Elasticity

DISCLAIMER

Elasticity

INTRODUCTION

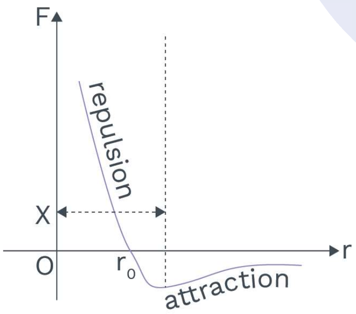
Elasticity deals with property of a material, such as its strength and ability to withstand against external forces which are acting on it.

KEY POINTS

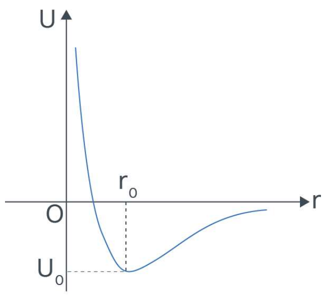
- Elasticity
- Plasticity
- Inter molecular force

Inter Molecular Forces:

- The forces between the molecules due to electrostatic interaction between the charges of the molecules are called intermolecular forces. Thus intermolecular forces are also electromagnetic in nature. These forces are active if the separation between two molecules is of the order of molecular size (» 10^{-9} m).The variation of intermolecular forces with distance is shown in figure.



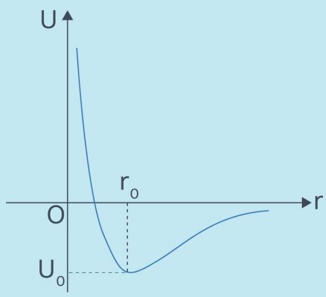
- The variation of potential energy versus distance (U-r) graph is.



- For large distance 'r', the intermolecular force is negligible.
- As the distance decreases, the force of attraction increases.

Concept Reminder

The variation of potential energy versus distance (U-r) graph is.



- At a particular distance x, the force of attraction becomes maximum. After this distance, the force of attraction decreases and becomes zero at a distance 'r_0'.
- Interatomic force is the force between two atoms or molecules due to electrostatic force of attraction between charges.
- **Rigid Body:** A body whose size and shape cannot be changed, however large the applied force may be is called rigid body.
  There is no perfectly rigid body in nature. The nearest approach to a perfect rigid body is diamond.
- **Deformation force:** The external force which changes the size (or) shape (or) both of a body without moving it as a whole is called deformation force.
- **Restoring force:** The internal force which restores the size and shape of the body when deformation force is removed is called restoring force.
- In equilibrium magnitude of restoring force is equal to the deformation force.
- The restoring force at a point do not form action-reaction pair with applied force. This force is responsible for the elastic nature of the body.

KEY POINTS

- Rigid body
- Deformation force
- Restoring force

Elastic Behaviour of Solids:

- In a solid, atoms (or) molecules are bonded together by the interatomic (or) intermolecular forces & stay in a stable equilibrium position.
- When a solid is deformed, the atoms (or) molecules are then displaced from their equilibrium positions causing a change in the interatomic distances. When the deforming force is removed, then the interatomic forces tend to drive them back to their original positions. Thus the body regains its original shape and size.

Definitions

The property of a material by virtue of which it regains its original size and shape when deformation force is removed is called elasticity.

Elasticity:

- The property of the material by virtue of which it regains its original size and shape when deformation force is removed is called elasticity.
  Ex.: Quartz, Diamond, Steel, Rubber etc.
  There is no perfectly elastic material exist in nature, but quartz is the nearest perfectly elastic material. Elasticity is molecular property of matter.

Concept Reminder

Examples of elasticity:
- Quartz
- Diamond
- Steel
- Rubber

Plasticity:

- The property of the material by virtue of which it does not regain its original size and shape after the deforming force is removed is called Plasticity.
  Ex.: Putty dough, Chewing gum, Soldering lead.
- No material is perfectly plastic but putty is nearest approach for perfect plastic material.

Definitions

The property of a material by virtue of which it does not regain its original size and shape after the deforming force is removed is called Plasticity.

STRESS

The restoring force which is acting per unit area of the deformed body is called stress.

$$\text{Stress} = \frac{\text{Internal restoring force}}{\text{Area}}$$

$$= \frac{F_{\text{internal}}}{A} = \frac{F_{\text{external}}}{A} \text{ (at equilibrium)}$$

SI Unit: N/m^2.
dimensions: $[M^1L^{-1}T^{-2}]$

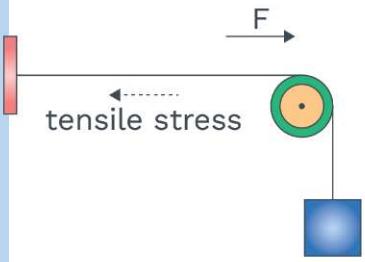
Concept Reminder

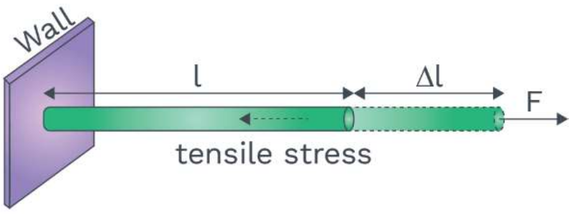
Examples of elasticity:
- Putty dough
- Chewing gum
- Soldering lead

Types of stress:

(i) Longitudinal Stress:
When stress is normal to the surface of the body, then it is known as longitudinal stress. There are two types of longitudinal stress:

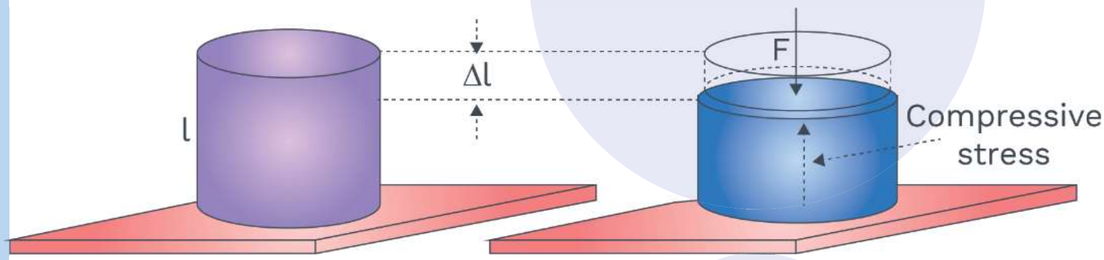
(a) Tensile Stress
The longitudinal stress, produced due to the increase in the length of a body, is defined as tensile stress.





(b) Compressive Stress

The longitudinal stress, produced due to the decrease in the length of a body, is defined as compressive stress.

