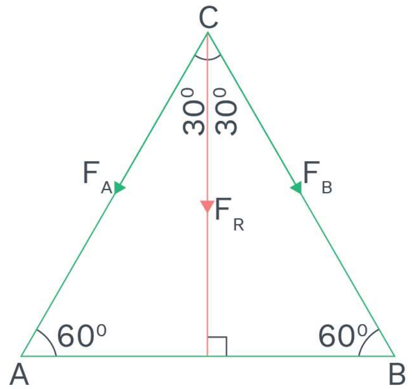
From Cavendish experiment the value of universal gravitational constant (G) can be calculated by

$$G = \frac{k\theta r^2}{Mml}$$

M = Mass of heavier sphere
m = Mass of lighter sphere
k = Torsion constant
θ = Angle of twist

Ex. If two particles each of mass 'm' are placed at the two vertices of an equilateral triangle of side 'a', then the resultant gravitational force on mass m placed at the third vertex is.

Sol.



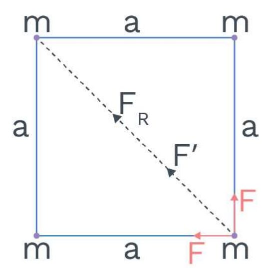
$$F_R = \sqrt{F_A^2 + F_B^2 + 2F_AF_B\cos 60^\circ}$$

$= \sqrt{3}F \qquad [\because F_A = F_B = F]$

$$F_R = \sqrt{3}\left[\frac{Gm^2}{a^2}\right]$$

Ex. If four identical particles each of mass m, are kept at the four vertices of a square of side length 'a', the gravitational force of attraction on any one of the particles is

Sol.

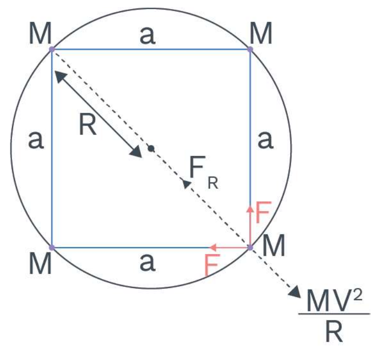


$$F_R = \sqrt{2}F + F' = \frac{\sqrt{2}Gm^2}{a^2} + \frac{Gm^2}{2a^2}$$

$F_R = \dfrac{Gm^2}{a^2}\left(\sqrt{2} + \dfrac{1}{2}\right)$ along the diagonal towards the opposite corner.

Ex. Four particles, each of mass M Kg and equidistant from each other, move along a circle of radius R under the action of their mutual gravitational attraction. Find out the speed of each particle.

Sol. Let 'a' be the distance between two particles.



The resultant gravitational force on any one of the particles is given by $F_R = \dfrac{Gm^2}{a^2}\left(\sqrt{2} + \dfrac{1}{2}\right)$.

Which provides necessary centripetal force for motion of mass M in circle, so

$$\left(\sqrt{2}+\frac{1}{2}\right)\left(\frac{GM^2}{a^2}\right)=\frac{Mv^2}{\frac{a}{\sqrt{2}}}$$

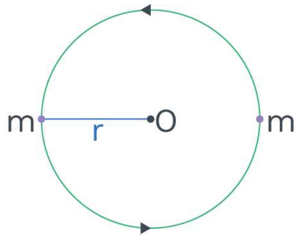
$$v^2=\left(\frac{2\sqrt{2}+1}{2\sqrt{2}}\right)\frac{GM}{a}=\left(\frac{2\sqrt{2}+1}{2\sqrt{2}}\right)\frac{GM}{\sqrt{2}R}$$

$$\Rightarrow v=\frac{1}{2}\sqrt{\frac{GM}{R}\left(1+2\sqrt{2}\right)}$$

Ex. Two particles of equal masses move in a circle of radius r under the action of their mutual gravitational attraction. Find the speed of each particle if the mass of each particle is m.

Sol. In this case the two particles maintain gravitational force of attraction diametrically. The gravitational force on one of the particles must be equal to the necessary centripetal force

$$\frac{mv^2}{r}=\frac{Gmm}{(2r)^2}\quad\Rightarrow v=\sqrt{\frac{Gm}{4r}}$$



Ex. Mass M is split into two parts m and (M-m), which are then separated by a certain distance. What is the ratio of (m/M) which maximises the gravitational force between the parts ?

Sol. If r is the distance between m and (M-m), the gravitational force between them will be

$$F=G\frac{m(M-m)}{r^2}=\frac{G}{r^2}(mM-m^2)$$

For F to be maximum $\frac{dF}{dm}=0$ as M & r are constants.

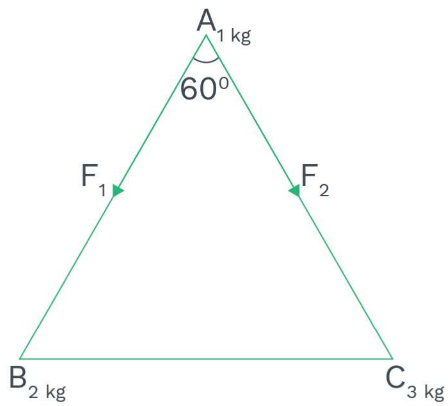
$$\Rightarrow \frac{d}{dm}\left[\frac{G}{r^2}(mM-m^2)\right]=0\ \Rightarrow M-2m=0$$

$$\Rightarrow \frac{m}{M}=\frac{1}{2}.$$

So, the force will be maximum when the parts are equal.

Ex. Three spherical balls of masses 1kg, 2kg and 3kg are placed at the corners of an equilateral triangle of side 1m. Find the magnitude of the gravitational force exerted by 2 kg and 3 kg masses on 1 kg mass.

Sol. If F_1 is the force of attraction between 1kg, 2kg masses, then, $F_1 = G \times \frac{1 \times 2}{(1)^2} \Rightarrow F_1 = 2G$



If F_2 is the force of attraction between 1kg, 3kg masses, then

$$F_2 = G \times \frac{1 \times 3}{(1)^2} \Rightarrow F_2 = 3G$$

The angle between the forces F_1 and F_2 is 60°. If 'F_R' is the resultant of these two forces, then

$$F_R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2 \cos\theta}$$

$$\Rightarrow F_R = \sqrt{(2G)^2 + (3G)^2 + 2 \times 2G \times 3G \times \cos 60°}$$

$$\Rightarrow F_R = \sqrt{19}\, G$$

Ex. In Cavendish's experiment, let each small mass be 20g and each large mass be 5 kg. The rod connecting the small masses is 50 cm long, while the small and the large spheres are separated by 10.0 cm. The torsion constant is $4.8 \times 10^{-8}\,\text{kgm}^2\text{s}^{-2}$ and the resulting angular deflection is 0.4°. Calculate the value of universal gravitational constant G from this data.

Sol. Here, m = 20g = 0.02kg, M = 5kg

r =10cm = 0.1m, l = 50cm = 0.5m

$$\theta = 0.4° = (0.4°)(2\pi / 360°) = 0.007\,\text{rad},$$

$$k = 4.8 \times 10^{-8}\,\text{kgm}^2\text{s}^{-2}$$

Thus, from $G = \frac{k\theta r^2}{Mml}$

$$G = \frac{\left(4.8\times10^{-8}\right)\left(0.007\right)\left(0.1\right)^2}{5\times0.02\times0.5}$$

$$= 6.72\times10^{-11}\,Nm^2kg^{-2}$$

Ex. The mean orbital radius of the Earth around the Sun is 1.5 $\times 10^8$km. Estimate the mass of the Sun.

Sol. As the centripetal force is provided by the gravitational pull of the Sun on the Earth

$$\frac{GM_sM_e}{r^2} = M_e r\omega^2 = M_e r\frac{4\pi^2}{T^2}$$

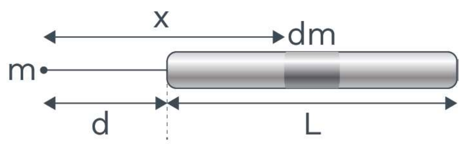
$$\Rightarrow \quad M_s = \frac{4\pi^2 r^3}{GT^2}$$

given, r = 1.5 × 10^8km = 1.5 × 10^{11}m

T = 365 days = 365 × 24 × 60 × 60 s

$$\therefore M_s = \frac{4\times(22/7)^2\times(1.5\times10^{11})^3}{(6.67\times10^{-11})\times(365\times24\times60\times60)^2} \approx 2\times10^{30}\text{kg}$$

Ex. A point mass 'm' is situated at a distance 'd' from one end of rod of mass 'M' and length 'L' as shown in fig. Calculate the gravitational force magnitude between them.



Sol. Consider an element of mass 'dm' and length 'dx' at a distance 'x' from the point mass.

Mass of the element $dm = \frac{M}{L}dx$.

Gravitational force on 'm' due to this element is

$$dF = \frac{Gm\left(\frac{M}{L}dx\right)}{x^2}$$

$$\Rightarrow F = \int_d^{(d+L)} \frac{Gm\left(\frac{M}{L}dx\right)}{x^2}$$

$$\Rightarrow F = \frac{GmM}{L}\int_d^{(d+L)} x^{-2}dx = \frac{GmM}{L}\left[\frac{x^{-1}}{-1}\right]_d^{(d+L)}$$

$$\Rightarrow F = \frac{GmM}{L}\left[\frac{-1}{x}\right]_d^{(d+L)} = \frac{GmM}{L}\left[\frac{1}{d} - \frac{1}{(d+L)}\right]$$

$$\Rightarrow F = \frac{GmM}{L}\left[\frac{d+L-d}{(d+L)d}\right] = \frac{GmM}{d(d+L)}$$

Gravitational Field:

- The space or region around a mass particle in which its gravitational effects are produced or experienced is known as gravitational field.

Gravitational field intensity ($\vec{I}$, $\vec{g}$ or $\vec{E}$):

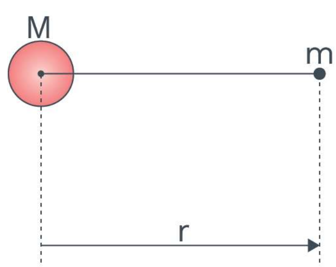
- The gravitational field intensity at a point is defined as the gravitational force exerted on unit mass placed at that point, in gravitational field.

$$\boxed{\vec{I} = \frac{\vec{F}}{m}}$$

- Gravitational field intensity is a vector quantity whose direction is same as that of gravitational force.
- Its SI unit is 'N/kg'.
- Dimensions of intensity

$$= \frac{[F]}{[m]} = \frac{[M^1L^1T^{-2}]}{[M^1]} = [M^0L^1T^{-2}]$$

Gravitational field intensity due to Point Mass:



$$\Rightarrow \quad \vec{I} = \frac{GM}{r^2}(-\hat{r})$$

$$\boxed{\vec{I} = \frac{GM}{r^2}(-\hat{r})}$$

where 'M' is the mass of that particle due to which we have to find intensity.

Definitions

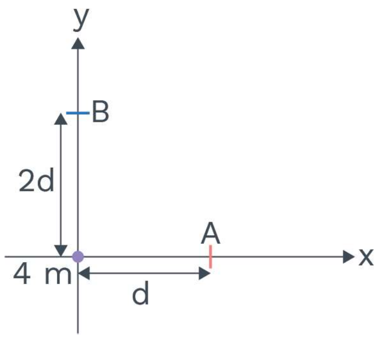
- The space or region around a mass particle in which its gravitational effects are produced or experienced is known as gravitational field.
- The gravitational field intensity at a point is defined as the gravitational force exerted on unit mass placed at that point, in gravitational field.

$$\boxed{\vec{I} = \frac{\vec{F}}{m}}$$

KEY POINTS

- Gravitational field
- Gravitational field intensity.

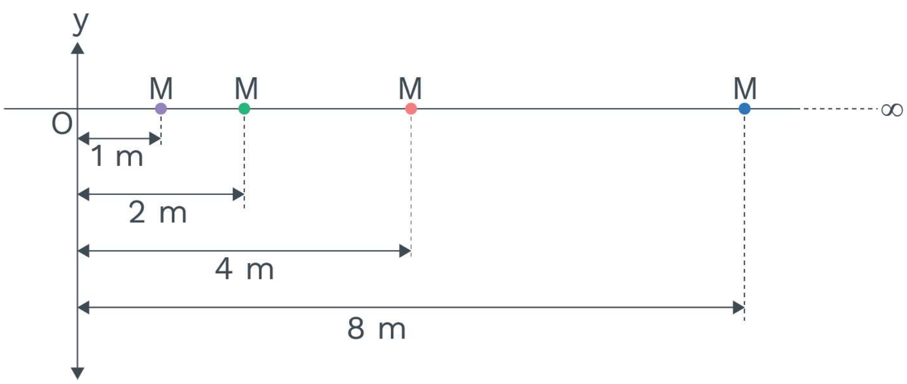
Ex. Calculate intensity of gravitational field at A and B.



Sol. $\vec{I}_A = \frac{G(4m)}{d^2}(-\hat{i})$

$\vec{I}_B = \frac{G(4m)}{(2d)^2}(-\hat{j})$

Ex. Find the net gravitational field intensity at the origin.



Sol. $\vec{I}_{net} = \vec{I}_1 + \vec{I}_2 + \vec{I}_3 + \vec{I}_4 + ...\infty$ terms

$$= \frac{GM}{(1)^2}\hat{i} + \frac{GM}{(2)^2}\hat{i} + \frac{GM}{(4)^2} + ...\infty \text{ terms}$$

$$= GM\hat{i}\left(1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + ...\infty\right)$$

$$\Rightarrow \vec{I}_{net} = \frac{4}{3}GM\hat{i}$$

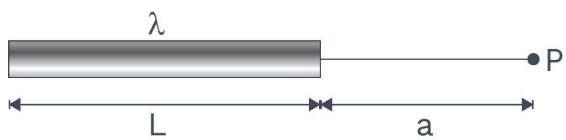
Gravitational Field (Due to Various Mass Distributions)

1. Gravitational field due to point mass

$$\vec{g} = \frac{-GM}{r^3}\vec{r} \quad \text{or} \quad g = \frac{GM}{r^2}$$

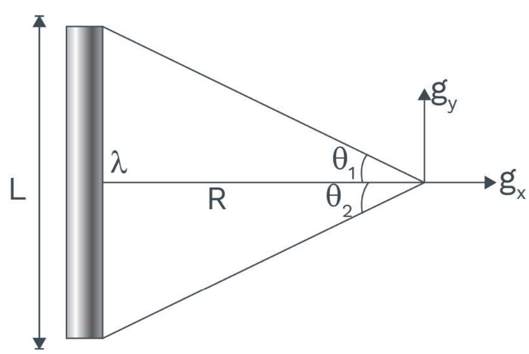
2. Gravitational field due to rod

(i) $g = \dfrac{G\lambda L}{a(a+L)}$



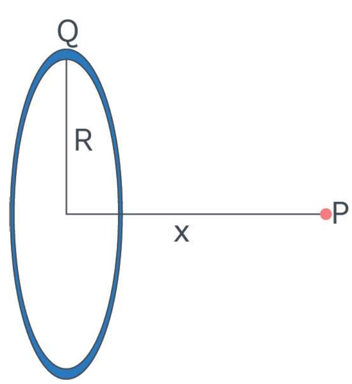
(ii) $g_x = \dfrac{G\lambda}{R}[\sin\theta_1 + \sin\theta_2]$

$g_y = \dfrac{G\lambda}{R}[\cos\theta_2 - \cos\theta_1]$



3. Gravitational field due to a ring at a point on the axis

$$g = \frac{GMx}{(x^2+R^2)^{3/2}}$$



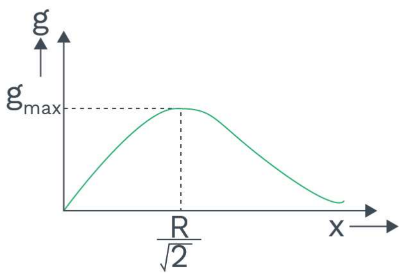
Rack your Brain

If force between two similar point masses at a distance of r from each other is F. If one of mass is doubled and distance between them will be halved, then find out percentage change in force between them.

Concept Reminder

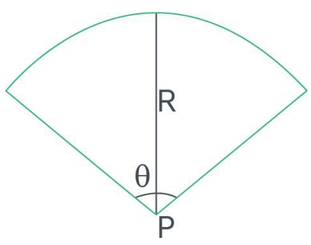
Gravitational field due to point mass M is

$$\vec{I} = \frac{-GM}{r^2}\hat{r}$$

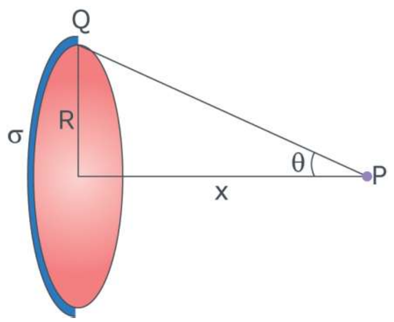


4. Gravitational field due to an arc with linear mass density λ at the centre of arc

$$g = \frac{2G\lambda}{R}\sin\frac{\theta}{2}$$



5. Gravitational field due to disc



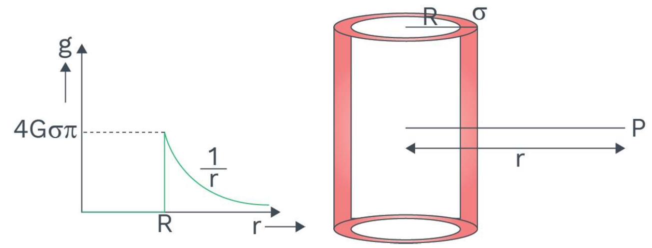
$$g = 2G\sigma\pi(1-\cos\theta)$$

$$= 2G\sigma\pi\left(1-\frac{x}{\sqrt{x^2+R^2}}\right)$$

6. Gravitational field due to infinitely long cylindrical shell

Concept Reminder

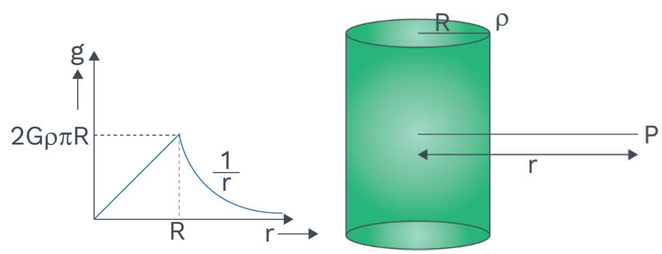
If two masses are released from rest to move towards each other under mutual gravitational attraction, then total mechanical energy and linear momentum of system remains conserved.



$$g = \frac{4G\sigma\pi R}{r}, \quad r \geq R$$

$$g_i = \text{zero}$$

7. Gravitational field due to infinitely long solid cylinder



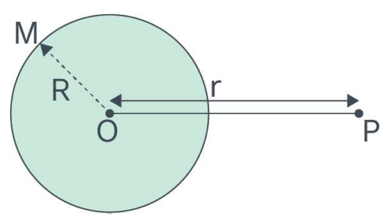
$$g_0 = \frac{2G\rho\pi R^2}{r} \qquad r \geq R$$

$$g_i = 2G\rho\pi r \qquad r < R$$

- **For solid sphere**
  Le 'M' be the mass of sphere, 'R' be the radius of sphere and 'r' the distance of observation points from the centre of sphere.