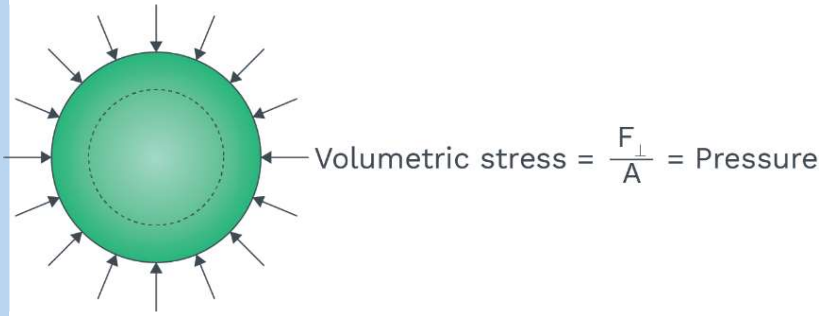


(ii) Volumetric Stress:

If equal normal forces are applied over every unit surface of a body, then it undergoes a certain change in volume. The force opposing this change in volume per unit area is defined as volume stress.

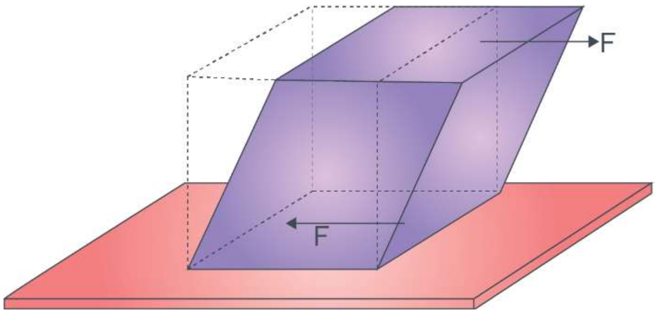
KEY POINTS

- Stress
- Strain



$$\text{Volumetric stress} = \frac{F_{\perp}}{A} = \text{pressure}$$

(iii) Shear Stress:



When the stress is tangential or it is parallel to the surface of a body then it is known as shear stress. Due to this stress, the shape of the body changes or it gets twisted but not its volume.

$$\text{Shear stress} = \frac{F_{||}}{A} = \frac{F_{tangential}}{A}$$

(iv) Breaking Stress:

The stress required to cause the actual fracture of a material is called the breaking stress or ultimate strength.

$$\text{Breaking stress} = \frac{F}{A};$$

F = force required to break the body.

Dependence of breaking stress:

(i) Nature of material
(ii) Temperature
(iii) Impurities.

Independence of breaking stress:

(i) Cross sectional area or thickness
(ii) Applied force.

Maximum load (force) which can applied on the wire depends on:

(i) Cross sectional area or thickness
(ii) Nature of material
(iii) Temperature
(iv) Impurities.

Concept Reminder

Types of stress:

- Longitudinal stress
- Volume stress
- Shear stress
- Breaking stress

Concept Reminder

Dependence of breaking stress:

(i) Nature of material
(ii) Temperature
(iii) Impurities.

Independence of breaking stress:

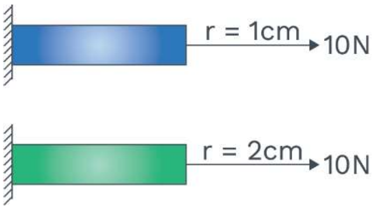
(i) Cross sectional area or thickness
(ii) Applied force.

Concept Reminder

Maximum load (force) which can applied on the wire depends on:

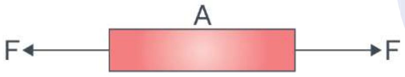
(i) Cross sectional area or thickness
(ii) Nature of material
(iii) Temperature
(iv) Impurities.

Ex. Find the ratio of stress in both wires A and B.



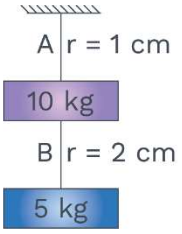
Sol. $\frac{S_A}{S_B} = \frac{F_A}{A_A} \times \frac{A_B}{F_B} = \frac{10}{\pi(1)^2} \times \frac{\pi(2)^2}{10} = \frac{4}{1}$

Ex. Find the stress in wire of cross-section A.



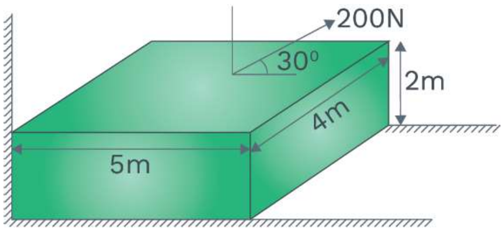
Sol. Stress = $\frac{F}{A}$

Ex. Find the ratio of stress in both wires A and B.



Sol. $\frac{S_A}{S_B} = \frac{F_A}{A_A} \times \frac{A_B}{F_B} = \frac{150}{\pi(1)^2} \times \frac{\pi(2)^2}{50} = \frac{12}{1}$

Ex. Find out the longitudinal stress and tangential stress on the fixed block shown in figure.



Rack your Brain

The following four wires are made of the same material. Which of these will have the largest extension when the same tension is applied?

(1) length = 100 cm, diameter = 1 mm

(2) length = 200 cm, diameter = 2 mm

(3) length = 300 cm, diameter = 3 mm

(4) length = 50 cm, diameter = 0.5 mm

Sol. Longitudinal stress

$$= \frac{F_{\perp}}{A} = \frac{200 \sin 30°}{5 \times 4} = 5 \text{ N/m}^2$$

Tangential stress

$$= \frac{F_{||}}{A} = \frac{200 \cos 30°}{5 \times 4} = 5\sqrt{3} \text{ N/m}^2$$

Ex. A bar of cross-section A is being subjected to equal & opposite tensile forces F at its ends. Consider a plane through the bar which is making an angle θ with a plane at right angles to the bar length.

(a) What is the value of tensile stress at this plane in the terms of F, A and θ ?

(b) What is the shearing stress at this plane, in terms of F, A and θ ?

Sol. (a) As tensile stress = (normal force/area)

Here, $A_N = \text{area} = \left(\frac{A}{\cos\theta}\right)$

and normal force $F_N = F \cos\theta$

So, tensile stress

$$= \frac{F\cos\theta}{\left(\frac{A}{\cos\theta}\right)} = \frac{F\cos^2\theta}{A}$$

(b) As shear stress = (tangential force/area)

Here, area = $\left(\frac{A}{\cos\theta}\right)$ and

Tangential force = $F \sin\theta$

So, shear stress

$$= \frac{F\sin\theta}{\left(\frac{A}{\cos\theta}\right)} = \frac{F\sin\theta\cos\theta}{A} = \frac{F\sin 2\theta}{2A}$$

Rack your Brain

A wire elongates by ℓ mm when a load W is hanged from it. If the wire goes over a pulley and two weights W each are hung at the two ends, the elongation of the wire will be (in mm).

Ex. 40 N force is required to break a wire. Find out the amount of force required to break the another wire of same material but two times of radius.

Sol. For wires of same material breaking stress will be same.

$\Rightarrow$ $B.S._1 = B.S._2$

$$\Rightarrow \frac{40}{\pi R^2} = \frac{F}{\pi(2R)^2} \Rightarrow F = 160 \text{ N}$$

Ex. The breaking stress of a wire is 6×10^8 N/m^2 and density is 3×10^3 Kg/m^3. What should be the maximum length of the wire so that it may not break under its own weight?

Sol. $B.S. = \frac{mg}{A} = \rho \ell g$

$$\Rightarrow \ell = \frac{B.S.}{\rho g} = \frac{6 \times 10^8}{3 \times 10^3 \times 10} = 2 \times 10^4 \text{ m}$$

$= 20$ km

Ex. A lift of mass 500 kg is connected with a wire of breaking stress $\frac{3}{\pi} \times 10^8 \text{ N/m}^2$. If the lift moves up with an acceleration of 5.2 m/s^2 then calculate the minimum radius of wire. [Take g = 9.8 m/s^2]

Sol. $B.S. = \frac{F}{A} = \frac{m(g+a)}{\pi r^2}$

$$\Rightarrow r^2 = \frac{m(g+a)}{\pi (B.S.)} = \frac{500 \times [9.8 + 5.2]}{\pi \times \frac{3}{\pi} \times 10^8}$$

$r = 5 \times 10^{-3}\text{m} = 5 \text{ mm}$

Note:

1. If the stress is normal to the surface it is called normal stress.

2. The stress is always normal to surface in case of change in length of a wire or volume of body.

3. When the external force compresses the body

Concept Reminder

1. If the stress is normal to surface called normal stress.

2. Stress is always normal to surface in case of change in length of a wire or volume of body.

3. When external force compresses the body $\Rightarrow$ Nature of atomic force will be repulsive.

4. When external forces expanses the body $\Rightarrow$ Nature of atomic force will be attractive.

⇒ Nature of the atomic force will be repulsive.

4. When the external forces expanses the body

⇒ Nature of atomic force will be attractive.

Difference between Pressure v/s Stress:

S. No.	PRESSURE	STRESS
1.	Pressure is always normal to the area.	Stress can be normal or tangential
2.	Always compressive in nature	May be compressive or tensile in nature.
3.	Scalar	Tensor

Ex. A copper wire which is 4.0 m long & cross-sectional area 1.2 cm^2 is stretched by a force of 4.8×10^3 N stress will be.

Sol. Stress $= \frac{F}{A} = \frac{4.8 \times 10^3 N}{1.2 \times 10^{-4} m^2} = 4.0 \times 10^7 \ N / m^2$

STRAIN

$$\text{Strain} = \frac{\text{Change in the dimension of the body}}{\text{Original dimension of the body}}$$

Definitions

Strain:

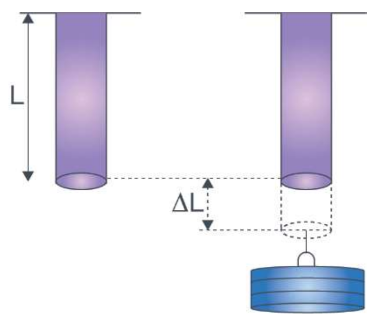
Strain = (Change in the dimension of the body) / (Original dimension of the body.)

There are three types of strains:

Types of strains depend upon the directions of applied force.

(i) Longitudinal strain:

$$= \frac{\text{Change in length of the body}}{\text{Initial length of the body}} = \frac{\Delta L}{L}$$



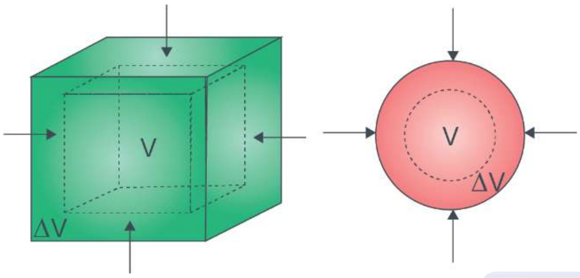
KEY POINTS

- Longitudinal strain
- Volume strain
- Shear strain

Longitudinal strain is possible only in solids.

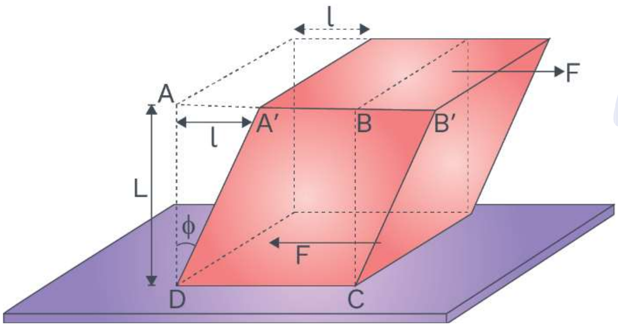
(ii) Volume strain:

$$= \frac{\text{Change in volume of the body}}{\text{Original volume of the body}} = \frac{\Delta V}{V}$$



(iii) Shear strain:

When a deforming force is being applied to a body parallel to its surface, then its shape (not size) changes. The strain produced in this manner is known as shear strain.



The strain produced due to a change in shape of the body is known as shear strain.

$\tan\phi = \frac{\ell}{L}$ (Here ϕ is very small)

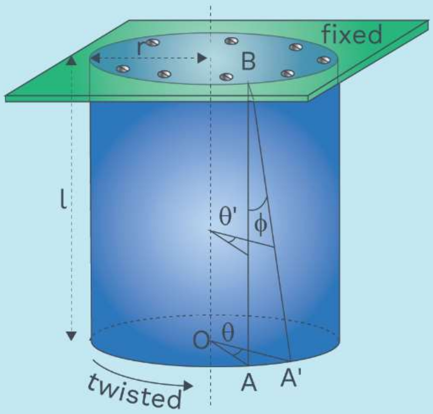
Shear strain $\phi = \frac{\ell}{L}$

$$F = \frac{\text{Displacement of upper face relative to the lower face}}{\text{Distance between two faces}}$$

Relation Between the angle of twist and Angle

Concept Reminder

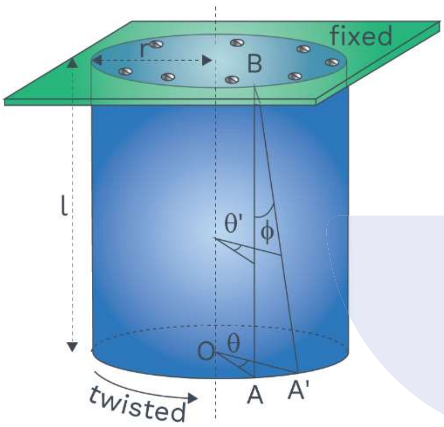
Relation Between angle of twist and Angle of shear:



$\phi = \frac{r\theta}{\ell}$

of shear:

When a cylinder of length 'l' and radius 'r' is fixed at one end and tangential force is applied at the other end, then the cylinder gets twisted. Figure shows the angle of shear ABA' and angle of twist AOA'.



$$\text{Arc } AA' = r\theta \text{ and Arc } AA' = \ell\phi$$

So, $r\theta = \ell\phi \Rightarrow \phi = \frac{r\theta}{\ell}$

θ = angle of twist, ϕ = angle of shear.