Case I: Outside the sphere i.e., r > R, then

$$\boxed{\vec{I}_{out} = \frac{GM}{r^2}(-\hat{r})}$$

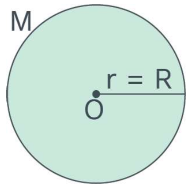


Concept Reminder

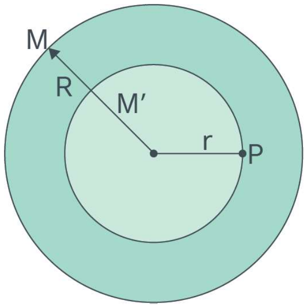
The force of attraction between a hollow spherical shell of uniform density and a point mass situated outside is just as if entire mass of shell is concentrated at centre of shell.

Case II: At the surface i.e., r = R, then

$$\vec{I}_{surface} = \frac{GM}{R^2}(-\hat{r})$$



Case III: Inside the sphere i.e., r < R, then

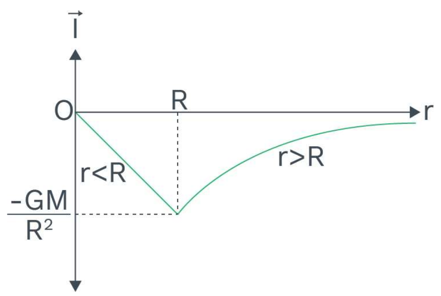


$$\vec{I}_{in} = \frac{GM'}{r^2} \qquad \text{..... (1)}$$

Now, we know, $\frac{M'}{M} = \frac{V' \times D'}{V \times D} = \frac{\frac{4}{3}\pi r^3 \times \rho}{\frac{4}{3}\pi R^3 \times \rho} = \frac{r^3}{R^3}$

Putting value of M' in e.g. (1), we get $\boxed{\vec{I}_{in} = \frac{GMr}{R^3}(-\hat{r})}$

⇒ Graph between 'I' and 'r' for a solid sphere:



$$I_{max} = I_{sur} = \frac{GM}{R^2}$$

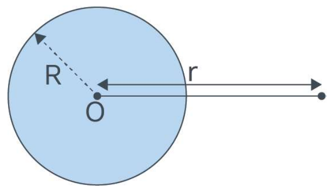
Concept Reminder

- For hollow sphere
  if r < R
  $\vec{I}_{inside} = 0$
  if $r \geq R$
  $\vec{I}_{outside} = \frac{-GM}{r^2}\hat{r}$
- For solid sphere
  if r < R
  $\vec{I}_{inside} = \frac{-GM\vec{r}}{R^3}$
  if $r \geq R$
  $\vec{I}_{outside} = \frac{-GM}{r^2}\hat{r}$

- **For hollow sphere**

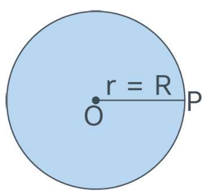
Case I: Outside the sphere i.e., $r > R$, then

$$\vec{I}_{out} = \frac{GM}{r^2}(-\hat{r})$$



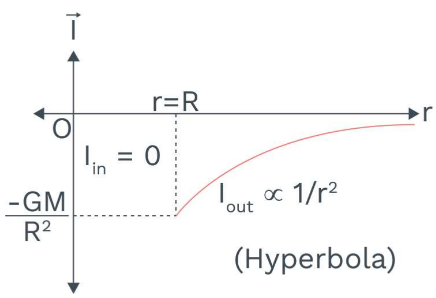
Case II: At the surface i.e., $r = R$, then

$$\vec{I}_{surface} = \frac{GM}{R^2}(-\hat{r})$$

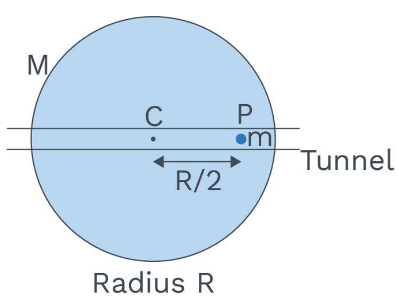


Case III: $I_{in} = 0$

$\Rightarrow$ I v/s r graph for hollow sphere:



Ex. Calculate the gravitational force between M and m.



Concept Reminder

The force of attraction due to hollow spherical shell of uniform density, on a point mass situated inside it is, zero.

Gravitation

Sol. **Step-1:** First find I at point P

$$I_P = \frac{GM(R/2)}{R^3}$$

Step-2:

$$F = mI_P = m\left[\frac{GM(R/2)}{R^3}\right] \Rightarrow \boxed{F = \frac{GmM}{2R^2}}$$

Gravitational Lines Of Forces

Gravitational field can also be represented by lines of force. A line of force is drawn in such a way that at each point the direction of field is tangent to line that passes through the point. Thus, tangent to any point on a line of force gives the direction of gravitational field at that point. By convention lines of force are drawn in such a way that their density is proportional to the strength of field. Figure shows the field of a point mass in its surrounding. We can see that the lines of force are radially inward giving direction of field and as we go closer to the mass the density of lines is more which shows that field strength is increasing.

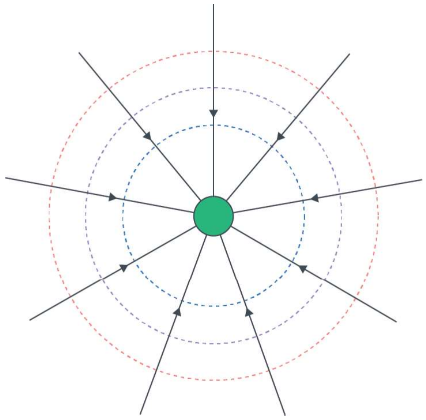
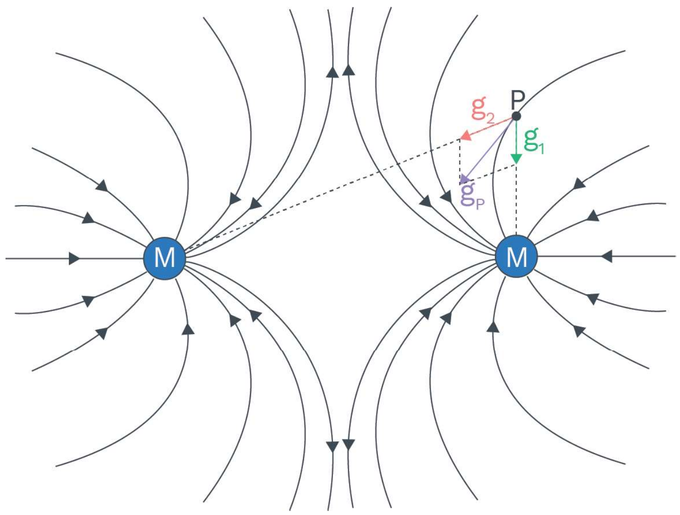


Figure shows the configuration of field lines for a system of two equal masses separated by a given distance. Here we can see that there is no point where any two lines of force intersects or meet. The reason is obvious that at one point in space there can never be two directions of gravitational

Rack your Brain

Two particles of equal mass m go around a circle of radius R under the action of their mutual gravitational attraction. Find speed of each particle.

fields. It should be noted that a line of force gives the direction of net gravitational field in the region. Like electric field, gravitational field never exists in closed loops.



- **Gravitational Flux:** $\phi = \int \vec{g}.\vec{ds}$
- **Gravitational Gauss law:** $\oint \vec{g}.\vec{ds} = -4\pi GM_{in}$

  Here $\vec{g}$ is the gravitational field due to all the masses. M_{in} is the mass inside the assumed Gaussian surface.

Acceleration Due To Gravity (g):

- Earth attracts other bodies by the gravitational force of attraction which is known as gravity force and the acceleration produced due to this force in a freely falling body is known as acceleration due to gravity.

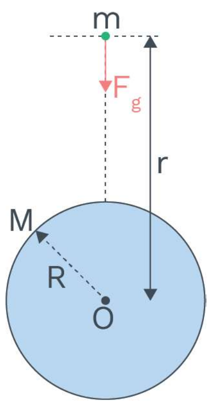
$$F = \frac{GMm}{r^2}$$

$$\text{Now, } a = \frac{F}{m} = g$$

$$\Rightarrow \boxed{g = \frac{GM}{r^2}}$$

Definitions

Earth attracts other bodies by the gravitational force of attraction which is known as gravity force and the acceleration produced due to this force in a freely falling body is known as acceleration due to gravity.



on the surface $\boxed{g_{surface} = \frac{GM}{R^2}}$

where, $\left[\begin{array}{l} G = 6.67 \times 10^{-11} Nm^2 / kg^2 \\ M = \text{Mass of earth} = 6 \times 10^{24} kg \\ R = \text{Radius of earth} = 6400\,km \end{array}\right]$

$\boxed{g = 9.81 m/s^2 \approx 10\,m/s^2}$ (In S.I unit)

$\boxed{g = 32\,feet/s^2}$

In terms of density of earth, (ρ):

$$g = \frac{GM}{R^2} = \frac{G \times \rho \times \frac{4}{3}\pi R^3}{R^2} \quad \boxed{g_{surface} = \frac{4}{3}\pi G R \rho}$$

Problem based on up to 5% variation:

(1) If mass (M) and radius (R) of a planet, if small change is occurring in (M) and (R) then $g = \frac{GM}{R^2}$

$$\boxed{\frac{\Delta g}{g} = \frac{\Delta M}{M} - 2.\frac{\Delta R_e}{R_e}}$$

(2) If M is constant $\therefore g \propto \frac{1}{R^2}$

$$\boxed{\frac{\Delta g}{g} = -2.\frac{\Delta R_e}{R_e}}$$

(3) If R is constant $\therefore g \propto M$

$$\frac{\Delta g}{g} = \frac{\Delta M}{M}$$

Concept Reminder

$$g_{surface} = \frac{Gm}{R^2} = \frac{4}{3}\pi G R \rho$$
$$= 9.8\,m/s^2$$

KEY POINTS

- Gravitational line of forces
- Acceleration due to gravity.

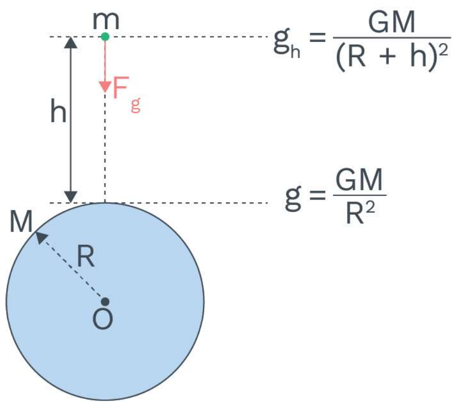
Concept Reminder

If variation is less than 5%

$$\frac{\Delta g}{g} = \frac{\Delta m}{M} - \frac{2\Delta R}{R}$$

Factors affecting acceleration due to gravity (g):

(1) Effect of Altitude (height):



$\because \quad g = \frac{GM}{R^2}$

$\Rightarrow \quad \therefore \quad \boxed{GM = gR^2}$ .... (1)

$g_h = \frac{GM}{(r+h)^2} = \frac{gR^2}{(R+h)^2}$ [From eq. (1)]

$\Rightarrow \quad g_h = \frac{gR^2}{R^2\left(1+\frac{h}{R}\right)^2} \qquad \boxed{g_h = \frac{g}{R^2\left(1+\frac{h}{R}\right)^2}}$

Now, if h <<< R, means h/R <<<1, then by using Binomial theorem

$g_h = g\left(1-2\left(\frac{h}{R}\right)\right) \; [\because (1+x)^{-n} = 1-nx]$

$\Rightarrow \quad \boxed{g_h = g\left(1-\frac{2h}{R}\right)}$

(Valid if h $\leq$ 5% of R, means h $\leq$ 320 km)

$\Rightarrow$ **Decrement in 'g' with height 'h':** $\left[(\Delta g)_h\right]$

$(\Delta g)_h = g - g_h$

$\Rightarrow \quad (\Delta g)_h = g - g\left(1-\frac{2h}{R}\right)$

Rack your Brain

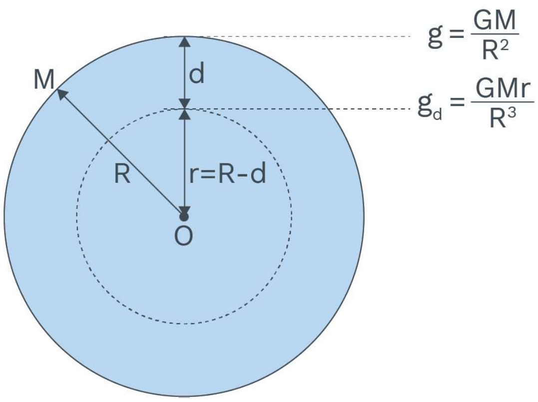
The density of newly discovered planet is twice that of earth. The acceleration due to gravity at the surface of the planet is equal to that at the surface of earth. If the radius of earth R then find radius of planet.

$$\Rightarrow \ (\Delta g)_h = g - g + \frac{g2h}{R}$$

Fractional decrement $\Rightarrow \boxed{\frac{(\Delta g)_h}{g} = \frac{2h}{R}}$ (valid up to 5%)

(2) Effect of Depth (d):

Acceleration due to gravity in form of density (at earth surface)



So, $g_{surface} = \frac{4}{3}\pi R\rho G$ .... (1)

$$g_d = \frac{4}{3}\pi G\rho r \Rightarrow g_d = \frac{4}{3}\pi(R-d)\rho G \quad .... (2)$$

Eq. (2) divided by (1)

$$\boxed{g_d = g\left(1 - \frac{d}{R}\right)}$$

(Valid for all depths or 100% depth)

Fractional decrement is $\boxed{\frac{(\Delta g)_d}{g} = \frac{d}{R}}$

(Valid for all depths)

(3) Shape of the earth

Earth is not a ideal sphere and its radius at equator is more than poles by 21 km.

Concept Reminder

(1) Effect of height

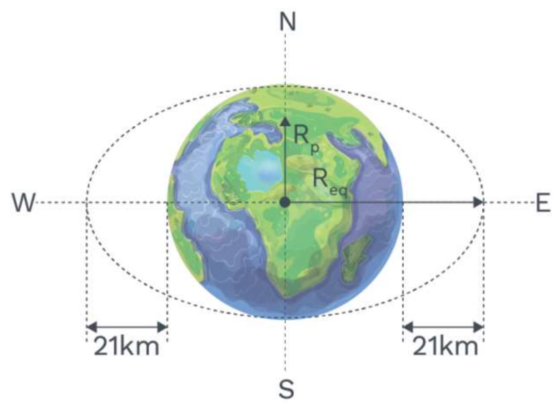
$$g' = \frac{GM}{(R+h)^2} = \frac{gR^2}{(R+h)^2}$$

if h << R

$$\Rightarrow \ g' = g\left(1 - \frac{2h}{R}\right)$$

(2) Effect of depth $g' = g\left(1 - \frac{d}{R}\right)$

Earth:



Now, $g_p = \dfrac{GM}{R_p^2}$, $\quad g_{eq} = \dfrac{GM}{R_{eq}^2}$

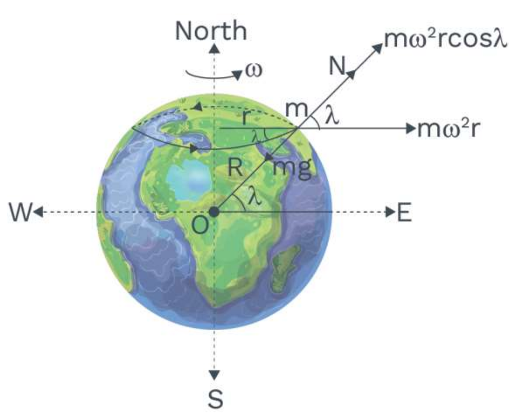
$\therefore \quad R_p < R_{eq} \qquad \therefore \quad g_p > g_{eq}$

Difference in 'g' at poles and equator due to shape.

$$\boxed{(\Delta g)_{shape} = g_p - g_{eq} = 0.02\, m/s^2}$$

(4) Rotation of earth:

$$\boxed{r = R\cos\lambda}$$



$\Rightarrow$ Net inward force or $W_{apparent}$

$N = mg' = mg - m\omega^2 r\cos\lambda$

$\Rightarrow \boxed{g' = g - \omega^2 R\cos^2\lambda}$

Concept Reminder

The acceleration due to gravity is maximum at earth's surface. It decreases either we go at higher attitudes, or we move below the surface.

KEY POINTS

- Equator
- Pole
- Apparent weight
- Angle of latitude

where,

λ = Angle of latitude; R = Radius of earth

r = radius of circular orbit of particle.

ω = Angular velocity of rotation of earth.

g = Acceleration due to gravity without considering the rotation of earth.

g' = Effective acceleration due to gravity after considering the rotation earth.

mg' = Apparent or effective weight.

- **Important Points:**

(1) At the equator $(\lambda = 0°)$:

$$g_{eq} = g - \omega^2 R\cos^2(0°)$$

$$\boxed{g_{eq} = g - \omega^2 R}$$

(2) At the poles $(\lambda = 90°)$:

$$g_{pole} = g - \omega^2 R\cos^2(90°) = g - \omega^2 R(0)$$

$$\boxed{g_{Pole} = g}$$

(3) If the earth suddenly stops rotating about its own axis, then apparent weight of bodies or effective acceleration due to gravity will increase at all the places except poles.

(4) If the angular velocity of rotation of earth is gradually increased, then apparent weight of bodies will decrease (except the poles). At a particular 'ω', the apparent weight will become zero. This condition is known as conditions of weightlessness.

Ex. Calculate that imaginary angular speed of Earth so that body placed at equator becomes weight less and what will be length of Day in this case.