Ex. A cylindrical wire having radius 1 cm and length 10 cm is twisted by an angle $\frac{\pi}{60}$ radian, then find shear strain produced in the wire.

Sol. $\phi = \frac{r\theta}{\ell} = \frac{(1)\left(\frac{\pi}{60}\right)}{10} = \frac{\pi}{600}$ radian

$= \left(\frac{\pi}{600} \times \frac{180}{\pi}\right) = 0.3°$

Hooke's law:

According to this law, the stress produced in a body within the elastic limit is directly proportional to the corresponding strain.

Stress $\propto$ Strain

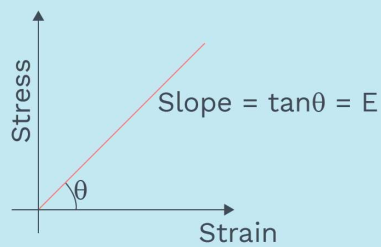
Rack your Brain

The pressure that has to be applied to the ends of a steel wire of length 10 cm to keep its length constant when its temperature is raised by 100°C is. (For steel Young's modulus is 2×10^{11} N m^{-2} and coefficient of thermal expansion is 1.1×10^{-5} K^{-1})

Concept Reminder

Hooke's law:

Stress $\propto$ Strain



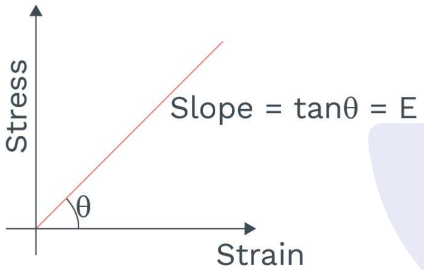
Note : angle is measured always from strain axis

Stress = E × strain

Here, E = coefficient of elasticity or modulus of elasticity

$$\frac{\text{Stress}}{\text{Strain}} = E$$

Hence, within the elastic limit the ratio of stress & strain remains constant which is equal to coefficient of elasticity.



Note : angle is measured always from strain axis

stress ∝ strain

$$\frac{\text{Stress}}{\text{Strain}} = E$$

$$\tan\theta = E$$

The slope of stress & strain graph gives coefficient of elasticity.

Important points:

E depends on:

1. Nature of material

2. Impurities

3. Temperature

E is independent from: stress & strain.

Types of modulus of elasticity:

YOUNG'S MODULUS

Suppose that a rod having length l and a uniform cross-sectional area A is subjected to a longitudinal pull. In other words, the two equal and opposite forces are applied at its ends.

$$\text{Stress} = \frac{F}{A}$$

Concept Reminder

Elasticity (E) depends on:

1. Nature of material
2. Impurities
3.Temperature

E is independent from: stress & strain.

KEY POINTS

- Young's modulus
- Bulk's modulus

The stress in the present case is called the linear stress, tensile stress, or extensional stress. If the direction of the force is then reversed so that ΔL is negative, we speak of the compressional strain & compressional stress. If the elastic limit is being not exceeded, then from Hooke's law.

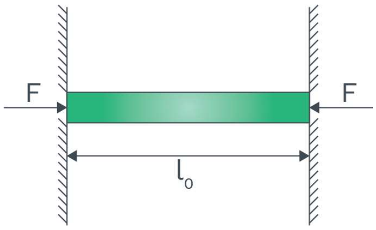
Stress $\propto$ strain

or Stress = Y × strain

or $$\boxed{Y = \frac{\text{stress}}{\text{strain}} = \frac{F}{A} \cdot \frac{L}{\Delta L}} \quad \text{...(i)}$$

Where Y is the constant of proportionality, is called the Young's modulus of the material of the rod and may be defined as the ratio of the linear stress to linear strain, provided that the elastic limit is not exceeded. Since the strain has no unit, the unit of Y is Nm^{-2}.

Consider a rod having length l_0 which is fixed in between the rigid end separated at a distance l_0 now if the temperature of the rod is increased by $\Delta\theta$, then the strain produced in the rod will be:



strain =

$$\frac{\text{length of the rod at new temperature} - \text{natural length of the rod at new temperature}}{\text{natural length of the rod at new temperature}}$$

$$= \frac{\ell_0 - \ell_0(1+\alpha\Delta\theta)}{\ell_0(1+\alpha\Delta\theta)} = \frac{-\ell_0\alpha\Delta\theta}{\ell_0(1+\alpha\Delta\theta)}$$

$\therefore$ α is very small so, strain = – $\alpha\Delta\theta$

(the negative sign in the answer represents that the length of the rod is less than its natural length that means is compressed by the ends.)

Definitions

Young's modulus may be defined as the ratio of the linear stress to linear strain, provided the elastic limit is not exceeded. Since strain has no unit, the unit of Y is Nm^{-2}.

Concept Reminder

Strain

$$= \frac{\ell_0 - \ell_0(1+\alpha\Delta\theta)}{\ell_0(1+\alpha\Delta\theta)} = \frac{-\ell_0\alpha\Delta\theta}{\ell_0(1+\alpha\Delta\theta)}$$

We know that $Y = \dfrac{\text{stress}}{\text{strain}}$

then F = YαΔTA

Note:

(A) For Loaded Wire:

$$\Delta L = \frac{FL}{\pi r^2 Y} \quad \left[\because\ Y = \frac{FL}{A\Delta L} \text{ and } A = \pi r^2\right]$$

For the rigid body

ΔL = 0 so Y = ∞

i.e. elasticity of the rigid body is infinite.

(B) If same stretching force is applied to different wire of same material.

$$\Delta L \propto \frac{L}{r^2} \quad \text{[As F and Y are constant]}$$

Greater the value ΔL, greater will be elongation.

(C) Elongation of wire by its own weight:

In this case F = Mg acts at CG of the wire so length of wire which is stretched will be L/2

$$\Delta L = \frac{FL}{AY} = \frac{(Mg)\times\frac{L}{2}}{\pi r^2 Y} = \frac{MgL}{2AY} = \frac{\rho g L^2}{2Y}$$

[M = ρAL]

$$\Delta L = \frac{\rho g L^2}{2Y}$$

Ex. A wire having length 1 m and area of cross section 4×10^{-8} m^2 increases in length by 0.2 cm when a force of 16 N is applied. Value of Y for the material of the wire material will be.