Sol. $mg' = mg - m\omega^2 R\cos^2\lambda$

But for weightlessness, $W_{app} = 0$

$\Rightarrow 0 = mg - m(\omega)^2 R\cos^2\lambda$

$$\Rightarrow (\omega')^2 = \frac{g}{R\cos^2\lambda}$$

$$\Rightarrow \boxed{\omega' = \frac{1}{\cos\lambda}\sqrt{\frac{g}{R}}}$$

At equator $(\lambda = 0°)$

$$\text{So, } \omega' = \frac{1}{\cos(0°)}\sqrt{\frac{10}{6400 \times 1000}} = \frac{1}{800}\text{rad/s} = 1.25 \times 10^{-3}\text{rad/s} \quad \boxed{\omega' \approx 17 \times \omega_{actual}}$$

$\Rightarrow$ Time period (T):

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{g/R}} = 2\pi\sqrt{\frac{R}{g}}$$

So, $T = 5063\,s = 84.6\,min. \simeq 1.4\,hrs.$

T = 1.4 hrs. = 84.6 min = 5063 sec.

Ex. A star 2.5 times large the mass of the sun is reduced to a size of 12 km and rotates with a speed of 1.5 rps. Will an object placed on its equator remain stuck to its surface due to gravity? (Mass of the sun = 2 × 10^{30} kg).

Sol. Acceleration due to gravity, $g = \frac{GM}{R^2}$

$$= \frac{6.67 \times 10^{-11} \times 2.5 \times 2 \times 10^{30}}{(12000)^2} = 2.3 \times 10^{12} ms^{-2}$$

Centrifugal acceleration

$= r\omega^2 = r(2\pi f)^2 = 12000(2\pi \times 1.5)^2 = 1.1 \times 10^6\ ms^{-2}$

Since, $g > r\omega^2$, the body will remain stuck with the surface of star.

Ex. The height at which the acceleration due to gravity becomes g/9 (where g is the acceleration due to gravity on the earth surface) in terms of the radius of the earth (R) is

Sol. Given $\frac{g}{9} = g\left(\frac{R}{R+h}\right)^2 \Rightarrow \frac{R}{R+h} = \frac{1}{3}$

$3R = R + h \Rightarrow 2R = h$

Ex. How much above the surface of earth does the acceleration due to gravity reduce by 36% of its value on the surface of earth.

Sol. Then g reduces by 36%, the value of g there is

100 – 36 = 64%. It means, $g' = \frac{64}{100} g$.

If 'h' is the height of location above the earth surface, then

$$g' = g\frac{R^2}{(R+h)^2} \Rightarrow \frac{64}{100} g = g\frac{R^2}{(R+h)^2}$$

$$\Rightarrow \frac{8}{10} = \frac{R}{R+h} \Rightarrow h = \frac{R}{4} = \frac{6.4 \times 10^6}{4} = 1.6 \times 10^6 m$$

Ex. Find out the percentage decrease in the weight of the object when taken to a depth of 32 Km below the earth surface.

Sol. Weight of the object at depth 'd' is

$$mg' = mg\left(1 - \frac{d}{R}\right)$$

$$\% \text{ Decrease in weight} = \frac{mg - mg'}{mg} \times 100$$

$$= \frac{d}{R} \times 100 = \frac{32}{6400} \times 100 = 0.5\%$$

Ex. A man can jump 1.5m on the Earth. Find out the approximate height he might be able to jump on a planet whose density is one-quarter that of the Earth and whose radius is one-third that of the Earth.

Sol. We know that, in case of Earth,

$$g = \frac{GM}{R^2} = G \times \frac{(4\pi/3)R^3\rho}{R^2} = \left(\frac{4}{3}\pi G\right)R\rho$$

Similarly, for the other planet whose radius $\frac{R}{3}$ and density $\frac{\rho}{4}$ is, $g' = \left(\frac{4\pi G}{3}\right)\left(\frac{R}{2}\right)\left(\frac{\rho}{4}\right)$

$$g' = \frac{1}{12}\left(\frac{4\pi G}{3}\right)R_{\rho} = \frac{1}{12}g \Rightarrow \frac{g}{g'} = 12$$

$$h_{max} = \frac{u^2}{2g} \Rightarrow h_{max} \propto \frac{1}{g} \text{ (Here u is constant)}$$

$$\frac{h'}{h} = \frac{g}{g'} = 12 \Rightarrow h' = 12h = 12 \times 1.5 = 18\,m$$

Work Done In Gravitational Field

Work done against Gravity by an External Agent:

Consider the gravitational field of a fixed mass M. Let us bring a particle of mass m from infinity (a very large distance from the fixed mass M) to any point P. While bringing the particle m slowly, its acceleration is zero. That means, the net force acting on the particle is zero. Hence the external agent will have to apply a force $\vec{F}_{ext}$ that must counter-balance the gravitational force $\vec{F}_g$ acting on the particle due to the fixed mass. Since the particle is displaced radially towards the fixed mass, work done by the force $\vec{F}_{ext}$ for an elementary displacement of m is given as $dW_{ext} = \vec{F}_{ext}.d\vec{s} = F_{ext} ds \cos 180°$.

Concept Reminder

If angular velocity of earth rotation about its own axis is increased by 17 times, then body placed at equator will be weightless.

$\Rightarrow$ The total work done by the external agent in bringing the test mass m, from infinity ($r_1 = \infty$) to any point 'P' which is at a distance r_2 = r from the fixed mass, is given as:

$$W_{ext} = \int dW_{ext}$$

$$= \int_{r_1}^{r_2} F_{ext} ds \cos 180°$$

$$= -\int_{r_1}^{r_2} F_{ext} (ds)$$

Putting $F_{ext} = F_g = \frac{GMm}{r^2}$ and ds = −dr, we obtain,

$$W_{ext} = \int_{\infty}^{r} \frac{GMm}{r^2} dr.$$

$$\Rightarrow W_{ext} = -\frac{GMm}{r}\Big|_{\infty}^{r}$$

$$\Rightarrow W_{ext} = -\frac{GMm}{r} \qquad \text{... (i)}$$

Therefore, the work done by gravity on the particle in bringing it from infinity to the point (P) of consideration is given as:

$$W_g = -W_{ext} \Rightarrow W_g = \frac{GMm}{r}.$$

Definitions

The work done by gravity on the particle in bringing it from infinity to the point (P) of consideration is given as:

$$W_g = -W_{ext} \Rightarrow W_g = \frac{GMm}{r}$$

Gravitational Potential (V):

The gravitational potential at a point in the gravitational field of a body is known as the quantity of work done by an external agent in bringing a body of unit mass from infinity to that point, slowly (no change in kinetic energy).

Definitions

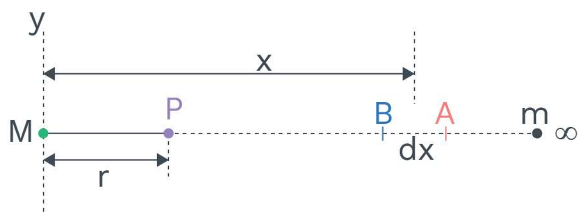
The gravitational potential at a point in the gravitational field of a body is defined as the amount of work done by an external agent in bringing a body of unit mass from infinity to that point, slowly (no change in kinetic energy).

Gravitational potential due to a point mass:

$$V_P = \frac{W_{\infty \to P}}{m}$$

The unit of gravitational potential is J kg^{-1}. Dimensional Formula of gravitational potential

$$= \frac{\text{Work}}{\text{mass}} = \frac{[ML^2T^{-2}]}{[M]} = [M^0L^2T^{-2}]$$



We know, $F = \frac{GM(m)}{x^2} = \frac{GMm}{x^2}$

So, work done for very small displacement (dx):

$dW = Fdx$

$\Rightarrow dW = \frac{GMm}{x^2}dx$

Now, integrating both sides, $\int_0^W dW = \int_\infty^r \frac{GMm}{x^2}dx$

$W_{\infty\to p} = -\frac{GMm}{r}$

$\Rightarrow \frac{W_{\infty\to p}}{m} = -\frac{GM}{r}$

$$\boxed{V_P = -\frac{GM}{r}}$$

The value of gravitational potential depends on the reference level chosen.

Here, negative sign means that this potential is due to an attractive force.

Note: The gravitational potential is maximum at infinite and its maximum value is 'zero'.

Potential Gradient:

The rate of change of potential w.r.t distance is known as potential gradient, means.

Potential gradient $= \frac{dV}{dr}$

Concept Reminder

If we choose $r = \infty$ as reference point, then potential due to point mass is given as $V = \frac{-GM}{r}$

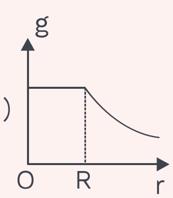
KEY POINTS

- Gravitational potential
- Potential gradient

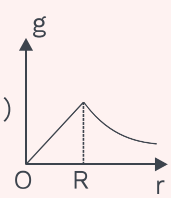
Rack your Brain

Starting from centre of earth having radius R, the variation of g is shown by which of following graph?

(i)



(ii)



Relation between intensity and potential gradient:

$$\boxed{I = -\frac{dV}{dr} = -\text{ve potential gradient}}$$

Note (1): $F = -\frac{dU}{dr}$ and $I = -\frac{dV}{dr}$ are valid for all conservative force.

Note (2): For hollow sphere,

$\because \quad I_{in} = 0$

$\therefore \quad -\frac{d}{dr}\left(V_{in}\right) = 0 \qquad \left[\because I = -\frac{dV}{dr}\right]$

$\therefore \quad V_{in}$ should be 'CONSTANT'

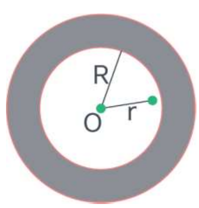
- The gravitational potential inside a hollow sphere remains constant and its value is same as that of surface.

Gravitational Potential due to a Spherical Shell:

Let M be the mass of spherical shell and R is its radius

- At a point inside the spherical shell, (If r < R)

$$V_{inside} = \frac{-GM}{R}$$



- At a point on the surface of the spherical shell,

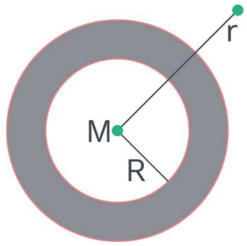
$$V_{surface} = \frac{-GM}{R} \qquad \text{(If } r = R\text{)}$$

$$V_{centre} = \frac{-GM}{R} \qquad (r = 0 \text{ at centre})$$

$$V_{inside} = V_{surface} = V_{centre} = -\frac{GM}{R}$$

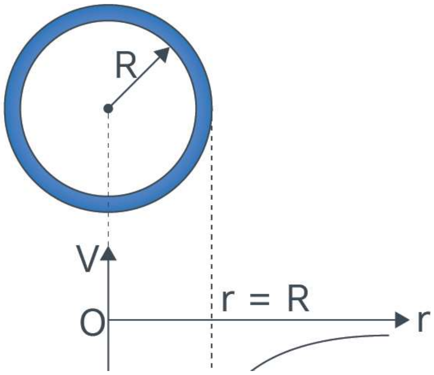
- At a point outside the spherical shell,

$$V_{outside} = \frac{-GM}{r} \qquad \text{(If } r > R\text{)}$$



At infinity, $V_{\infty} = 0$

The variation of magnitude of V with r is as shown (For a spherical shell)



Gravitational Potential due to a Solid Sphere:

- At a point inside the solid sphere,

$$V_{inside} = \frac{-GM}{2R^3}(3R^2 - x^2)$$

$$V_{inside} = -GM\left(\frac{3}{2R} - \frac{x^2}{2R^3}\right) \qquad \text{(If } x < R\text{)}$$

- At a point on the surface of the solid sphere,

$$V_{surface} = \frac{-GM}{R} \qquad \text{(If } x = R\text{)}$$

- At a point outside the solid sphere,

$$V_{outside} = \frac{-GM}{x} \qquad \text{(If } x > R\text{)}$$

Concept Reminder

(i) For spherical shell

if $r \le R$

$$V_{inside} = \frac{-GM}{R} = \text{constant if } r >$$

R,

$$V_{outside} = \frac{-GM}{R}$$

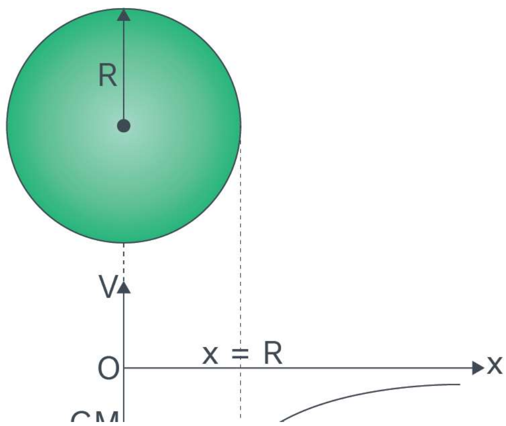
(ii) For solid sphere

if $r \le R$ $V_{inside} = \frac{-GM}{2R^3}\left(3R^2 - r^2\right)$

if $r > R$,

$$V_{outside} = \frac{-GM}{r}$$

- At the centre, $x = 0 \Rightarrow V_c = -\frac{3}{2}\frac{GM}{R} = \frac{3}{2}V_{surface}$
- **The variation of V with x is shown:**



- In case of solid sphere magnitude of potential is maximum at centre.
- **Gravitational potential difference:** The amount of work done in bringing a unit mass between two points in the gravitational field is called as the gravitational potential difference between the two points.

$$\Delta V = V_b - V_a = -\left(\frac{W_b - W_a}{m_0}\right)$$

$$W_{ab} = -m_0(V_b - V_a) = -Gmm_0\left(\frac{1}{r_b} - \frac{1}{r_a}\right)$$

Relation between gravitational field and potential:

- Gravitational field and the gravitational potential are related by $\vec{E} = -\text{gradient}V = -\text{grad}V$

$$\vec{E} = -\left[\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right]$$

Here, $\frac{\partial V}{\partial x}$ = Partial derivative of potential function V with respect to x, i.e., differentiate V w.r.t. x assuming y and z to be constant.

Concept Reminder

Relation between gravitation field and potential

- $\vec{I} = -\left[\frac{\partial V}{\partial x}\hat{i} - \frac{\partial V}{\partial y}\hat{j} - \frac{\partial V}{\partial z}\hat{k}\right]$
- $V = -\int_{\infty}^{r} \vec{I}.d\vec{r}$

The above equation can be written in the following forms.

- $E = \frac{-dV}{dx}$, If gravitational field is along x-direction only.
- $dV = -\vec{E}.\overrightarrow{dr}$, (where $\overrightarrow{dr} = dx\hat{i} + dy\hat{j} + dz\hat{k}$ and $\vec{E} = E_x\hat{i} + E_y\hat{j} + E_z\hat{k}$)

Note: a) If $\vec{E}$ is given V can be calculated by the formula $V = \int_{\infty}^{r} dV = -\int_{\infty}^{r} \vec{E}.\overrightarrow{dr}$

(b) The negative slope of V-r curve gives E

Gravitational Potential energy:
The quantity of work done by the gravitational force in bringing a body from infinity to any point in the gravitational field is specify as the gravitation potential energy at that point.

For a conservative field, $F = -\frac{dU}{dr}$

$$\Rightarrow dU = -\vec{F}.\overrightarrow{dr}$$

$$\Rightarrow \int_{U_0}^{U} dU = -\int_{r_0}^{r} \vec{F}.\overrightarrow{dr}$$

$$\Rightarrow U - U_0 = -\int_{r_0}^{r} \vec{F}.\overrightarrow{dr}$$

We generally select the reference point at infinity and consider potential energy to be zero there. If we take $r_0 = \infty$ and $U_0 = 0$ then,

$$\Rightarrow U = -\int_{\infty}^{r} \vec{F}.\overrightarrow{dr} = -W\left[\text{as } \int_{\infty}^{r} \vec{F}.\overrightarrow{dr} = W\right]$$

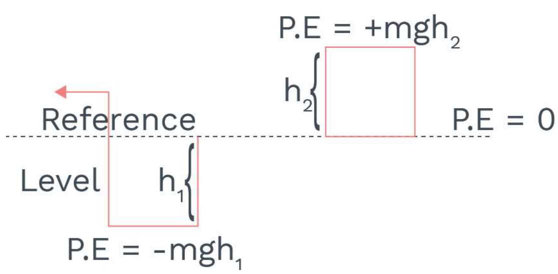
Potential energy of a body or a system is the negative work done by the conservative forces in bringing it from infinity to present position.

Definitions

The amount of work done by the gravitational force in bringing a body from infinity to any point in the gravitational field is defined as the gravitation potential energy at that point.

KEY POINTS

- Gravitational potential difference
- Gravitational potential energy



- If a particle moves opposite to the field direction, then work done by the field will be negative. So potential energy will increase and change in potential energy will be positive.
- If a particle moves in the direction of the field work done is positive, so potential energy decreases and change in potential energy is negative.
- Potential energy exists for only conservative forces and it does not exist for non-conservative forces.
- By the definition of gravitational potential,

$$V = -\frac{W}{m} = \frac{U}{m} \Rightarrow U = mV$$

Gravitational Potential Energy of Two Particle System:

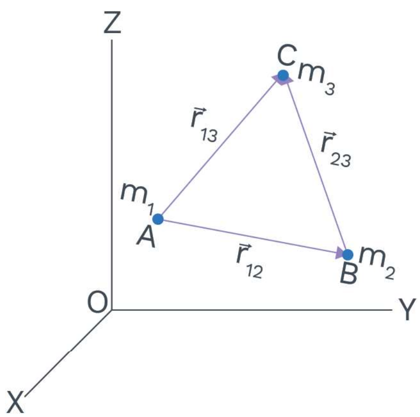
- The gravitational potential energy of two particles of masses m_1 and m_2 separated by a distance r is given by $U = -\frac{Gm_1m_2}{r}$

Gravitational Potential Energy of Three Particle System:

Consider a system consists of three particles of masses m_1, m_2 and m_3 located at A, B and C respectively. Total potential energy 'U' of the system is $U = -G\left[\frac{m_1m_2}{|\vec{r}_{12}|} + \frac{m_2m_3}{|\vec{r}_{23}|} + \frac{m_1m_3}{|\vec{r}_{13}|}\right]$

Concept Reminder

The gravitational potential energy of two particles of masses m_1 and m_2 separated by a distance r is given by $U = -\frac{Gm_1m_2}{r}$



- If a body is moving only under the influence of gravitational force, from law of conservation of mechanical energy $U_1 + K_1 = U_2 + K_2$.

Gravitational Potential Energy of a System of Particles:-

- The gravitational potential energy for a system of n particles is given by

$$U = \sum U_i = -\left[\frac{Gm_1m_2}{r_{12}} + \frac{Gm_2m_3}{r_{23}} + \ldots\ldots\right]$$

For n particle system there are $\frac{n(n-1)}{2}$ pairs and the potential energy is determined for each pair and added to get total potential energy of the system.