Sol. By Hook's law

$$Y = \frac{F/A}{\Delta L/L} = \frac{FL}{A\Delta L}$$

$$Y = \frac{16\times 1}{(4\times 10^{-8})(0.2\times 10^{-2})} = 2\times 10^{11}\ N/m^2$$

Note:

(i) Y is only defined for solids

(ii) For liquid and gases Y = 0

Concept Reminder

For Loaded Wire:

$$\Delta L = \frac{FL}{\pi r^2 Y}\left[\because\ Y = \frac{FL}{A\Delta L}\ \&\ A = \pi r^2\right]$$

For rigid body

ΔL = 0 so Y = ∞

i.e. elasticity of rigid body is infinite.

Concept Reminder

Elongation of wire by its own weight:

In this case F = Mg acts at CG of the wire so length of wire which is stretched will be L/2

$$\Delta L = \frac{FL}{AY} = \frac{(Mg)\times\frac{L}{2}}{\pi r^2 Y}$$

$$\frac{MgL}{2AY} = \frac{\rho g L^2}{2Y} \quad [M = \rho AL]$$

$$\Delta L = \frac{\rho g L^2}{2Y}$$

Concept Reminder

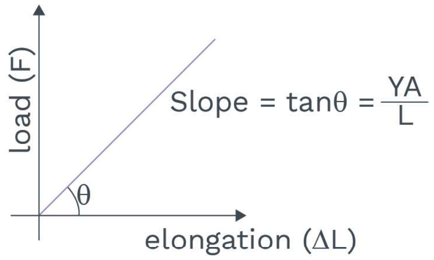
(i) Y is only defined for solids

(ii) For liquid and gases Y = 0

(iii) For ideal rigid body Y = ∞

(iii) For ideal rigid body $Y = \infty$

Graph between load [F] and elongation [ΔL]:



Note : angle is measured always from strain axis

$$Y = \frac{FL}{A\Delta L}$$

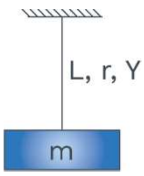
$$F = \frac{YA}{L}\Delta L$$

$$F = k.\Delta L \quad ...(i)$$

$$\text{Spring force } F = k.x \quad ...(ii)$$

$$\Rightarrow \boxed{k = \frac{YA}{L}}$$

When a block of mass 'm' is hanged vertically with a massless wire of length 'L', radius 'r' and young's modules 'Y', then increment in the length of wire.

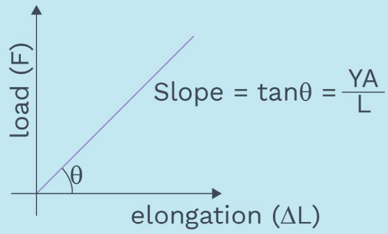


$$\Delta L = \frac{FL}{AY} = \frac{mgL}{\pi r^2 Y}$$

Increment of length of a wire due to its own weight:

Concept Reminder

Graph between load [F] and elongation [ΔL]:



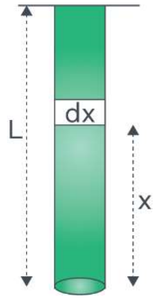
Note : angle is measured always from strain axis

Rack your Brain

Two Young's modulus of steel is twice that of brass. Two wires of same length and of same area of cross section, one of steel and another of brass are suspended from the same roof. If we want the lower ends of the wires to be at the same level, then the weights added to the steel and brass wires must be in the ratio of:

(1) 2 : 1 (2) 4 : 1

(3) 1 : 1 (4) 1 : 2



Consider a rope of mass M and length L hanging vertically. As the tension at different points on the rope is different, stress as well as strain will be different at different points.

(i) Maximum stress will be at the point of suspension

(ii) Minimum stress will be at the lower end.

Consider an element of rope of length dx at x distance from the lower end, then tension there

$$T = \left(\frac{M}{L}\right)xg$$

So stress $= \frac{T}{A} = \left(\frac{M}{L}\right)\frac{xg}{A}$

Let increase in length of this element be dy then strain $= \frac{dy}{dx}$.

So, Young modulus of elasticity $Y = \frac{\text{stress}}{\text{strain}} = \frac{\frac{M}{L}\frac{xg}{A}}{\frac{dy}{dx}}$

$$\Rightarrow \left(\frac{M}{L}\right)\frac{xg}{A}dx = Ydy$$

Summing up the expression for full length of the rope,

$$\frac{Mg}{LA}\int_0^L xdx = Y\int_0^{\Delta\ell} dy$$

Concept Reminder

Increment of length of wire due to its weight:

$$Y = \frac{Mg \times \frac{\ell}{2}}{A \times \Delta\ell} \Rightarrow \Delta\ell = \frac{Mg\ell}{2AY}$$

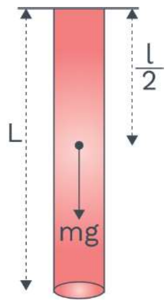
But M = (lA)ρ

$$\therefore \quad \Delta\ell = \frac{\ell A\rho g\ell}{2AY} \text{ or } \Delta\ell = \frac{\rho g\ell^2}{2Y}$$

$$\Rightarrow \frac{Mg}{LA}\frac{L^2}{2} = Y\Delta\ell \quad \Rightarrow \Delta\ell = \frac{MgL}{2AY}$$

[Since the stress is varying linearly we may apply the average method to evaluate strain.]

Alternate Method: Since the, weight acts at the centre of gravity, therefore



$\therefore$ The original length will be taken as $\frac{\ell}{2}$

$$\therefore \quad Y = \frac{Mg \times \frac{\ell}{2}}{A \times \Delta\ell} \quad \Rightarrow \Delta\ell = \frac{Mg\ell}{2AY}$$

But M = (lA)ρ

$$\therefore \quad \Delta\ell = \frac{\ell A\rho g\ell}{2AY} \quad \text{or} \quad \Delta\ell = \frac{\rho g\ell^2}{2Y}$$

Ex. Find the force required to generate 0.1% change in length of a wire whose cross-sectional area is 1 cm^2 and young's modulus is 5×10^{10} N/m^2.

Sol. $F = AY\left(\frac{\Delta\ell}{\ell}\right)$

$$F = (10^{-4})(5 \times 10^{10})\left(\frac{0.1}{100}\right)$$

$F = 5 \times 10^3$ N

Ex. Two wires P and Q are made of same material and have same dimensions. Find the ratio of change in length in wire P and Q.

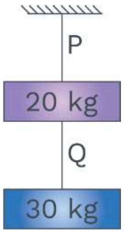
Rack your Brain

Two wires are made of the same material and have the same volume. However wire 1 has cross-sectional area A and wire 2 has cross-sectional area 3A. If the length of wire 1 increases by Δx on applying force F, how much force is needed to stretch wire 2 by the same amount?

(1) 4F (2) 6F

(3) 9F (4) F

Sol. $\Delta\ell = \frac{F\ell}{AY}$



same dimensions → L and A are same

same material → same Y

$\therefore$ $\Delta l \propto F$

$\therefore$ $\frac{\Delta\ell_P}{\Delta\ell_Q} = \frac{T_P}{T_Q} = \frac{50g}{30g} = \frac{5}{3}$

Ex. Cross sectional area of a wire ($Y = 5 \times 10^{11}$ N/m^2) is 0.1 cm^2. Find the force required to make its length double.

Sol. $F = AY\left(\frac{\Delta\ell}{\ell}\right)$ $(\ell_i = \ell, \ell_f = 2\ell)$

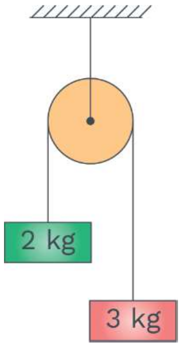
$F = (0.1 \times 10^{-4})(5 \times 10^{11})(1)$

$\therefore$ $\Delta\ell = \ell$

$F = 5 \times 10^6$ N

Ex. Two blocks of masses 2 kg and 3 kg are connected through a wire which passes over a massless and frictionless pulley. If the young's modulus of wire is 12×10^{11} N/m^2 and cross-sectional area is 0.1 cm^2 then find strain in wire ($g = 10$ m/s^2)

Sol. $T = \left(\frac{2m_1m_2}{m_1 + m_2}\right)g$

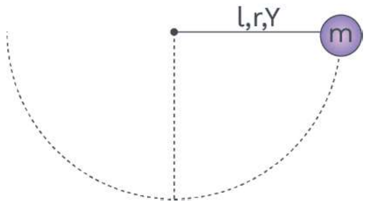


$$T = \frac{2(2)(3)(10)}{2+3} = 24\,N$$

$$\frac{\Delta\ell}{\ell} = \frac{F}{AY} = \frac{(24)}{(0.1\times10^{-4})(12\times10^{11})}$$

$$\frac{\Delta\ell}{\ell} = 2\times10^{-6}$$

Ex. An object of mass 'm' is connected to a wire of length 'l', radius 'r' and Young's modulus 'Y' as shown in figure. If the object is released from the horizontal position of the wire, then find out the maximum increment in length of the wire.



Sol. $$\frac{\Delta\ell}{\ell} = \frac{F\ell}{AY}$$

$$T_{max} = \frac{mv^2}{\ell} + mg$$

$$T_{max} = \frac{m}{\ell}(\sqrt{2g\ell})^2 + mg = 3mg$$

$$\therefore \quad \Delta\ell_{max} = \frac{(3mg)(\ell)}{(\pi r^2)(Y)}$$

Rack your Brain

The bulk modulus of a spherical object is B. If it is subjected to uniform pressure p, the fractional decrease in radius is:

(1) $\frac{p}{B}$ (2) $\frac{B}{3p}$

(3) $\frac{3p}{B}$ (4) $\frac{p}{3B}$

Ex. Two wire P and Q are made of same material and have same volume. The length of P is 3 times that of Q. If they are stretched by same force then find out the ratio of increament in their lengths.

Sol. Given,

$Y_P = Y_Q$

$V_P = V_Q$

$l_P = 3\ l_Q$

$F_P = F_Q$

$$\Delta\ell = \left(\frac{F\ell}{AY}\right) \times \frac{\ell}{\ell} = \frac{F\ell^2}{VY}$$

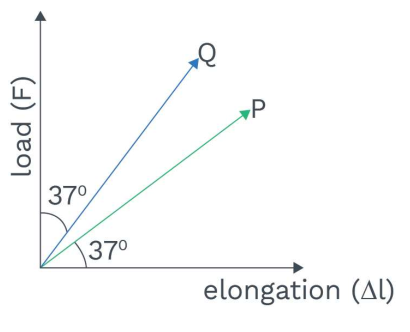
$\therefore \quad \Delta\ell \propto \ell^2$

$$\therefore \quad \frac{\Delta\ell_P}{\Delta\ell_s} = \left(\frac{\ell_P}{\ell_Q}\right)^2 = \left(\frac{3}{1}\right)^2 = \frac{9}{1}$$

Ex. A wire has initial length 8m and density 1 kg/m^3, then find extension in the wire due to its own weight (Y = 2 × 10^8 N/m^2)

Sol. $\Delta\ell = \dfrac{\rho g \ell^2}{2Y} = \dfrac{1 \times 10 \times 64}{2 \times 2 \times 10^8} = 16 \times 10^{-7}\text{m}$

Ex. A graph is plotted between load v/s elongation for 2 wires P and Q which are made of same material and have same cross-sectional area. Find ratio of initial length of wire P and Q.



Sol. $\Delta\ell = \dfrac{F\ell}{AY}$

Rack your Brain

A solid sphere of radius r made of a soft material of bulk modulus K is surrounded by a liquid in a cylindrical container. A massless piston of area a floats on the surface of the liquid, covering entire cross-section of cylindrical container. When a mass m is placed on the surface of the piston to:

(1) $\dfrac{mg}{3Ka}$ (2) $\dfrac{mg}{Ka}$

(3) $\dfrac{Ka}{mg}$ (4) $\dfrac{Ka}{3mg}$

$$\frac{F}{\Delta\ell} = \frac{AY}{\ell} = \tan\theta \quad \text{(slope)}$$

$$\therefore \quad \frac{\tan\theta_1}{\tan\theta_2} = \frac{\left(\frac{AY}{\ell_1}\right)}{\left(\frac{AY}{\ell_2}\right)} = \left(\frac{\ell_2}{\ell_1}\right)$$

$$\therefore \quad \frac{\ell_2}{\ell_1} = \frac{\tan 53^\circ}{\tan 37^\circ} = \frac{\frac{4}{3}}{\frac{3}{4}} = \frac{16}{9}$$

Ex. When a mass of 4 kg is hanged from a wire, then the length of wire becomes 10m and when 4 kg and 6 kg both are hanged from the same wire, then its length becomes 13m. Find out the natural length of the wire.

Sol. $F \propto \Delta L$

$$\Rightarrow \quad \frac{F_1}{F_2} = \frac{\Delta L_1}{\Delta L_2}$$

$$\Rightarrow \quad \frac{40}{100} = \frac{10 - L}{13 - L}$$

$$\Rightarrow \quad L = 8 \text{ m}$$

Bulk's modulus of elasticity 'K' or 'B':
Within the elastic limit, the ratio of the volume stress (i.e., change in pressure) to the volume strain is called bulk's modulus of elasticity.

$$K \text{ or } B = \frac{\text{Volume stress}}{\text{Volume strain}} = \frac{\frac{F}{A}}{\frac{-\Delta V}{V}} = \frac{\Delta P}{\frac{-\Delta V}{V}}$$

Here the minus sign indicates a decrease in the the volume with an increase in stress and vice-versa.
Unit of K: N/m^2 or pascal.