Gravitational Potential Energy of a body in Earth's Gravitational Field:

- If a point mass 'm' is at a distance r from the centre of the earth, then, $U = -\frac{GMm}{r}$
- On the surface of earth,

$$U_{surface} = -\frac{GMm}{R} = -mgR\left(\because\ g = \frac{GM}{R^2}\right)$$

- At a height 'h' above the surface of earth, $U_h = -\frac{GMm}{R+h}$

Concept Reminder

The gravitational potential energy for a system of n particles is given by

$$U = \sum U_i$$
$$= -\left[\frac{Gm_1m_2}{r_{12}} + \frac{Gm_2m_3}{r_{23}} + \ldots\ldots\right]$$

Rack your Brain

A particle of mass M is situated at the centre of a spherical shell of some mass and radius a. Find the magnitude of gravitational potential at a point situated at a/2 distance from centre.

- The difference in potential energy of the body of mass m at a height h and on the surface of earth is

$$\Delta U = U_h - U_{surface}$$

$$= -\frac{GMm}{R+h} - \left(-\frac{GMm}{R}\right) = GMm\left(\frac{1}{R} - \frac{1}{R+h}\right)$$

$$= \frac{GMmh}{(R+h)R} = \frac{GMmh}{R^2\left(1+\frac{h}{R}\right)} \Rightarrow \Delta U = \frac{mgh}{1+\frac{h}{R}}$$

$h << R, \Delta U \approx mgh$

- Work done in lifting a body of mass m from earth surface to a height h above the earth's surface is

$$W = U_h - U_{surface}; W = GMm\left(\frac{1}{R} - \frac{1}{R+h}\right) = \frac{mgh}{1+\frac{h}{R}}$$

- Gravitational potential energy at centre of the earth is given by

$$U_c = mV_c = -\frac{3}{2}\frac{GMm}{R}$$

Here, $V_c = \frac{3}{2}V_s = \frac{-3GM}{2R}$ (It is minimum but not zero. However, 'g' at centre of earth is zero)

- **Self-potential energy of a uniform sphere of mass 'M' and radius 'R':**

It is the amount of work done to bring identical massive particles to construct a sphere of mass M, radius R and density ρ.

For a sphere of radius 'x', mass of the sphere $= \frac{4}{3}\pi x^3 \rho$, Gravitational potential on the surface = $\frac{-4}{3}\pi G\rho x^2$

(Since gravitational potential

$$= \frac{-G}{x} \times \frac{4}{3}\pi x^3 \rho = \frac{-4}{3}\pi G x^2 \rho \;)$$

Work done by agent in increasing the surface from x to x + dx is

Concept Reminder

Work done in lifting a body of mass m from earth surface to a height h above earth's surface is

$$W = \frac{mgh}{1+\frac{h}{R}}$$

Definitions

Self-potential energy

It is the amount of work done to bring identical massive particles to construct a sphere of mass M, radius R and density ρ.

$$\frac{-Gm(dm)}{x} = \text{Gravitational potential} \times dm$$

$$= \left(\frac{-4}{3}\pi G x^2 \rho\right)\left(4\pi x^2 dx \rho\right) = \frac{16\pi^2}{3} G\rho^2 x^4 dx$$

Therefore, total work done

$$= \frac{-16\pi^2}{3} G\rho^2 \int_0^R x^4 dx = \frac{-16\pi^2 G\rho^2 R^5}{15}$$

$$= \frac{-16\pi^2 G R^5}{15}\left(\frac{M}{\frac{4}{3}\pi R^3}\right)^2 = \frac{-3}{5}\frac{GM^2}{R}$$

= Gravitational self-potential energy of a sphere.

- **Self-potential energy of a thin uniform shell of mass 'm' and radius 'R' is** $-\frac{GM^2}{2R}$
- Change in the gravitational potential energy in lifting a body from the surface of the earth to a height equal to 'nR' from the surface of the earth

$$\Delta U = \frac{GMmh}{R(R+h)} = \frac{GMm(nR)}{R(R+nR)} = \frac{GMmn}{R(n+1)} = \frac{mgRn}{n+1}$$

Concept Reminder

Self-Potential Energy

(i) Spherical shell

$$U = \frac{-GM^2}{2R}$$

(ii) Solid sphere

$$U = \frac{-3GM^2}{5R}$$

Ex. The gravitational field due to a mass distribution is given by $E = -K/x^3$ in x-direction. Suppose the gravitational potential to be zero at infinity, find out its value at a distance x.

Sol. The potential at a distance x is

$$V = -\int E dx = \int_{\infty}^{x} \frac{K}{x^3} dx = \left[\frac{-K}{2x^2}\right]_{\infty}^{x} = \frac{-K}{2x^2}$$

Ex. A particle of mass m is placed at the centre of a uniform spherical shell of equal mass and radii a. Find out the gravitational potential at a point 'P' at a distance a/2 from the centre of spherical shell.

Sol. The gravitational potential at 'P' due to shell at centre is $V_1 = \frac{-Gm}{a/2} = \frac{-2Gm}{a}$

The potential at 'P' due to shell is $V_2 = \frac{-Gm}{a}$

The net potential at P is $V_1 + V_2 = \frac{-3Gm}{a}$

Ex. The gravitational field in a region is given by the equation E = (5i + 12j) N/kg. If a particle of mass 2kg is moved from the origin to the point (12m, 5m) in this region, the change in the gravitational potential energy is

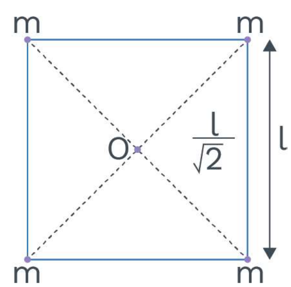
Sol. $dV = -\vec{E}.\overrightarrow{dr}$

$$= -(5\hat{i} + 12\hat{j}).(12\hat{i} + 5\hat{j}) = -(60 + 60) = -120 \text{ J/Kg}$$

Change in gravitational potential energy

dU = mdV= 2 (–120) = –240 J

Ex. Find the gravitational potential energy of a system of four particles, each of mass m placed at the vertices of as square of side l. Also obtain the gravitational potential at centre of the square.



Sol. The system has four pairs with distance l and two diagonal pairs with distance $\sqrt{2}l$.

$$U = -4\frac{Gm^2}{l} - 2\frac{Gm^2}{\sqrt{2}l} = -\frac{2Gm^2}{l}\left(2 + \frac{1}{\sqrt{2}}\right)$$

The gravitational potential at the centre of the square is

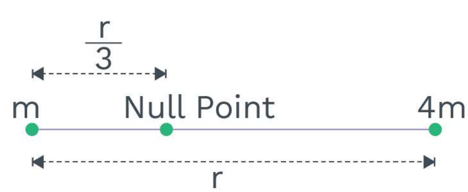
V = Algebraic sum of potential due to each particle

$$V = -\frac{4Gm}{r} = -\frac{4\sqrt{2}Gm}{l} \qquad \left(\because r = \frac{\sqrt{2}l}{2}\right)$$

Ex. Two bodies of masses m and 4m are placed at a distance r. The gravitational potential at a point on the line joining them where the gravitational field is zero is

Sol. Position of null point from m is

$$x = \frac{r}{\sqrt{\frac{4m}{m}} + 1} = \frac{r}{3}$$



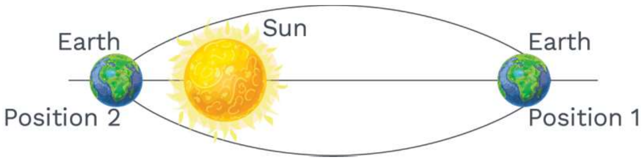
$$\therefore \quad \text{Potential } V = -Gm\left(\frac{3}{r} + \frac{12}{2r}\right) = \frac{-9Gm}{r}$$

Important Points:

(1) Potential energy is a scalar quantity.

(2) Its SI unit is Joule and Dimension are $[M^1L^2T^{-2}]$

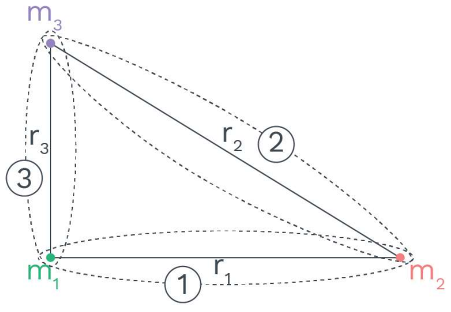
(3) It is a –ve quantity whose maximum value is zero at infinite separation. As r increases, U also increases.



at position 2 the potential energy of sun + earth system is minimum (as r is minimum)

(4) If there are more than two particles in a system, then the net gravitational potential energy of whole system is sum of gravitational potential energy of all the possible pairs of that system.

For e.g.:



$$U_{system} = \left(-\frac{Gm_1m_2}{r_1}\right) + \left(-\frac{Gm_2m_3}{r_2}\right) + \left(-\frac{Gm_1m_3}{r_3}\right)$$

$$\boxed{U_{system} = -\frac{Gm_1m_2}{r_1} - \frac{Gm_2m_3}{r_2} - \frac{Gm_1m_3}{r_3}}$$

Note. Work done by external agent $W_{ext} = \Delta U = U_f - U_i$ without change in K.E. To find the work done **by external agent** to raise a particle of mass 'm' to 'h' height above the surface of earth. Without change in KE

Rack your Brain

A body of mass m is placed on earth's surface which is taken from earth surface to a height of h = 3R, then change in gravitational potential energy is

1. $\frac{mgR}{4}$
2. $\frac{2}{3}mgR$
3. $\frac{3}{4}mgR$
4. $\frac{mgR}{2}$

To find the velocity required at the surface of earth to project it to 'h' height from the surface of earth. Applying 'Conservation of mechanical energy' on surface and at 'h' height.

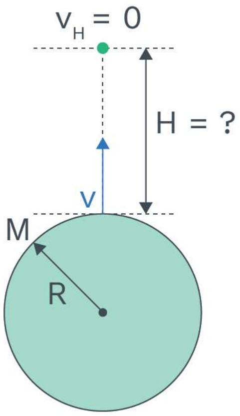
$(K.E. + U)_{surface} = (K.E. + U)_{final}$

Concept Reminder

Velocity of object on surface of earth when it is dropped from height h above the surface is

$$\Rightarrow v^2 = \frac{2gh}{1+\frac{h}{R}} \Rightarrow \boxed{v = \sqrt{\frac{2gh}{1+\frac{h}{R}}}}$$

To find the maximum height attained by a body when it is projected with velocity 'v' from the surface of earth.



$$\text{as} \quad v^2 = \frac{2gh}{1+\frac{h}{R}}$$

$$\Rightarrow v^2 + \frac{v^2 h}{R} = 2gh$$

$$\Rightarrow v^2 = 2gh - \frac{v^2 h}{R}$$

$$\Rightarrow v^2 = h\left(2g - \frac{v^2}{R}\right)$$

$$\Rightarrow h = \frac{v^2}{2g - \frac{v^2}{R}} = \frac{v^2 R}{2gR - v^2}$$

$$H = \frac{v^2 R}{2gR - v^2} \Rightarrow \boxed{H = \frac{v^2}{2g - \frac{v^2}{R}}}$$

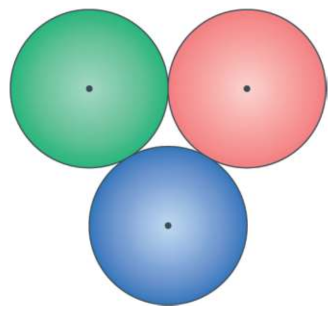
- **Problem based on work done by external agent / change in potential energy / find potential energy of system**

Concept Reminder

Maximum height reached by an object when it is thrown with velocity v is $H = \frac{v}{2g - \frac{v^2}{R}}$

Ex. Three solid spheres of mass M and radii R are placed in contact as shown in figure. Find the potential energy of the system?

Sol.



$PE = PE_{12} + PE_{23} + PE_{31}$

$$= -\frac{GM^2}{2R} - \frac{GM^2}{2R} - \frac{GM^2}{2R} \Rightarrow PE = -\frac{3GM^2}{2R}$$

Problems based on conservation of energy

Ex. A particle is projected vertically upward from the surface of the earth with a speed of $\sqrt{\frac{3}{2}gR}$, R being the radius of the earth and g is the acceleration due to gravity on the surface of the earth. Then the maximum height ascended is (neglect cosmic dust resistance)

Sol. $\frac{1}{2}mv_0^2 - \frac{GMm}{R} = -\frac{GMm}{R+h}$ and $g = \frac{GM}{R^2}$

Therefore $\frac{1}{2} \times \frac{3}{2} \times gR - Rg = -\frac{R^2 g}{R+h}$

We get h = 3R

Ex. A body of mass m kg starts falling from a distance 2R above the earth's surface. What is its kinetic energy when it has fallen to a distance 'R' above the earth's surface? (Where R is the radius of Earth)

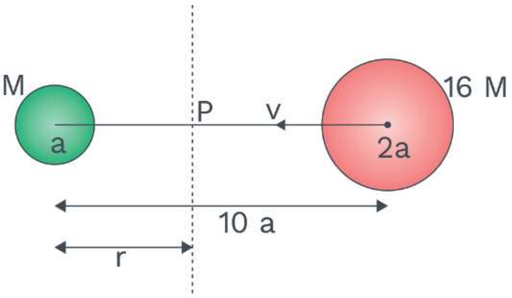
Sol. By conservation of mechanical energy,

$$-\frac{GMm}{3R} + 0 = -\frac{GMm}{2R} + K.E.$$

$$\Rightarrow \quad K.E. = \frac{GMm}{R}\left[\frac{1}{2} - \frac{1}{3}\right] = \frac{1}{6}\frac{GMm}{R} = \frac{1}{6}\frac{(gR^2)m}{R} = \frac{1}{6}mgR$$

Ex. Distance between centres of two stars is 10 a. The masses of these stars are M and 16 M and their radii are a & 2a respectively. A body is fired straight from the surface of the larger star towards the smaller star. What should be its minimum initial speed to reach the surface of the smaller star?

Sol. Let P be the point on the line joining the centres of the two planets that the net field at it is zero



Then, $\frac{GM}{r^2} - \frac{G.16M}{(10a-r)^2} = 0$

$\Rightarrow \quad (10a-r)^2 = 16r^2$

$\Rightarrow \quad 10a - r = 4r$

$\Rightarrow \quad r = 2a$

Potential at point P,

$$V_P = \frac{-GM}{r} - \frac{G.16M}{(10a-r)} = \frac{-GM}{2a} - \frac{2GM}{a} = \frac{-5GM}{2a}$$

Now if the particle projected from the larger planet has enough energy to cross this point, it will reach the smaller planet.

For this, the K.E. imparted to the body must be just enough to raise its total mechanical energy to a value which is equal to P.E. at point P.

i.e. $\frac{1}{2}mv^2 - \frac{G(16M)m}{2a} - \frac{GMm}{8a} = mV_P$

or, $\frac{v^2}{2} - \frac{8GM}{a} - \frac{GM}{8a} = \frac{-5GMm}{2a}$

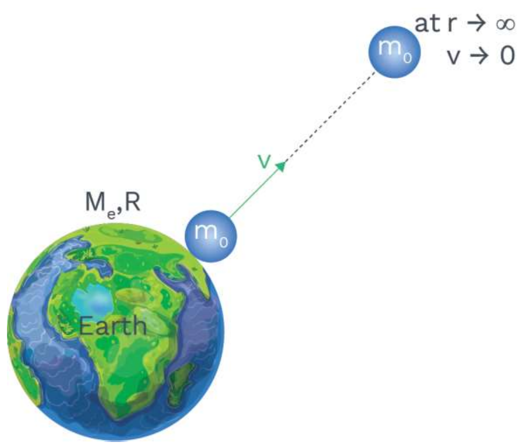
or $v^2 = \frac{45GM}{4a}$

or, $v_{min} = \frac{3}{2}\sqrt{\frac{5GM}{a}}$

Escape speed (v_e)

Minimum speed required for an object at Earth's surface so that it just escapes the Earth's gravitational field.

Suppose a particle of mass m is on earth's surface We project it with a velocity v from the earth's surface, so that it just reaches $r \to \infty$ (at $r \to \infty$ its velocity become zero)



Applying energy conservation between initial position (when the particle was at earth's surface) and find positions

(When the particle just reaches to $r \to \infty$)

$$K_i + U_i = K_f + U_f$$

$$\frac{1}{2}mv^2 + m\left(-\frac{GM_e}{R}\right) = 0 + m\left(-\frac{GM_e}{(r \to \infty)}\right)$$

$$\Rightarrow v = \sqrt{\frac{2GM_0}{R}}$$

Escape speed from earth is surface $v_e = \sqrt{\frac{2GM_e}{R}}$

Escape Energy

Minimum energy given to a particle in form of kinetic energy so that it can just escape from Earth's gravitational field.

Definitions

Escape speed (v_e)

Minimum speed required for an object at Earth's surface so that it just escapes the Earth's gravitational field.

KEY POINTS

- Escape Speed
- Escape Energy

Concept Reminder

Escape speed from earth is surface $v_e = \sqrt{\frac{2GM_e}{R}}$

Escape energy $= \frac{GM_e m}{R_e} = \left(\frac{1}{2}mv^2{}_{escape}\right)$

$v_e = \sqrt{\frac{2GM_e}{R_e}}$ (In form of mass)

If M = constant $v_e \propto \frac{1}{\sqrt{R_e}}$

- $v_e = \sqrt{2gR_e}$ (In form of g)

If g = constant $v_e \propto \sqrt{R_e}$

- $v_e = R_e\sqrt{\frac{8\pi G.\rho}{3}}$ (In form of density)

If $\rho =$ constant $v_e \propto R_e$

Concept Reminder

Escape Energy

Minimum energy given to a particle in form of kinetic energy so that it can just escape from Earth's gravitational field.

Escape Velocity of a Body From Certain height Above The Surface of a Planet:

- At a height 'h' above the surface of a planet

$$PE_{body} = \frac{-GMm}{R+h}$$

For object to escape $TE = PE + KE = 0$

$$KE_{body} = -PE_{body} = \frac{GMm}{R+h} \Rightarrow \frac{1}{2}mv_e^2 = \frac{GMm}{R+h}$$

$$\Rightarrow v_e = \sqrt{\frac{2GM}{R+h}} = \sqrt{2g_h(R+h)}$$

Here, g_h is acceleration due to gravity at height h.