Note: B is defined for solids, liquids & gases.
Compressibility 'C': The reciprocal of

Definitions

The reciprocal of bulk's modulus of elasticity is defined as compressibility.

$$C = \frac{1}{K}$$

SI unit of C: m^2/N or $pascal^{-1}$.

bulk's modulus of elasticity is defined as compressibility.

$$C = \frac{1}{K}$$

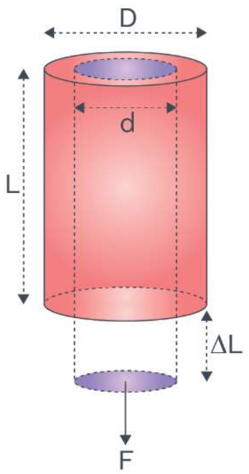
SI unit of C: m^2/N or $pascal^{-1}$.

Modulus of Rigidity 'η':
Within the elastic limit, the ratio of shearing stress to shearing strain is called modulus of rigidity of a material.

$$\eta = \frac{\text{shearing stress}}{\text{shearing strain}} = \left(\frac{\frac{F_{tangential}}{A}}{\phi}\right) = \frac{F_{tangential}}{A\phi}$$

Note: Angle of shear 'ϕ' is always taken in radians. It is only defined for solids, for liquids and gases $\eta = 0$.

Poisson's Ratio (σ):
Within elastic limit, the ratio of lateral strain to the longitudinal strain is called poisson's ratio.



$$\sigma = \frac{\text{lateral strain}}{\text{longitudinal strain}} = \frac{\beta}{\alpha}$$

$$\beta = \frac{-\Delta D}{D} = \frac{d - D}{D} \text{ and } \alpha = \frac{\Delta L}{L}$$

$-1 \le \sigma \le 0.5$ (theoretical limit)

Definitions

Within elastic limit, the ratio of lateral strain to the longitudinal strain is called poisson's ratio.

$$\sigma = \frac{\text{lateral strain}}{\text{longitudinal strain}} = \frac{\beta}{\alpha}$$

Concept Reminder

Relation between Y, K, η and σ: (To be remembered)

$$Y = 3K(1 - 2\sigma)$$
$$Y = 2\eta(1 + \sigma)$$
$$\frac{9}{Y} = \frac{3}{\eta} + \frac{1}{K}$$

$\sigma \approx 0.2$ to 0.4 (experimental limit)

Relation between Y, K, η and σ: (To be remembered)

$$Y = 3K(1 - 2\sigma)$$

$$Y = 2\eta(1 + \sigma)$$

$$\frac{9}{Y} = \frac{3}{\eta} + \frac{1}{K}$$

Interatomic Force Constant

$$k \text{ or } k_a = Y.r_0$$

Y = Young's modulus; r_0 = interatomic distance under normal circumstances.

Ex. Find the increase in the pressure which required to decrease the 100 litres volume of a liquid by 0.002% in container (Bulk modulus of the liquid = 2100 MPa)

Sol. $B = \dfrac{\Delta P}{-\dfrac{\Delta V}{V}}$

$$\Delta P = B\left(\frac{-\Delta V}{V}\right)$$

$$\Delta P = (2100 \times 10^6)\left(\frac{0.002}{100}\right)$$

$$\Delta P = 42 \times 10^3 \text{Pa} = 42 \text{ kPa}$$

Ex. The bulk modulus of a spherical object is 'K'. If it is subjected to uniform pressure 'P'. Then find:

(1) Fractional decrease in its volume

(2) Fractional decrease in its radius

(3) Fractional change in its density

Sol. (1) Bulk modulus

$$K = \frac{\Delta P}{-\dfrac{\Delta V}{V}}$$

$$-\frac{\Delta V}{V} = -\frac{\Delta P}{K}$$

$$-\frac{\Delta V}{V} = -\frac{P}{K}$$

(2) For spherical body

$$V = \frac{4}{3}\pi r^3$$

$\Rightarrow \quad \frac{\Delta V}{V} = 3\frac{\Delta r}{r} \Rightarrow -\frac{\Delta r}{r} = \frac{P}{3K}$

(3) $\quad \rho = \frac{M}{V} = MV^{-1}$

$$\frac{\Delta \rho}{\rho} = \frac{\Delta M}{M} - \frac{\Delta V}{V}$$

$\Rightarrow \quad \frac{\Delta \rho}{\rho} = \frac{P}{K} \left(\frac{\Delta M}{M} = 0\right)$

Ex. The bulk modulus of rubber is 9 × 10^8 N/m^2. To what depth a rubber ball be taken in a lake so that its volume is decreased by 0.1%?

Sol. $B = \frac{\Delta P}{-\Delta V / V}$

$$\Delta P = B\left(\frac{-\Delta V}{V}\right)$$

$$\rho g h = (9 \times 10^8)\left(\frac{0.1}{100}\right)$$

$$h = \frac{(9 \times 10^8)(10^{-3})}{(10^3)(10)} = 90 \text{ m}$$

Ex. The approximate depth of an ocean is 3000m. The compressibility of water is 5 × 10^{-10} Pa^{-1} and density of water is 10^3 kg/m^3. What fractional compression of water will be obtained at the bottom of the ocean.

Sol. $B = -\frac{\Delta P}{\Delta V / V} \Rightarrow -\frac{\Delta V}{V} = \Delta P \times \frac{1}{B}$

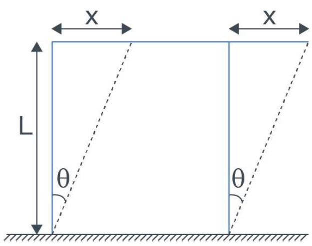
$= \rho g h \times C$

$= 10^3 \times 10 \times 3000 \times 5 \times 10^{-10}$

$= 1.5 \times 10^{-2}$

Ex. A rubber cube having side 5 cm has one side fixed while a tangential force equal to 1800 N is being applied to opposite face. Find the shearing strain & the lateral displacement of the strained face. The modulus of rigidity for rubber is 2.4 × 10^6 N/m^2.

Sol. Stress = $\frac{F}{A} = \eta \frac{x}{L}$



Strain = $\theta = \frac{F}{A\eta} = \frac{1800}{25 \times 10^{-4} \times 2.4 \times 10^{6}}$

$= \frac{180}{25 \times 24} = \frac{3}{10} = 0.3$ radian

As $\frac{x}{L} = 0.3$

$\Rightarrow \quad x = 0.3 \times 5 \times 10^{-2} = 1.5 \times 10^{-2} m = 1.5 \text{ cm}$

Concept Reminder

Work done in stretching a wire (Potential energy stored in a wire):

$$\text{E.P.E.} = \frac{1}{2} F \Delta \ell$$

$$\text{E.P.E.} = \frac{1}{2}(\text{stress})(\text{strain})(\text{volume})$$

$$\text{E.P.E.} = \frac{1}{2} Y(\text{strain})^2(\text{volume})$$

$$\text{E.P.E.} = \frac{1}{2}\frac{(\text{stress})^2}{Y}(\text{volume})$$

Ex. The increase in length of a wire on stretching is 0.025%. If its Poisson's ratio is 0.4, then the percentage decrease in diameter.