Behaviour of a Body Projected Vertically Up with Different Velocities from the Surface of a Planet:

- Consider a body of mass 'm' projected with a velocity 'v' from the surface of a planet of mass 'M' and radius 'R'

Case-I: If the velocity of projection $v < v_e$ then, Total energy is negative. The body goes to certain maximum height and then falls back. To find this maximum height, we use law of conservation of energy. $TE_{surface} = TE_{max.height}$

Concept Reminder

Escape velocity of a body from certain height above the surface of a Planet:

$$\Rightarrow v_e = \sqrt{\frac{2GM}{R+h}} = \sqrt{2g_h(R+h)}$$

$$-\frac{GMm}{R}+\frac{1}{2}mv^2=-\frac{GMm}{R+h}$$

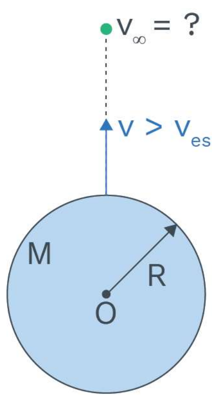
$$\frac{1}{2}mv^2=\frac{GMmh}{R(R+h)}\Rightarrow\frac{R+h}{h}=\frac{2GM}{Rv^2}$$

$$\frac{R}{h}+1=\frac{v_e^2}{v^2}\Rightarrow\frac{R}{h}=\left(\frac{v_e^2}{v^2}-1\right)$$

$$\Rightarrow\ h=\frac{R}{\left(\frac{v_e^2}{v^2}-1\right)}$$

Case-II: -If the velocity of projection $v = v_e$ then, total energy of the body just becomes zero, so that the body just escapes from the planet and goes to infinity and the body possess zero velocity at infinity.

Case-III: - If a body is projected with a velocity greater than the escape velocity ($v > v_e$), total energy is positive, the body escapes from gravitational influence of the planet and enter into inter stellar space with certain velocity.
By law of conservation of energy



$$\frac{1}{2}mv^2-\frac{GMm}{R}=\frac{1}{2}mv_\infty^2$$

$$\Rightarrow v_\infty^2=v^2-\frac{2GM}{R}=v^2-v_e^2\qquad\left[\because v_e^2=\frac{2GM}{R}\right]$$

Rack your Brain

A particle of mass m is kept at rest at a height 3R from surface of earth where 'R' is radius of earth and 'M' is mass of earth. Find the minimum speed with which it should be projected, so that it does not return back.

Concept Reminder

Inter planetary or inter stellar speed $v_\infty=\sqrt{v^2-v_e^{\,2}}$

$\Rightarrow v_{\infty} = \sqrt{v^2 - v_e^2}$

i.e., the body will move in interplanetary or interstellar space with a velocity $\sqrt{v^2 - v_e^2}$

Salient features regarding escape velocity:

- Escape velocity depends on the mass, density and radius of the planet from which the body is projected.
- Escape velocity independent on the mass of the projected body and its direction of projection and the angle of projection.
- Escape velocity from the surface of earth=11.2Km/s
- Escape velocity from the surface of moon=2.31 Km/s
- There is no atmosphere on moon, because r.m.s. velocity of molecules is greater than the escape velocity on the moon (i.e., $v_{rms} > v_e$)
- Escape velocity on sun is maximum. As $v_{rms} < v_e$, hence, even the lightest molecules cannot escape from there. Sufficient amount of hydrogen is present in the atmosphere of the sun since the escape velocity on the sun is very high.
- If a body falls freely from infinity, then, it reaches the earth with a velocity of 11.2 Km/s.

Concept Reminder

Condition of black hole for plane/star is v_e > speed of light (c)

Definitions

Kepler's first law or law of orbits: Every planet revolves around the sun in elliptical orbit with the sun at one of its foci.

Ex. If Earth has mass nine times and radii twice that of the planet mars, calculate the velocity required by a rocket to pull out of the gravitational force of Mars. Take escape speed on surface of Earth to be 11.2 km/s

Sol. Here, $M_e = 9M_m$, and $R_e = 2R_m$

v_e (escape speed on surface of Earth) =11.2 km/s

Let V_m be the speed required to pull out of the gravitational force of mars.

We know that $v_e = \sqrt{\dfrac{2GM_e}{R_e}}$ and $v_m = \sqrt{\dfrac{2GM_m}{R_m}}$

Dividing, we get $\dfrac{v_m}{v_e} = \sqrt{\dfrac{2GM_m}{R_m} \times \dfrac{R_e}{2GM_e}} = \sqrt{\dfrac{M_m}{M_e} \times \dfrac{R_e}{R_m}} = \sqrt{\dfrac{1}{9} \times 2} = \dfrac{\sqrt{2}}{3}$

$\Rightarrow v_m = \dfrac{\sqrt{2}}{3}(11.2\,\text{km/s}) = 5.3\,\text{km/s}$

Ex. A rocket is fired with a speed $v = 2\sqrt{gR}$ near the earth's surface and directed upwards.

(a) Show that it will escape from the earth.

(b) Show that in interstellar space its speed is $v = \sqrt{2gR}$

Sol. (a) As PE of the rocket at the surface of the earth is (–GMm/R) and at infinity is zero, energy required for escaping from earth

$$= 0 - \left(\frac{GMm}{R}\right) = mgR \left[\because g = \frac{GM}{R^2}\right]$$

And as initial KE of the rocket $\frac{1}{2}mv^2 = 2mgR$ is greater than the energy required for escaping (= mg R), the rocket will escape.

(b) If is the velocity of the rocket in interstellar space (free from gravitational effects) then by conservation of energy,

$$\frac{1}{2}m\left(2\sqrt{gR}\right)^2 - \frac{1}{2}m\left(\sqrt{2gR}\right)^2 = \frac{1}{2}mv^2$$

$$v^2 = 4gR - 2gR \quad \text{or} \quad v = \sqrt{2gR}$$

Ex. A planet in a distant solar system is 10times more massive than the earth and its radii is 10 times smaller. Given that escape velocity from the earth surface is 11 km/s, the escape velocity from the surface of the planet is

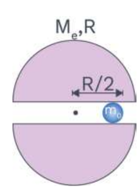
Sol. Given $M_P = 10M_e ; R_P = \frac{R_e}{10}$

We know that $v_e = \sqrt{\frac{2GM}{R}}$

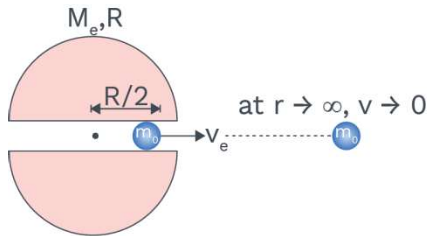
$$\therefore \quad v_P = \sqrt{\frac{2GM_P}{R_P}} = \sqrt{\frac{100 \times 2GM_e}{R_e}} = 10v_e$$

$$= 10 \times 11 = 110\,km/s$$

Ex. A narrow tunnel is dug along the diameter of the earth, and a particle of mass m_0 is placed at $\frac{R}{2}$ distance from the centre. Find the escape speed of the particle from that place.



Sol. Suppose we project the particle with speed v_e, so that it just reaches infinity ($r \rightarrow \infty$). Applying energy conservation principle



$$K_i + U_i = K_f + U_f$$

$$\frac{1}{2}m_0v_e^2 + m_0\left[-\frac{GM_e}{2R^3}\left\{3R^2 - \left(\frac{R}{2}\right)^2\right\}\right] = 0$$

$$\Rightarrow v_e = \sqrt{\frac{11GM_e}{4R}}$$

Ex. If 'M' is mass of a planet and R is its radii then in order to become black hole [c is speed of light].

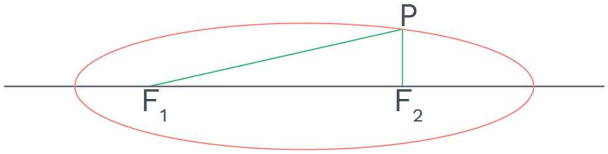
Sol. A planet can become a black hole if its mass and radius are such that it has an immense force of gravity on its surface. The force of attraction has to be so large that even light cannot escape from its surface. Speed of light = c

$$v_e = \sqrt{\frac{2GM}{R}}$$

If $v_e \geq c \Rightarrow$ Even light can't escape from the surface of such planet making it appear black.

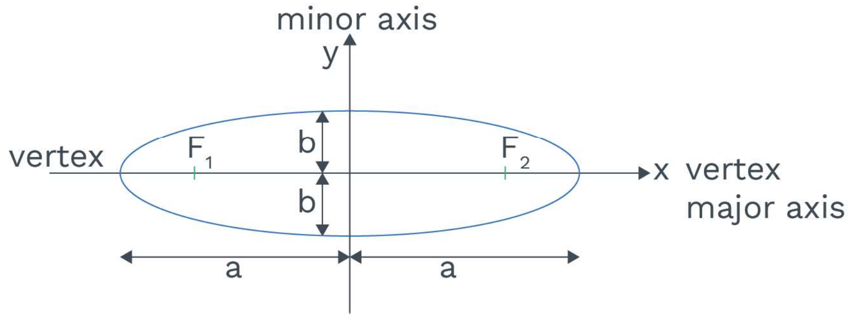
Ellipse

Ellipse is the set of all points such that sum of the distance from the foci (plural of focus) and any point on ellipse is constant.



$$\boxed{F_1P + F_2P = 2a = \text{constant}}$$

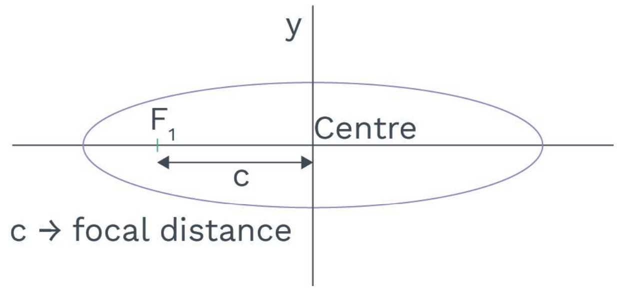
Basic



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

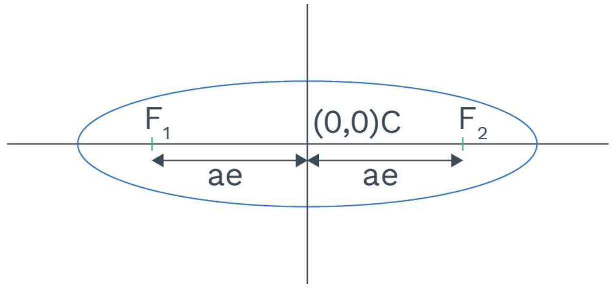
Major axis - It is the segment that runs through the elongated portion of the ellipse and intersect the vertices and foci

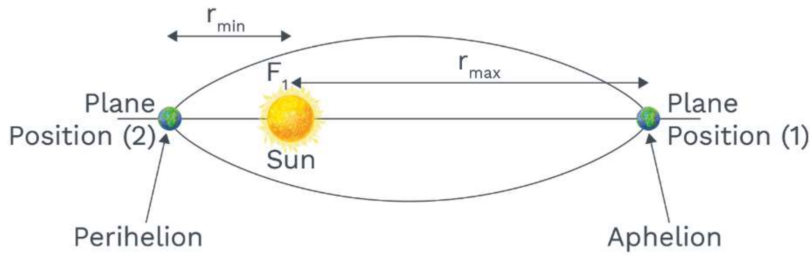
Centre (C) $\rightarrow$ Point of intersection of major and minor axis.



$$e = \frac{c}{a}$$

For ellipse $0 < e < 1$ for circle $e = 0$

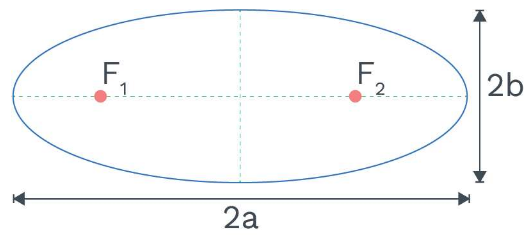




Kepler's Laws

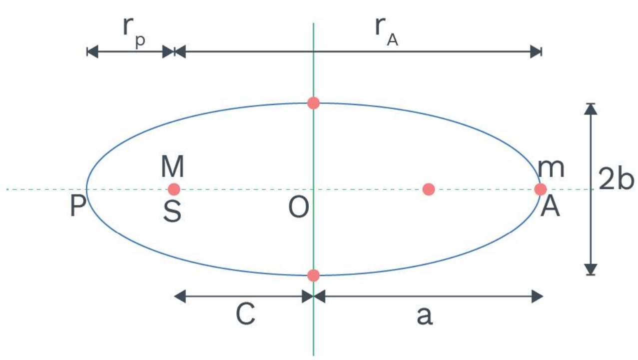
Kepler found important regularities in the motion of the planets. These regularities are known as 'Kepler's three laws of planetary motion'.

- **Kepler's first law or law of orbits:-** Every planet revolves around the sun in elliptical orbit path with the sun at one of its foci.



As shown in the fig., sun may be at F_1 or F_2. Here a and b denote the lengths of semi major and semi minor axes.

- The closest position of the planet from the sun is called perihelion.
- The extreme position of the planet from the sun is called aphelion.
- A 'm' mass planet is moving in an elliptical orbit around the sun of mass 'M', at one of its foci.



- Eccentricity of the elliptical path $e = \frac{SO}{OA}$

$$e = \frac{c}{a} \Rightarrow c = ea$$

- From fig, $r_p = a - c = a - ea = a(1 - e)$

Definitions

Kepler's first law or law of orbits: Every planet revolves around the sun in elliptical orbit with the sun at one of its foci.

KEY POINTS

- Kepler's law
- Semi-major axis

Similarly, $r_a = a + c = a + ea = a(1+e)$

- From conservation of angular momentum at A and P, we have $mv_p r_p = mv_A r_A$

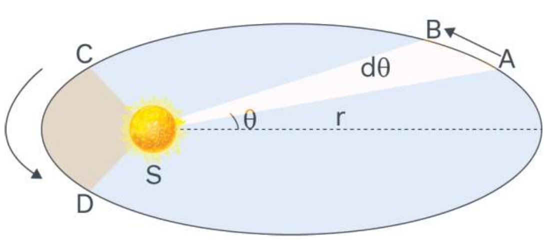
$$\frac{v_p}{v_A} = \frac{r_A}{r_p} = \frac{1+e}{1-e}$$

- From conservation of energy, we have

$$v_A = \sqrt{\frac{GM}{a}\left(\frac{1-e}{1+e}\right)} \text{ and } v_p = \sqrt{\frac{GM}{a}\left(\frac{1+e}{1-e}\right)}$$

- If e > 1 and total energy (P.E. + K.E) > 0, the path of the satellite is hyperbolic and it escapes from its orbit.
- If e < 1 and the total energy is negative value, it moves in an elliptical path.
- If e = 0 the total energy is negative value, it moves in circular path.
- If e = 1 the total energy is zero, it will take parabolic path.
  The way of the projectile thrown to lower heights is parabolic and thrown to greater heights is elliptical.
- **Kepler's second law:**
  Corresponding to the second law, A line joining any planet to the Sun sweeps out equal areas in equal time intervals i.e., the areal speed of the planet remains constant.
  If a planet moves from A to B in a given time interval, and from C to D in the same time interval, then the areas ASB and CSD will be equal.
  dA = area of the curved triangle

$$SAB = \frac{1}{2}(AB \times SA) = \frac{1}{2}(rd\theta \times r) = \frac{1}{2}r^2 d\theta$$



Concept Reminder

- If $e < 1 \Rightarrow$ elliptical path
- If $e = 0 \Rightarrow$ circular path
- If $e = 1 \Rightarrow$ parabolic path

Definitions

Kepler's second law: - The line that joins any planet to the sun sweeps out equal area in equal interval of time $\frac{dA}{dt} = \text{constant}$

Thus, the instantaneous areal speed of the planet

is $\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}r^2\omega = \frac{1}{2}rv$ ... (i)

where ω is the angular speed of the planet.

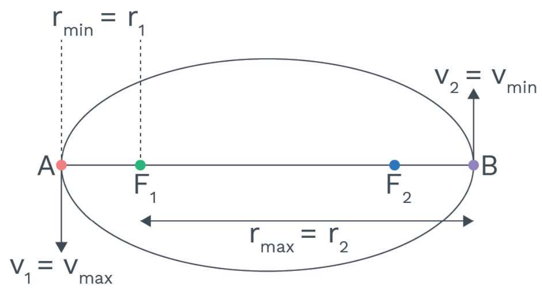
Let L be the angular momentum of the planet about the Sun S and m the mass of the planet.

Then $L = I\omega = mr^2\omega = mvr$

where $I(=mr^2)$ is the instantaneous moment of inertia of the planet about the Sun S.

From eq. (i) & (ii) $\frac{dA}{dt} = \frac{L}{2m}$

Now, As there is no external torque on the planet. Its angular momentum remains constant so as areal speed also remain constant which justify the Kepler's II law. Thus, Kepler's second law is equivalent to conservation of angular momentum. Applying conservation of angular momentum between points A and B



$v_1 = v_{max}$ $\quad L_A = L_B \Rightarrow mv_{max}r_{min} = mv_{min}r_{max}$

$\Rightarrow v_{max}r_{min} = v_{min}r_{max}$

- **Kepler's third law or Law of periods:-** The square of period of revolution of a planet around sun is proportional to cube of the average distance of planet (i.e., semi major axis of elliptical orbit) from the sun.

$$r_{mean} = \frac{r_{max} + r_{min}}{2} = \frac{(1+e)a + (1-e)a}{2} = a$$

Hence, $T^2 \propto a^3$

where ' a ' is length of semi major axis of ellipse

Concept Reminder

Kepler's second law is basically an alternative statement of law of conservation of angular momentum.

Definitions

Kepler's third law or Law of periods: The square of period of revolution of a planet around the sun is proportional to cube of the average distance of planet (i.e., semi major axis of elliptical orbit) from the sun.

- The gravitational force between the planet and the Sun provides the necessary centripetal force for the planet to go round the Sun.
  If M = mass of Sun, m = mass of planet and
  r = average distance of the planet from the Sun, then $F = \frac{GMm}{r^2} = mr\omega^2$

$$\frac{GM}{r^3} = \frac{4\pi^2}{T^2} \qquad \left(\text{as } \omega = \frac{2\pi}{T}\right)$$

$$T^2 = 4\pi^2 \frac{r^3}{GM} \Rightarrow T^2 \propto r^3$$

Ex. The mean distance of a planet from the sun is approximately 1/4 times that of earth from the sun. Find the number of years required for planet to make one revolution about the sun.

Sol. Given $r_P = \frac{1}{4} r_E$ and $T_E = 1\text{yr}$

From Kepler's third law, $T^2 \propto r^3$

$$\left(\frac{T_P}{T_E}\right)^2 = \left(\frac{r_P}{r_E}\right)^3 \Rightarrow T_P = T_E\left(\frac{r_P}{r_E}\right)^{\frac{3}{2}}$$

$$T_P = (1)\left(\frac{r_E}{4r_E}\right)^{\frac{3}{2}} = \left(\frac{1}{4}\right)^{\frac{3}{2}} = 0.125 \text{ Yrs}$$

Ex. A planet revolves across the sun in elliptical orbit. The Arial velocity $\left(\frac{dA}{dt}\right)$ of the planet is 4.0×10^{16} m²/s. The least distance between the planet and the sun is 2×10^{12} m. Then find the maximum speed of the planet in km/s is:

Sol. $\frac{dA}{dt} = \frac{r^2\omega}{2}$ is constant

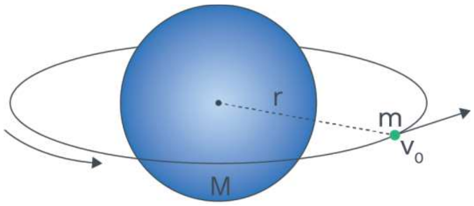
$$\therefore \quad \frac{dA}{dt} = \frac{r_{max}^2\omega_{min}}{2} = \frac{r_{min}^2\omega_{max}}{2}$$

$$\Rightarrow \quad \omega_{max} = \frac{2dA/dt}{r_{min}^2} = \frac{2 \times 4 \times 10^{16}}{\left(2 \times 10^{12}\right)^2} = 2 \times 10^{16} \times 10^{-24} = 2 \times 10^{-8} m/s = 2 \times 10^{-11} Km/s$$

Earth Satellites

Satellites: The bodies revolving round a planet in its gravitational field are defined as satellites.

Orbital speed of Satellites: The velocity of a satellite revolving around the earth of mass M and radius R in a circular orbit of radius 'r' at a height 'h' from the surface of earth is called orbital velocity.



$$v_0 = \sqrt{\frac{GM}{r}} = \sqrt{\frac{GM}{(R+h)}} = \sqrt{\frac{gR^2}{(R+h)}}$$

Angular velocity $\omega = \sqrt{\frac{GM}{(R+h)^3}}$

Time period $T = 2\pi\sqrt{\frac{(R+h)^3}{GM}} = 2\pi\sqrt{\frac{(R+h)^3}{gR^2}}$

- For a satellite orbiting very close to earth.

  $h << R$ then, $v_0 = \sqrt{\frac{GM}{R}} = \sqrt{gR}$

  $\omega^2 \propto \frac{1}{R^3} \Rightarrow T^2 \propto R^3$

- For two satellites revolving around the earth in different circular orbits of radii r_1 and r_2 at vertical heights h_1 and h_2, $\frac{v_1}{v_2} = \sqrt{\frac{r_2}{r_1}} = \sqrt{\frac{R+h_2}{R+h_1}}$
- Orbital velocity for a satellite close to the surface of earth $v_0 = 7.92\,\text{kms}^{-1} \approx 8\,\text{kms}^{-1}$
- Orbital velocity is independent of mass of the satellite.
- It is always along the tangent to the orbit.

Definitions

Satellites: The bodies revolving round a planet in its gravitational field are defined as satellites.

KEY POINTS

- Satellite
- Orbital speed

Concept Reminder

For satellite orbiting very close to earth, orbital speed is

$$v_0 = \sqrt{\frac{GM}{R}}$$

i.e., $v_0 = \frac{v_e}{\sqrt{2}}$

- Relation between escape and orbital velocities is $v_e = \sqrt{2}v_0$
- If the speed of the orbiting body 'v' is such that $v_o < v < v_e$ then its orbit changes from circle to ellipse.
- If the satellite revolves close to the earth surface, (h<<R), then, Time period of revolution, $T = 2\pi\sqrt{\frac{R}{g}} = 84.6\,\text{min} = 1.41\text{hr}$

Frequency of Revolution(n): The number of revolutions made by the satellite in one second is called the frequency of revolution(n).

$$n = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{GM}{r^3}} = \frac{1}{2\pi}\sqrt{\frac{GM}{(R+h)^3}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{gR^2}{(R+h)^3}} \qquad \left[\because g = \frac{GM}{R^2}\right]$$

- If the satellite revolves close to the earth surface, (h << R) then $n = \frac{1}{2\pi}\sqrt{\frac{GM}{r^3}} = \frac{1}{2\pi}\sqrt{\frac{g}{R}}$

Angular Momentum: The angular momentum of the satellite is given by

$$L = mv_0 r = mr\sqrt{\frac{GM}{r}} = \sqrt{GMm^2r}$$

- For satellites $\tau = 0$ and L = constant in a given orbit because gravitational force is central force.
- Angular momentum of the satellite depends on mass of the satellite, mass of the planet and radius of the orbit.
- A satellite behaves like a freely falling body.
- When a satellite revolves round a planet in an elliptical orbit, then its orbital speed is not uniform.
- The mass of a planet can be determined with the help of its satellite.

Rack your Brain

A geostationary satellite is orbiting the earth at a height of 5R above the surface of 5R above the surface of earth, R being radius of earth. Find the time period of another satellite in hours at a height 2R from surface of earth.

- If the satellite is travelling in the same direction as the rotation of earth i.e., west to east, the time interval between two successive times at which it will appear vertically overhead to an observer at a fixed point on the equator is

$$\left\{\text{Since } \omega_{rel} = \omega_{sat} - \omega_{earth} \Rightarrow \frac{2\pi}{T} = \frac{2\pi}{T_s} - \frac{2\pi}{T_e}\right\}$$

Energy of Orbiting Satellite:

- The potential energy of the system is $U = \frac{-GMm}{r}$
- The kinetic energy of the satellite is, $K = \frac{1}{2}mv_0^2 = \frac{1}{2}m\left(\frac{GM}{r}\right)$ or $K = \frac{GMm}{2r}$
- The total energy is $E = K + U = -\frac{GMm}{2r}$
- When the satellite revolves in an orbit of radius 'r' (here r = R+h) potential energy is negative, means that the satellite is moving in the gravitational field of the planet. The planet and the satellite form a bound system. If it is to be escaped out of the gravitational field, some additional energy $\frac{GMm}{2(R+h)}$ will have to be given to it.

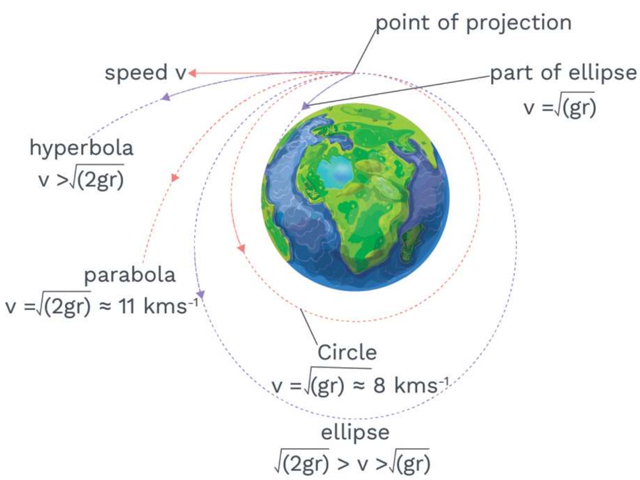
Trajectories of a body projected with different velocities:

An object revolves around a planet only when it is projected with sufficient velocity in a direction perpendicular to gravitational force of attraction of the planet on the object.

Concept Reminder

Energy of satellite

- $E = K + U = \frac{-GMm}{2r}$
- $K = |E|$
- $E = \frac{U}{2}$



If $v < \sqrt{gr}$ object falls on the surface of earth

If $v = \sqrt{gr}$ object revolve in a circular orbit.

If $\sqrt{gr} < v < \sqrt{2gr}$ object revolves in an elliptical orbit.

If $v = \sqrt{2gr}$ object escapes from the field and follows parabolic path.

If $v > \sqrt{2gr}$ object escapes from the field and follows hyperbolic path.

Definitions

The minimum energy required to break a bounded system, means to convert it into an unbounded system is known as binding energy.

Concept Reminder

- If $v = \sqrt{2gr}$ object escapes from the field and follows parabolic path.
- If $v > \sqrt{2gr}$ object escapes from the field and follows hyperbolic path.

Ex. A satellite orbits the earth at a height of 400 km above the surface. How much energy must be expended to rocket the satellite out of the gravitational influence of earth? Mass of the satellite is 200 kg, mass of earth =6.0×10^{24} kg, radius of earth = 6.4×10^6m, G = 6.67×10^{-11} Nm^2kg^{-2}.

Sol. Total energy of orbiting satellite at a height $h = \dfrac{-GMm}{2(R+h)}$

Energy expended to rocket the satellite out of the earth's gravitational field= – (Total energy of the satellite) = $\dfrac{GMm}{2(R+h)}$

$$= \frac{(6.67 \times 10^{-11}) \times (6 \times 10^{24}) \times 200}{2(6.4 \times 10^6 + 4 \times 10^5)} = 58.85 \times 10^8\,J$$

Ex. A body is projected vertically upwards from the surface of the earth with a velocity equal to half of escape velocity of the earth. If R is radius of the earth, maximum height attained by the body from the surface of the earth is

Sol. From law of conservation of energy,

$$TE_{surface} = TE_{Max\,Height}$$

$$-\frac{GMm}{R}+\frac{1}{2}m\left(\frac{v_e}{2}\right)^2 = -\frac{GMm}{R+h}+\frac{1}{2}m(0)^2$$

$$-\frac{GMm}{R}+\frac{1}{2}m\left(\frac{2GM}{4R}\right) = -\frac{GMm}{R+h}\quad\left(\because v_e^2=\frac{2GM}{R}\right)$$

On simplifying, we get h = R/3

Ex. A particle is fired vertically upwards from the surface of earth reaches a height 6400 Km. Find the initial velocity of the particle.

Sol. TE. on the surface of the earth

= TE. at the highest point

$$\frac{1}{2}mv^2-\frac{GMm}{R}=0-\frac{GMm}{R+h}$$

$$\Rightarrow\frac{1}{2}mv^2=\frac{mgh}{\left[1+\frac{h}{R}\right]}$$ given, h = R = 6400 Km

$$\text{So, } v^2 = gh \Rightarrow v=\sqrt{gh}$$

$$\Rightarrow v=\sqrt{10\times6400\times10^3}=8\,Km/s$$

Ex. Two heavy spheres each of mass 100 Kg and radius 0.1 m are placed 1 m apart on a horizontal table. What is the gravitational field and potential at the midpoint of the line joining their centres?

Sol. Gravitational field at the midpoint of the line joining their centres is given by

$$\vec{E}=\frac{Gm}{(r/2)^2}\left(-\vec{r}\right)+\frac{GM}{\left(r/2\right)^2}\left(\vec{r}\right)=\bar{0}$$

Gravitational potential at the midpoint of the line joining their centres is given by

$$V=\frac{-GM}{\left(r/2\right)}+\frac{-GM}{\left(r/2\right)}=\frac{-4GM}{r}$$

$$=-\frac{4\times6.67\times10^{-11}\times100}{1}=-2.7\times10^{-8}\,J/kg$$

Ex. The mass of a spaceship is 1000kg. It is to be launched from the earth surface out into free space. The value of 'g' and 'R' are $10ms^{-2}$and 6400km respectively. The required energy for this work will be

Sol. To launch the spaceship out into free space, from energy conservation, $-\frac{GMm}{R}+E=0$

$$E=\frac{GMm}{R}=\left(\frac{GM}{R^2}\right)mR=mgR=6.4\times10^{10}\text{ J}$$

Ex. The minimum energy required to launch a satellite of mass m from the surface of a planet of mass M and radius R in a circular orbit at an altitude of 2R is

Sol. From the law of conservation of energy

$$\frac{-GMm}{R}+KE_{imparted}=\frac{-GMm}{2\times3R}$$

$$KE_{imparted}=\frac{GMm}{R}\left(1-\frac{1}{6}\right)=\frac{5}{6}\left(\frac{GMm}{R}\right)$$

Ex. Two uniform solid spheres of equal radii R, but mass M and 4M have a centre to centre partition 6R The two spheres are held fixed. A projectile of mass 'm' is projected from the sphere's surface of mass 'M' directly towards the centre of the second sphere. Obtain an expression for the minimum speed 'v' of the projectile so that it reaches the surface of the second sphere.

Sol. The neutral point N is at a distance 'x' from mass M, given by $\frac{G\times4m}{(6R-x)^2}=\frac{G\times m}{x^2}\Rightarrow x=2R$

The mechanical energy at surface of 'M' is

$$E_t=\frac{1}{2}mv^2-\frac{GMm}{R}-\frac{4GMm}{5R}$$

At neutral point N, the speed approaches zero. The mechanical energy at point 'N' is purely potential.

$$E_N=-\frac{GMm}{2R}-\frac{GMm}{R}$$

From the principle of conservation of mechanical energy

$E_t=E_n$

$$\frac{1}{2}mv^2-\frac{GMm}{R}-\frac{4GMm}{5R}=-\frac{GMm}{2R}-\frac{GMm}{R}$$

or $$v^2=\frac{2GM}{R}\left(\frac{4}{5}-\frac{1}{2}\right)\Rightarrow v=\left(\frac{3GM}{5R}\right)^{1/2}$$

Ex. A ball is moving with a uniform speed in a circular orbit of radius 'R' in a central force inversely proportional to n^{th} power of R. If the period of rotation of the ball is T, then

Sol. $m\omega^2 R \propto \frac{1}{R^n}$

$\Rightarrow \quad \omega \propto \frac{1}{R^{\frac{n+1}{2}}}$

$\Rightarrow \quad T \propto R^{\frac{n+1}{2}}$

Bounded System:

- For a bounded system, nature of force between the particles is attractive and total energy of the system is '–ve'.

Binding energy:

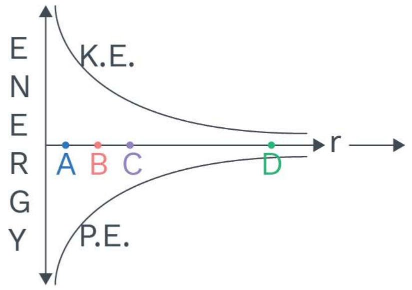
- The minimum energy required to break a bounded system, means to convert it into an unbounded system is known as binding energy.

So, $\because$ TE + BE = 0

$\therefore$ $\boxed{BE = -TE}$

Escape energy and ionisation energy are the practical examples of binding energy.

Here, at point A:



At A, B & C System is Bounded.

At point D: $\because$ |PE| = K.E. $\because$ TE = KE + PE = 0

So, system is unbounded.

Relation between the energies:

$$\boxed{KE = BE = -TE = \frac{PE}{2} = \frac{1}{2}mv_0^2 = \frac{GMm}{2r} = \frac{J^2}{2mr^2}}$$

Definitions

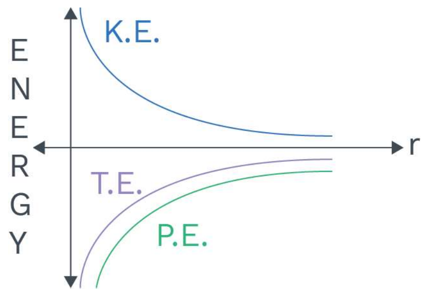
The minimum energy required to break a bounded system, means to convert it into an unbounded system is known as binding energy.

Concept Reminder

$$KE = BE = -TE = \frac{PE}{2}$$

$$= \frac{1}{2}mv_0^2 = \frac{GMm}{2r} = \frac{J^2}{2mr^2}$$

Graph:

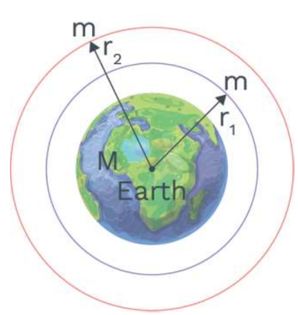


$$KE = \frac{GMm}{2r}$$

$$TE = -\frac{GMm}{2r}$$

$$PE = -\frac{GMm}{r}$$

Work done required to shift a satellite to a higher orbit:



Work = change in total energy.

$W = \Delta TE$

$\Rightarrow \; W = (TE)_f - (TE)_i$

$$\Rightarrow \; W = -\frac{GMm}{2r_2} - \left(-\frac{GMm}{2r_1}\right)$$

$$\Rightarrow \; \boxed{W = \frac{GMm}{2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)}$$

Concept Reminder

Work done required to shift a satellite to a higher orbit:

$$\Rightarrow \; \boxed{W = \frac{GMm}{2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)}$$

When Shift a satellite to a higher orbit then:

QUANTITY	VARIATION	RELATION WITH r
Orbital Velocity	Decreases	$v \propto \frac{1}{\sqrt{r}}$
Time period	Increases	$T \propto r^{3/2}$
Linear momentum	Decreases	$p \propto \frac{1}{\sqrt{r}}$
Angular momentum	Increases	$L \propto \sqrt{r}$
Kinetic energy	Decreases	$K \propto \frac{1}{r}$
Potential energy	Increases	$U \propto -\frac{1}{r}$
Total energy	Increases	$E \propto -\frac{1}{r}$
Binding energy	Decreases	$BE \propto \frac{1}{r}$

Weightlessness

When the apparent weight of a body becomes zero, the body is said to be in a state of weightlessness. In a satellite around earth, every part and particle of the satellite has an acceleration towards the centre of the earth which is exactly the value of earth's acceleration due to gravity at that position. Thus, in the satellite everything inside it is in a state of free fall. If a body is in a satellite (which does not produce its own gravity) orbiting the Earth at a height h above its surface, then

True weight $= mg_h = \frac{mGM}{(R+h)^2} = \frac{mg}{\left(1+\frac{h}{R}\right)^2}$ and

Concept Reminder

An astronaut experiences weightlessness in a space satellite. This is not because the gravitational force is very small. It is because both astronaut and satellite are in free fall towards the earth.

Apparent weight = $m(g_h - a)$

$$\text{But } a = \frac{v_0^2}{r} = \frac{GM}{r^2} = \frac{GM}{(R+h)^2} = g_h$$

$\Rightarrow$ Apparent weight = $m(g_h - g_h) = 0$.

Note: Condition of weightlessness can be overcome by creating artificial gravity by rotating the satellite in addition to its revolution.

Ex. Estimate the mass of the sun, assuming the orbit of the earth round the sun to be a circle. The distance between the sun and earth is 1.49×10^{11} m and $G = 6.66 \times 10^{-11}$ Nm^2/kg^2

Sol. Here the revolving speed of earth can be given as

$$v = \sqrt{\frac{GM}{r}} \qquad \text{[orbital speed]}$$

Where M is the mass of sun and r is the orbit radius of earth.

We know time period of earth around sun is T = 365 days, thus we have

$$T = \frac{2\pi r}{v} \text{ or } T = 2\pi r\sqrt{\frac{r}{GM}} \text{ or } M = \frac{4\pi^2 r^3}{GT^2}$$

$$= \frac{4 \times (3.14)^2 \times (1.49 \times 10^{11})^3}{(365 \times 24 \times 3600)^2 \times (6.66 \times 10^{-11})} = 1.972 \times 10^{22} \text{kg}$$

Ex. If the earth be one-half of its present distance from the sun, how many days will be in one year?

Sol. If orbit of earth's radius is R, in previous example we've discussed that time period is given as

$$T = 2\pi r\sqrt{\frac{r}{Gm}} = \frac{2\pi}{\sqrt{GM}} r^{3/2}$$

If radius changes or $r' = \frac{r}{2}$, new time period becomes

$$T' = \frac{2\pi}{\sqrt{GM}} r'^{3/2}$$

From above equations, we have

$$\frac{T}{T'} = \left(\frac{r}{r'}\right)^{3/2} \text{ or } T' = T\left(\frac{r'}{r}\right)^{3/2}$$

$$= 365\left(\frac{1}{2}\right)^{3/2} = \frac{365}{2\sqrt{2}} \text{days}$$

Ex. A satellite revolving in a circular orbit (equatorial) of radius r = 2.00 × 10^4km from west to east appear over a certain point at equator every t = 11.6 hours. Using this data, calculate the mass of the earth. The gravitational constant is supposed to be known.

Sol. Here the absolute angular velocity of satellite is given by $\omega = \omega_s + \omega_E$

Where ω_E is the angular velocity of earth, which is from west to east.

or $\omega = \frac{2\pi}{t} + \frac{2\pi}{T}$ [Where t = 11.6 hr. and T = 24 hr.]

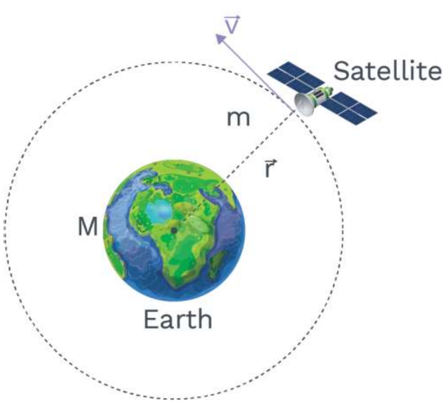
From Kepler's III law, we have $\omega = \frac{\sqrt{GM}}{r^{3/2}}$

Thus, we have $\frac{\sqrt{GM}}{r^{3/2}} = \frac{2\pi}{t} + \frac{2\pi}{T}$

or $M = \frac{4\pi^2 r^3}{G}\left[\frac{1}{t} + \frac{1}{T}\right]^2 = \frac{4\pi(2\times10^7)^3}{(6.67\times10^{-11})}\left[\frac{1}{11.6\times3600} + \frac{1}{24\times3600}\right]^2 = 6.0\times10^{24}$ kg

Ex. A satellite of mass m is moving in a circular orbit of radius r. Calculate its angular momentum with respect to the centre of the orbit in terms of the mass of the earth.

Sol. The situation is shown in figure



The angular momentum of the satellite with respect to the centre of orbit is given by

$$\vec{L} = \vec{r} \times m\vec{v}$$

Where $\vec{r}$ is the position vector of satellite with respect to the centre of orbit and $\vec{v}$ is its velocity vector of satellite.

In case of circular orbit, the angle between $\vec{r}$ and $\vec{v}$ is 90°. Hence

$L = mvr \sin 90° = mvr$ ... (1)

The direction is perpendicular (⊥) to the plane of the orbit.

We know orbital speed of satellite is

$$L = m\sqrt{\frac{GM}{r}} \Rightarrow L = (GMm^2r)^{1/2}$$

Now we will understand the concept of double star system through an example.

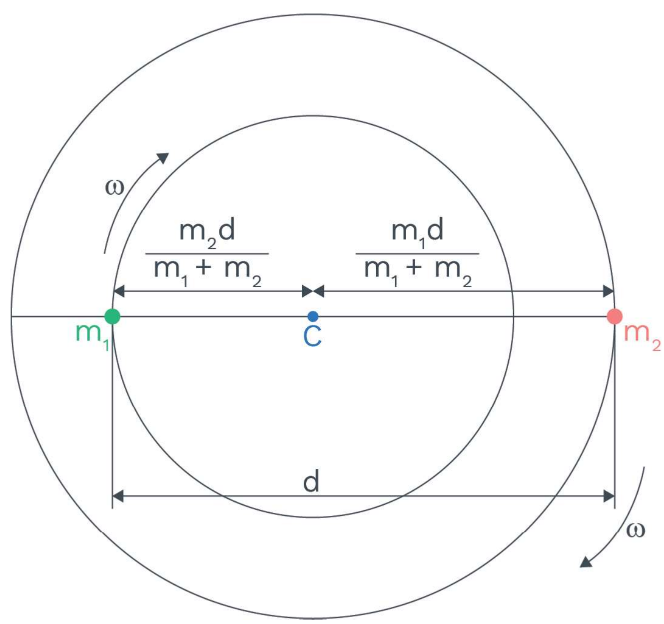
Ex. In a double star, two stars of masses $m_1$and m_2. distance d apart revolves about their common centre of mass under the influence of their mutual gravitational attraction. Find an expression for the period T in terms of masses m_1, $m_2$and d. Calculate the ratio of their angular momenta about centre of mass and also the ratio of their kinetic energies.

Sol. The centre of mass of double star from mass $m_1$is given by

$$r_{cm,} = \frac{m_1r_1 + m_2r_2}{m_1 + m_2} = \frac{m_1 \times 0 + m_2d}{m_1 + m_2} = \frac{m_2d}{m_1 + m_2}$$

$\therefore$ Distance of centre of mass from $m_2$is

$$r'_{cm} = d - r_{cm} = d - \frac{m_2d}{m_1 + m_2} = \frac{m_1d}{m_1 + m_2}$$



Both the stars rotate around centre of mass in their own circular orbits with the same angular speed ω the gravitational force acting on each star provides the necessary centripetal force, if we consider the rotation of mass m_1, then

$$m_1(r_{cm})\omega^2 = \frac{Gm_1m_2}{d^2}$$

or $\quad m_1\left(\dfrac{m_2 d}{m_1+m_2}\right)\omega^2 = \dfrac{Gm_1m_2}{d^2}$

This gives $\omega = \dfrac{2\pi}{T} = \sqrt{\left(\dfrac{G(m_1+m_2)}{d^3}\right)}$

or $\quad$ Period of revolution $T = 2\pi\sqrt{\left(\dfrac{d^3}{G(m_1+m_2)}\right)}$

Ratio of Angular Momenta is

$$\frac{J_1}{J_2} = \frac{I_1\omega}{I_2\omega} = \frac{I_1}{I_2} = \frac{m_1\left(\dfrac{m_2 d}{m_1+m_2}\right)^2}{m_2\left(\dfrac{m_1 d}{m_1+m_2}\right)^2} = \frac{m_2}{m_1}$$

Ratio of Kinetic energies is

$$\frac{K_1}{K_2} = \frac{\frac{1}{2}I_1\omega^2}{\frac{1}{2}I_1\omega^2} = \frac{I_1}{I_2} = \frac{m_2}{m_1}$$

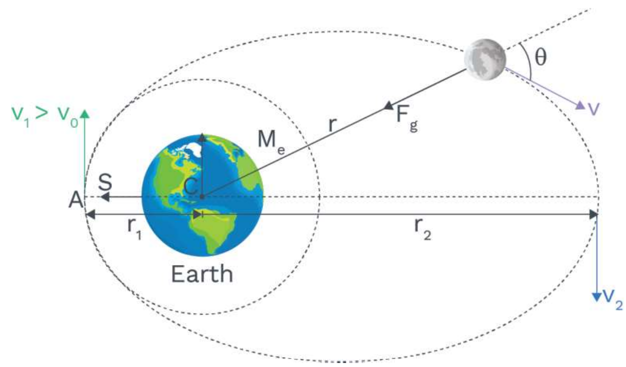
Satellite Motion and Angular Momentum Conservation

We've discussed that when a body is in bounded orbit around a planet it can be in elliptical or circular path depending on its kinetic energy at the time of launching. Let's consider a case when a satellite is launched in an orbit around the earth.

A satellite S is first fired away from earth source in vertical direction to penetrate the earth's atmosphere. When it reaches point A, it is imparted a velocity in tangential direction to start its revolution around the earth in its orbit.

Rack your Brain

The mean radius of earth is R, its angular speed on its own axis is ω and the acceleration due to gravity at earth's surface is g. What will be the radius of the orbit of a geostationary satellite?



This velocity is termed as insertion velocity, if the velocity imparted to satellite is $v_0 = \sqrt{\frac{GM_e}{r_1}}$ starts following the circular path shown in figure. If velocity imparted is $v_1 > v_0$ then it will trace the elliptical path shown. During this motion the only force acting on satellite is the gravitational force due to earth which is acting along the line joining satellite and centre of earth. As the force on satellite always passes through centre of earth during motion, we can say that on satellite there is no torque acting about centre of earth thus total angular momentum of satellite during orbital motion remains constant about earth's centre. As no external force is involved for earth-satellite system, no external work is being done here so we can also state that total mechanical energy of system also remains conserved.

In the elliptical path of satellite shown in figure if r_1 and r_2 are the shortest distance (perigee) and farthest distance (apogee) of satellite from earth and at points, satellite velocities are v_1 and v_2 then we have according to conservation of angular momentum, the angular momentum of satellite at a general point is given as

$$L = mv_1r_1 = mv_2r_2 = mvr\sin\theta$$

During motion the total mechanical energy of satellite (kinetic + potential) also remains conserved.

Thus the total energy of satellite can be given as

$$E = \frac{1}{2}mv_1^2 - \frac{GM_em}{r_1} = \frac{1}{2}mv_2^2 - \frac{GM_em}{r_2} = \frac{1}{2}mv^2 - \frac{GM_em}{r}$$

Using the above relations in equation written above we can find velocities v_1 and v_2 of satellite at nearest and farthest position in terms of r_1 and r_2.

Ex. A spaceship is launched into a circular orbit close to the earth's surface. What additional velocity has now to be imparted to the spaceship in the orbit of overcome the gravitational pull. (Radius of the earth = 6400 km and g = 9.8 m/sec.)

Sol. In an orbit close to earth's surface velocity of spaceship is $v = \sqrt{\frac{GM}{R}} = \sqrt{gR}$

We know escape velocity is $v_e = \sqrt{2gR}$

Hence additional velocity required to be imparted is $\Delta v = v_e - v = (\sqrt{2} - 1)\sqrt{gR}$

$$= (\sqrt{2} - 1)\sqrt{9.8 \times 6400 \times 10^3} = 3.28 \times 10^3 \text{ m / s}$$

Ex. A ball is fired vertically upward with a speed of 9.8 km/s. Calculate the maximum height attained by the ball. Radius of the earth = 6400 km and g at the surface = 9.8 m/s^2. Consider only earth's gravitation.

Sol. Initial energy of particle on earth's surface is

$$E_r = \frac{1}{2}mu^2 - \frac{GMm}{R}$$

If the particle reaches up to a height h above the earth, then its final energy will only the gravitational potential energy.

$$E_f = -\frac{GMm}{R+h}$$

According to energy conservation, we have

$$E_t = E_f$$

or $$\frac{1}{2}mu^2 - \frac{GMm}{R} = -\frac{GMm}{R+h}$$

or $$\frac{1}{2}u^2 - gR = -\frac{gR^2}{R+h}$$

or $$h = \frac{2gR^2}{2gR - u^2} - R = \frac{2 \times 9.8 \times (6400 \times 10^3)^2}{2 \times 9.8 \times 6400 \times 10^3 - (9.8)^2} - 5400 \times 10^3$$

$$= (27300 - 6400) \times 10^3 = 20900 \text{ km}$$

Ex. A satellite of mass m is orbiting the earth in a circular orbit of radius r. It starts losing energy slowly at a constant rate C due to friction. If M_e and R_e denote the mass and radii of the earth respectively, show the satellite falls on the earth surface in a time t represent by

$$t = \frac{GmM_e}{2C}\left(\frac{1}{R_e} - \frac{1}{r}\right)$$

Sol. Let velocity of satellite in its orbit of radius r be v then we have $v = \sqrt{\frac{GM_e}{r}}$

When satellite approaches earth's surface, if its velocity becomes v', then it is given as $v' = \sqrt{\frac{GM_e}{R_e}}$

The total initial energy of satellite at a distance r is

$$E_{T_i} = K_i + U_i = \frac{1}{2}mv^2 - \frac{GM_em}{r} = -\frac{1}{2}\frac{GM_em}{r}$$

The total final energy of satellite at a distance R_e is

$$E_{T_f} = K_f + U_f = \frac{1}{2}mv'^2 - \frac{GM_em}{R_e} = -\frac{1}{2}\frac{GM_em}{R_e}$$

As satellite is losing energy at rate C, if it takes a time t in reaching earth, we have

$$Ct = E_{T_i} - E_{T_f} = \frac{1}{2}GM_em\left[\frac{1}{R_e} - \frac{1}{r}\right]$$

$$\Rightarrow \quad t = \frac{GM_em}{2C}\left[\frac{1}{R_e} - \frac{1}{r}\right]$$

Ex. An (artificial) satellite is moving in a circular orbit around the earth with a speed equal to half the magnitude of escape velocity from the earth.

(i) Determine the height of the satellite above earth's surface.

(ii) If the satellite is stopped suddenly in its orbit and allowed to fall freely onto the earth, find the speed with which it hits the surface of the earth.

Sol. (i) Assume mass and radius be the mass and radius of the earth respectively. If m be the mass of satellite, then escape velocity from earth $v_c = \sqrt{2gR_e}$

velocity of satellite $v_s = \sqrt{\frac{gR_e}{2}}$

Further we know orbital speed of satellite at a height h is

$$v_s = \sqrt{\left(\frac{GM_e}{r}\right)} = \sqrt{\left(\frac{R_e^2 g}{R_e + h}\right)} \text{ or } v_s^2 = \frac{R^2 g}{R + h}$$

From equation written above, we get

$$h = R = 6400\,\text{km}$$

(ii) At this movement total energy at height h = total energy at earth's surface (principle of conservation of energy)

$$\text{or} \quad 0 - GM_e\frac{m}{R + h} = \frac{1}{2}mv^2 - GM_e\frac{m}{R_e}$$

$$\text{or} \quad \frac{1}{2}mv^2 = \frac{GM_em}{R_e} - \frac{GM_em}{2R_e} \quad [\text{As } h = R]$$

Solving we get $v = \sqrt{gR_e}$

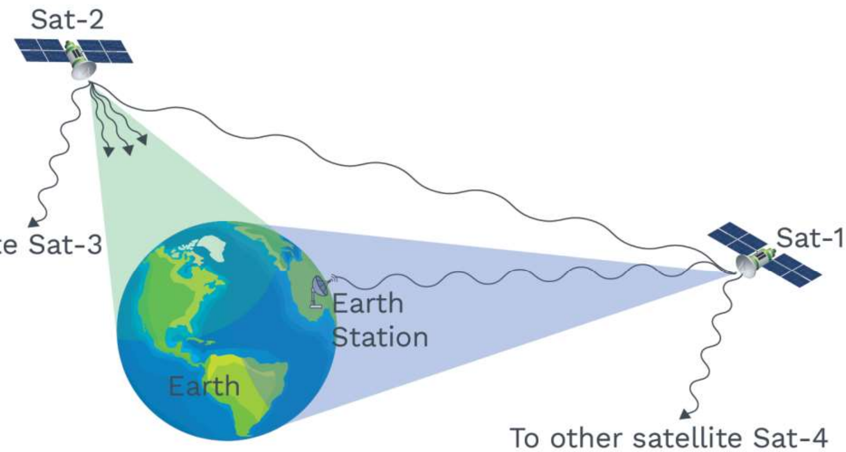
$$= \sqrt{9.8 \times 6400 \times 10^3} = 7.919\,\text{km/s}$$

Communication Satellites

Communication satellite around the earth are utilized by Information Technology for propagating information throughout the globe.

Figure appears as to how using satellites an information from an earth station, established at a point on earth's surface can be sent all over the world.

> **Concept Reminder**
>
> Communication satellite around the earth is used by Information Technology for spreading information throughout the globe.



First the information is sent to the nearest satellite in the range of earth station by means of electromagnetic waves then that satellite broadcasts the signal to the region of earth exposed to this satellite and also send the same signal to other satellite for broadcasting in other parts of the globe.

Geostationary Satellite and Parking Orbit

There are so many kinds of communication satellites revolving around the earth in different orbits at different heights depending on their utility. Some of which are Geostationary satellites, which looks at rest relative to earth or which have even angular velocity as that of earth's rotation i.e., with a time period of 24 hour such satellite should be orbiting in an orbit of specific radii. This orbit is known as parking orbit.

KEY POINTS

- Communication satellite
- Geostationary satellite
- Parking orbit

If a Geostationary satellite is at a peak h above the earth surface, then its orbiting speed is assume as

$$v_{gs} = \sqrt{\frac{GM_e}{(R_e + h)}}$$

The time period of its revolution can be as a result of Kepler's third law as

$$T^2 = \frac{4\pi^2}{GM_e}(R_e + h)^3$$

$$\text{or} \quad T^2 = \frac{4\pi^2}{g_s R_e^2}(R_e + h)^3$$

$$\text{or} \quad h = \left(\frac{g_s R_e^2}{4\pi^2} T^2\right)^{1/3} - R_e$$

$$\text{or} \quad h = \left[\frac{9.8 \times [6.4 \times 10^6] \times [86400]^2}{4 \times (3.14)^2}\right]^{1/3} - 6.4 \times 10^6$$

$$= 35954.6\,\text{km} \simeq 36000\,\text{km}$$

Thus, when a satellite is introduced in an orbit at a height of about 36000 km above the equator then it will appear to be at rest w.r.t a point on Earth's surface. A Geostationary satellite must have in orbit in equatorial plane due to the geographic limitation because of irregular geometry of earth (ellipsoidal shape).

Special Points about Geo-Stationary Satellite

- It moves in equatorial plane.
- Its height from Earth surface is 36000 km. (~ 6Re)
- Its angular velocity and time period should be equal as that of Earth.
- Its rotating direction should be same as that of Earth (West to East).
- Its orbit is called parking orbit and its orbital velocity is 3.1 km./sec.

Concept Reminder

Time period of satellite revolution is $T = 2\pi\sqrt{\frac{(R+h)^3}{gR^2}}$

Rack your Brain

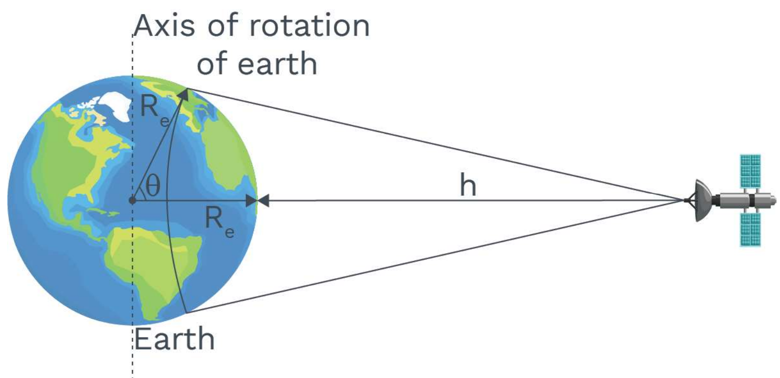
A satellite of mass m is orbiting the earth (if radius R) at a height h from its surface. If acceleration of gravity at earth's surface is g_o then find total energy of satellite in terms of g_o.

Broadcasting Region of a Satellite

Now as we identified the height of a geostationary satellite, we can positively find out the area of earth exposed to the satellite or area of the zone in which the communication can be made using this satellite.

Figure display earth and its exposed area to a geostationary satellite. Here the angle θ can be given as

$$\theta = \cos^{-1}\left(\frac{R_e}{R_e + h}\right)$$



Now we can find the solid angle Ω which the exposed area subtends on earth's centre as

$$\Omega = 2\pi(1 - \cos\theta)$$

$$= 2\pi\left(1 - \frac{R_e}{R_e + h}\right) = \frac{2\pi h}{R_e + h}$$

Thus, area of earth's surface to geostationary satellite is

$$S = \Omega R_e^2 = \frac{2\pi h R_e^2}{R_e + h}$$

Ex. A satellite is revolving around the earth in an orbit of radii double that of the parking orbit and revolving in even sense. Find out the periodic time duration between two instants when this satellite is closest to a geostationary satellite.

Sol. We realise that time period of revolution of a satellite is shown as

$$T^2 = \frac{4\pi^2}{GM_e} r^3 \qquad \text{[Kepler's III law]}$$

For satellite given in problem and for a geostationary satellite we have

$$\frac{T_1}{T_2} = \left(\frac{r_1}{r_2}\right)^3 \text{ or } T_1 = \left(\frac{r_1}{r_2}\right)^3 \times T_2 = (2)^3 \times 24 = 192\,\text{hr}$$

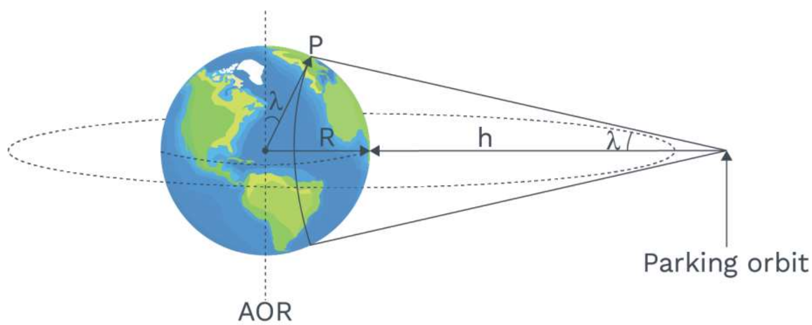
If Δt be the time between two successive instants when the satellite is closed then we must have

$$\Delta t = \frac{\theta}{\omega_1} = \frac{2\pi + \theta}{\omega_2} = \frac{2\pi}{\omega_2 - \omega_1}$$

Where ω_1 and ω_2 are the angular speeds of the planets

Ex. Find out the minimum colatitude which can directly obtain a signal from a geostationary satellite.

Sol. The farthest point on earth, which can receive signals from the parking orbit is the point where a length is described on earth surface from satellite as shown in figure. The colatitude λ of point P can be found from figure as



$$\sin\lambda = \frac{R_e}{R_e + h} \simeq \frac{1}{7}$$

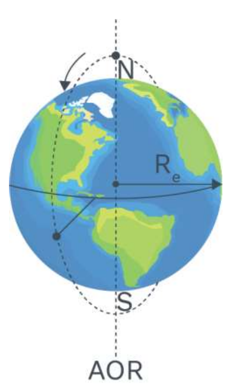
We know for a parking orbit $h \simeq 6R_e$

Thus, we have $\lambda = \sin^{-1}\left(\frac{1}{7}\right)$

Ex. If a artificial satellite is revolving around the earth surface in a circular orbit in a plane containing earth's axis of rotation. If angular speed of satellite is same to that of earth, find out the time it takes to move from a point over north pole of a point above the equator.

Sol. A satellite which rotates with angular speed equal to earth's rotation has an orbit radius 7 R_e and the angular speed of revolution is

$$\omega = \frac{2\pi}{86400} = 7.21 \times 10^{-5} \text{rad / s}$$

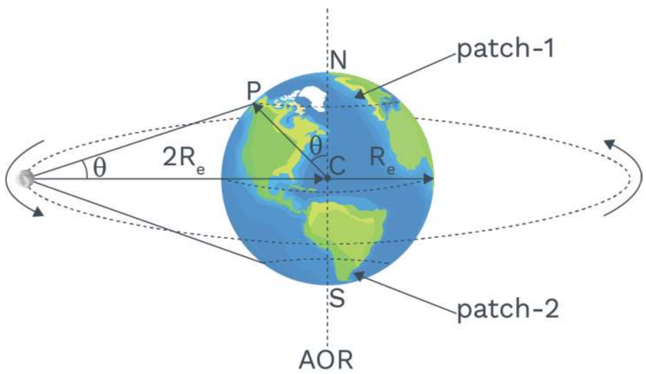


When satellite moves from a point above north pole to a point above equator, it traverses an angle $\pi / 2$, this time taken is

$$t = \frac{\pi / 2}{\omega} = 21600\,s = 6\,hrs.$$

Ex. An artificial satellite is orbiting around the earth in an orbit in equatorial plane of radii $2R_e$ where R_e is the radius of earth. Find out the area on earth, this satellite covers for communication purpose in its complete revolution.

Sol. As shown in figure when satellite S revolves, it contains a complete circular belt on earth's surface for communication. If the colatitude of the furthest point on surface up to which signals' can be received(point P) is θ then we have



$$\sin\theta = \frac{R_e}{2R_e} = \frac{1}{2} \quad \text{or} \quad \theta = \frac{\pi}{6}$$

During revolution satellite drops two spherical patches 1 and 2 on the earth surface at north and south poles where signals not can be transmitted due to curvature. The areas of these patches can be attained by solid angles.

Solid angle subtended by a patch on earth's centre is

$$\Omega = 2\pi(1 - \cos\theta) = \pi(2 - \sqrt{3})\,st.$$

Area of patch 1 & 2 is

$$A_P = \Omega R_e^2 = \pi(2 - \sqrt{3})R_e^2$$

Thus, total area on earth's surface to which communication can be made is

$$A_C = 4\pi R_e^2 - 2A_P 4\pi R_e^2 - 2\pi(2 - \sqrt{3})R_e^2$$

$$= 2\pi R_e^2(2 - 2 + \sqrt{3}) = 2\sqrt{3}\pi R_e^2$$

Examples

Q1 **An artificial satellite is in an elliptical orbit around the earth with aphelion of 6R and perihelion of 2R where R is Radius of the earth = 6400 Km. Calculate the eccentricity of the elliptical orbit.**

Sol: We know that

Perigee $(r_p) = a(1-e) = 2R$ ...(1)

Apogee $(r_a) = a(1+e) = 6R$ ...(2)

Solving (1) & (2), eccentricity (e) = 0.5

Q2 **The mean distance of a planet from the sun is approximately ¼ times that of earth from the sun. Find out the number of years required for planet to make one revolution about the sun.**

Sol: Given $r_p = \frac{1}{4}r_E$ and $T_E = 1\,yr$

From Kepler's third law, $T^2 \propto r^3$

$$\left(\frac{T_P}{T_E}\right)^2 = \left(\frac{r_P}{r_E}\right)^3 \Rightarrow T_P = T_E\left(\frac{r_P}{r_E}\right)^{\frac{3}{2}}$$

$$T_P = (1)\left(\frac{r_E}{4r_E}\right)^{\frac{3}{2}} = \left(\frac{1}{4}\right)^{3/2} = 0.125 \text{ Yrs}$$

Q3 **Let us consider that our galaxy consists of 2.5×10^{11} stars each of one solar mass. How long will this star at a distance of 50,000 light years from the galactic entre take to complete one revolution? Assume the diameter of the Milky way to be 10^5ly. $G = 6.67\times10^{-11}Nm^2Kg^{-2}$. $(1\text{ ly} = 9.46\times10^{15}\text{ m})$**

Sol: Here $M = 2.5\times10^{11}$ solar mass

$$= 2.5\times10^{11}\times(2\times10^{30})kg = 5.0\times10^{41}kg$$

$$r = 50,000\,\text{ly} = 50,000 \times 9.46 \times 10^{15}\ \text{m} = 4.73 \times 10^{20}\ \text{m}$$

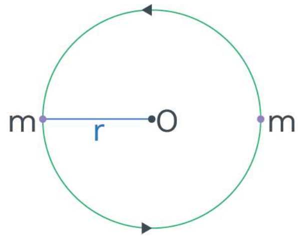
We know that $M = \frac{4\pi^2 r^3}{GT^2}$

$$T = \left(\frac{4\pi^2 r^3}{GM}\right)^{\frac{1}{2}} = \left[\frac{4 \times (22/7)^2 \times (4.73 \times 10^{20})^3}{(6.67 \times 10^{-11}) \times (5.0 \times 10^{41})}\right]^{\frac{1}{2}} = 3.53 \times 10^{14}\ \text{S}.$$

Q4 **Two particles of equal masses move in a circle of radius r under the action of their mutual gravitational attraction. Find the speed of each particle if the mass of each particle is m.**

Sol: In this case the two particles maintain gravitational force of attraction diametrically. The gravitational force on one of the particles must be equal to the necessary centripetal force

$$\frac{mv^2}{r} = \frac{Gmm}{(2r)^2} \Rightarrow v = \sqrt{\frac{Gm}{4r}}$$



Q5 **Mass M is split into two parts m and (M-m), which are then separated by a certain distance. What is the ratio of (m/M) which maximises the gravitational force between the parts?**

Sol: If r is the distance between m and (M-m), the gravitational force between them will be

$$F = G\frac{m(M-m)}{r^2} = \frac{G}{r^2}(mM - m^2)$$

For F to be maximum dF/dm = 0 as M & r are constants.

$$\Rightarrow \frac{d}{dm}\left[\frac{G}{r^2}\left(mM - m^2\right)\right] = 0; \quad \Rightarrow M - 2m = 0$$

$$\Rightarrow \frac{m}{M} = \frac{1}{2}.$$ So, the force will be maximum when the parts are equal.

Q6 **Two particles of masses 1 Kg and 2 Kg are placed at a distance of 50 cm. Find out the initial acceleration of the first particle due to gravitational force.**

Sol: Gravitational force between two particles is

$$F = \frac{Gm_1m_2}{r^2} = \frac{6.67\times10^{-11}\times1\times2}{(0.5)^2} = 5.3\times10^{-10}\text{ N}$$

The acceleration of 1 Kg particle is

$$a_1 = \frac{F}{m_1} = \frac{5.3\times10^{-10}}{1} = 5.3\times10^{-10}\text{ ms}^{-2}$$ towards the 2 Kg mass.

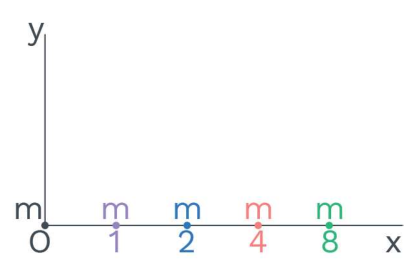
Q7 **An infinite number of particles each of mass m are placed on the positive X-axis at 1m, 2m, 4m, 8m,..... from the origin. Find out the magnitude of the resultant gravitational force on mass 'm' kept at the origin.**

Sol: The resultant gravitational force

$$F = \frac{Gm^2}{1} + \frac{Gm^2}{4} + \frac{Gm^2}{16} + \ldots\ldots$$

$$= Gm^2\left(1 + \frac{1}{4} + \frac{1}{16} + \ldots\ldots\right)$$

$$= Gm^2\left(\frac{1}{1-\frac{1}{4}}\right) = \frac{4}{3}Gm^2 \quad \left(\because S_\infty = \frac{a}{1-r}\right)$$



Q8 **What is the time period of rotation of the earth around its axis so that the object at the equator becomes weightless? ($g = 9.8\ m/s^2$, Radius of earth = 6400 km).**

Sol: When earth is rotating the apparent weight of a body at the equator is given by

$$W_{app} = mg - mR\omega^2$$

If bodies are weightless at the equator

$$0 = mg - mR\omega \Rightarrow g = R\omega^2$$

$$\Rightarrow \omega = \sqrt{\frac{g}{R}}$$

$$\text{Time period,} \quad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R}{g}}$$

$$T = 2\pi\sqrt{\frac{6.4\times10^6}{9.8}} = 5078\,s = 84 \text{ minute } 38\,s$$

Q9 **The height at which the acceleration due to gravity becomes g/9 (where g is the acceleration due to gravity on the earth surface) in terms of the radius of the earth (R) is**

Sol: Given $\frac{g}{9} = g\left(\frac{R}{R+h}\right)^2 \Rightarrow \frac{R}{R+h} = \frac{1}{3}$

$$3R = R + h \Rightarrow 2R = h$$

Q10 **How much above the surface of earth does the acceleration due to gravity reduce by 36% of its value on the earth surface.**

Sol: Since g reduces by 36%, the value of g there is

100 – 36 = 64%. It means, $g' = \frac{64}{100}g$.

If h is height of location above the earth surface then,

$$g' = g\frac{R^2}{(R+h)^2} \Rightarrow \frac{64}{100}g = g\frac{R^2}{(R+h)^2}$$

$$\Rightarrow \frac{8}{10} = \frac{R}{R+h} \Rightarrow h = \frac{R}{4} = \frac{6.4 \times 10^6}{4} = 1.6 \times 10^6 \text{ m}$$

Q11 **A particle of mass m is set at the centre of a uniform spherical shell of equal mass and radii a. Find out the gravitational potential at a point P at a distance a/2 from the centre.**

Sol: The gravitational potential at P due to particle at centre is $V_1 = \frac{-Gm}{a/2} = \frac{-2Gm}{a}$

The potential at P due to shell is $V_2 = \frac{-Gm}{a}$

The net potential at P is $V_1 + V_2 = \frac{-3Gm}{a}$

Q12 **If Earth has mass nine times and radius twice that of the planet mars, calculate the velocity required by a rocket to pull out of the gravitational force of Mars. Take escape speed on surface of Earth to be 11.2 km/s.**

Sol: Here, $M_e = 9M_m$, and $R_e = 2R_m$

v_e (escape speed on surface of Earth) = 11.2 km/s

Let V_m be the speed required to pull out of the gravitational force of mars.

We know that

$$v_e = \sqrt{\frac{2GM_e}{R_e}} \text{ and } v_m = \sqrt{\frac{2GM_m}{R_m}}$$

Dividing, we get $\frac{v_m}{v_e} = \sqrt{\frac{2GM_m}{R_m} \times \frac{R_e}{2GM_e}}$

$$= \sqrt{\frac{M_m}{R_e} \times \frac{R_e}{R_m}} = \sqrt{\frac{1}{9} \times 2} = \frac{\sqrt{2}}{3}$$

$$\Rightarrow v_m = \frac{\sqrt{2}}{3}(11.2\,\text{km/s}) = 5.3\,\text{km/s}$$

Q13 **A planet in a distant solar system is 10 times more massive than the earth and its radii is 10 times smaller. Given that escape velocity from the earth surface is 11 km/s, the escape velocity from the surface of the planet is**

Sol: Given $M_P = 10M_e$; $R_P = \frac{R_e}{10}$

We know that $v_e = \sqrt{\frac{2GM}{R}}$

$$\therefore \quad v_P = \sqrt{\frac{2GM_P}{R_P}} = \sqrt{\frac{100 \times 2GM}{R_e}} = 10V_e$$

$= 10 \times 11 = 110$ km/s.

Q14 **A body is projected vertically upwards from the surface of the earth with a velocity equal to half of escape velocity of the earth. If R is radius of the earth, maximum height attained by the body from the surface of the earth is**

Sol: From law of conservation of energy,

$$TH_{surface} = TE_{Max\ Height}$$

$$-\frac{GMm}{R} + \frac{1}{2}m\left(\frac{v_e}{2}\right)^2 = -\frac{GMm}{R+h} + \frac{1}{2}m(0)^2$$

$$-\frac{GMm}{R} + \frac{1}{2}m\left(\frac{2GM}{4R}\right) = -\frac{GMm}{R+h} \quad \left(\because v_e^2 = \frac{2GM}{R}\right)$$

On simplifying, we get $h = R/3$.

Q15 **The mass of a spaceship is 1000 kg. It is to be launched from the earth surface out into free space. The value of 'g' and 'R' are 10 ms⁻² and 6400 km respectively. The required energy for this work will be**

Sol: To launch the spaceship out into free space,

from energy conservation, $-\frac{GMm}{R} + E = 0$

$$E = \frac{GMm}{R} = \left(\frac{GM}{R^2}\right)mR = mgR = 6.4 \times 10^{10}\ J$$

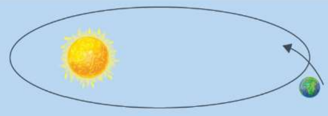
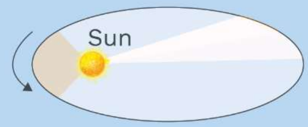
Q16 **The minimum energy needed to launch a satellite of mass m from the surface of a planet of mass M and radius R in a circular orbit at an altitude of 2R is**

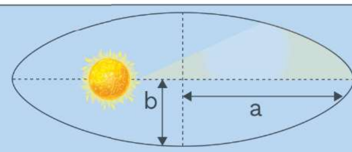
Sol: From the law of conservation of energy

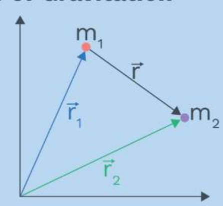
$$\frac{-GMm}{R} + KE_{imparted} = \frac{-GMm}{2 \times 3R}$$

$$KE_{imparted} = \frac{GMm}{R}\left(1 - \frac{1}{6}\right) = \frac{5}{6}\left(\frac{GMm}{R}\right).$$

Mind Map

- **GRAVITATION**
  - **Kepler's Laws of Planetary Motion**
    - **Laws of Orbits**
      Elliptical orbits
      
    - **Laws of Areas**
      Equal area in equal interval of time
      

      i.e., $\frac{dA}{dt}$ = Constant
    - **Laws of Periods**
      $T^2 \propto a^3$
      

  - **Newton's Law of Gravitation**
    r
    $F = \frac{GMm}{r^2}$
    - **Vector form of Newton's law of Gravitation**
      $\vec{F}_{12} = \frac{Gm_1m_2}{r^3}\vec{r}$
      

    - **Superposition Principle**
      $\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 .... + \vec{F}_n$