Sol. $\sigma = \frac{-\frac{\Delta D}{D}}{\frac{\Delta L}{L}}$

$-\frac{\Delta D}{D} = \sigma \times \frac{\Delta L}{L}$

= 0.4 × (0.025%) = 0.01%

Ex. Bulk modulus of a metal is 10^{10} N/m^2 and poisson's ratio 0.20. If the average distance between the molecules is 3 Å, then find inter atomic force constant.

Sol. Y = 3K(1 – 2σ) = 3 × 10^{10} (1 – 2 × 0.2)

Force constant k = Y.r_0

= 1.8 × 10^{10} × 3 × 10^{-10}

= 5.4 N/m

Ex. The bulk modulus of elasticity of a material is 2 × 10^8 N/m^2 and shear modulus of elasticity is 6 × 10^8 N/m^2.Then find out the poisson's ratio for this material.

Sol. Y = 3K (1 – 2σ) ...(i)

Y = 2η (1 + σ) ...(ii)

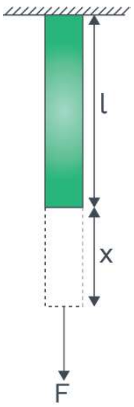
From (1) and (2)

$3K(1-2\sigma) = 2\eta(1+\sigma)$

$3 \times 2 \times 10^8 (1-2\sigma) = 2 \times 6 \times 10^8 (1+\sigma)$

$\Rightarrow \quad \sigma = -0.25$

Work done in stretching a wire
(Potential energy stored in a wire):



$$F = \frac{YA}{\ell} \times x$$

$$dW = Fdx$$

$$\int_0^W dW = \int_0^{\Delta\ell} \frac{YA}{\ell} x dx$$

$$W = \frac{YA}{\ell} \times \frac{(\Delta\ell)^2}{2}$$

$$W.D. = P.E. = \frac{1}{2}\frac{YA}{\ell}(\Delta\ell)^2$$

$$Y = \frac{F\ell}{A\Delta\ell} \Rightarrow \frac{YA}{\ell} = \frac{F}{\Delta\ell} \quad \text{...(i)}$$

$$E.P.E. = \frac{1}{2}\frac{F}{\Delta\ell}(\Delta\ell)^2$$

$$\boxed{E.P.E. = \frac{1}{2}F\Delta\ell} \quad \text{...(ii)}$$

$$E.P.E. = \frac{1}{2}\left(\frac{F}{A}\right)\left(\frac{\Delta\ell}{\ell}\right)(A\ell)$$

$$\boxed{E.P.E. = \frac{1}{2}(\text{stress})(\text{strain})(\text{volume})} \quad \text{...(iii)}$$

Concept Reminder

Potential energy density:

- $\frac{1}{2} \times \text{stress} \times \text{strain}$
- $\frac{1}{2} \times Y \times (\text{strain})^2$
- $\frac{1}{2} \times \frac{(\text{stress})^2}{Y}$

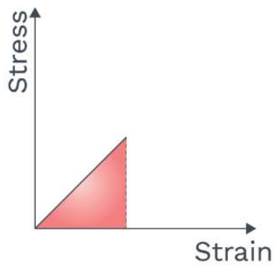
$$\boxed{\text{E.P.E.} = \frac{1}{2}Y(\text{strain})^2(\text{volume})} \quad \text{...(iv)}$$

$$\boxed{\text{E.P.E.} = \frac{1}{2}\frac{(\text{stress})^2}{Y}(\text{volume})} \quad \text{...(v)}$$

Potential energy density: Potential energy per unit volume stored in the wire.

$$\text{P.E.D.} = \frac{\text{P.E.}}{\text{Volume}}$$

$$\text{P.E.D.} \Rightarrow \begin{bmatrix} \frac{1}{2}\times \text{stress}\times\text{strain} & \text{...(i)} \\ \frac{1}{2}\times Y\times(\text{strain})^2 & \text{...(ii)} \\ \frac{1}{2}\times\frac{(\text{stress})^2}{Y} & \text{...(iii)} \end{bmatrix}$$



Note: Slope = tan θ = E = coefficient of elasticity

$$\text{Area} = \frac{1}{2}\times\text{stress}\times\text{strain} = \text{PE density}$$

Ex. Length of a wire increased by 0.04% then find out potential energy density stored in the wire. [$Y = 2 \times 10^9$ N/m^2]

Sol. Given,

$$\frac{\Delta\ell}{\ell}\times 100 = 0.04$$

$$\frac{\Delta\ell}{\ell} = 4\times10^{-4}$$

$$\text{P.E.D.} = \frac{1}{2}(Y)(\text{strain})^2$$

Rack your Brain

If S is stress and Y is Young's modulus of material of a wire, the energy stored in the wire per unit volume is:

(1) $2S^2Y$ (2) $\frac{S^2}{2Y}$

(3) $\frac{2Y}{S^2}$ (4) $\frac{S}{2Y}$

$$= \frac{1}{2}(2 \times 10^9)(4 \times 10^{-4})^2 = 160 \text{ J/m}^3$$

Ex. Two wires P and Q are made of same material and are of same length. The ratio of radius of both wires is 1 : 2. When they are stretched by same force then find out the ratio of energy stored per unit volume in both wires P and Q.

Sol. $$\text{P.E.D.} = \frac{1}{2} \times \frac{(\text{stress})^2}{Y} = \frac{\left(\frac{F}{A}\right)^2}{2Y} = \frac{F^2}{2\pi^2 r^4 Y}$$

$\because$ For both wires P and Q

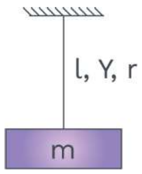
$Y \rightarrow$ Same

$F \rightarrow$ Same

$$\Rightarrow \quad \text{P.E.D.} \propto \frac{1}{r^4}$$

$$\Rightarrow \quad \frac{\text{P.E.D}_P}{\text{P.E.D}_Q} = \left(\frac{r_Q}{r_P}\right)^4 = \left(\frac{2}{1}\right)^4 = \frac{16}{1}$$

Ex. Find the elastic potential energy stored in wire.

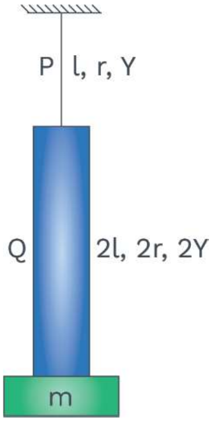


Sol. $$\text{E.P.E.} = \frac{1}{2} \times \frac{(\text{stress})^2}{Y} \times \text{volume}$$

$$= \frac{1}{2} \times \frac{\left(\frac{mg}{A}\right)^2}{Y} \times A\ell = \frac{1}{2} \times \frac{(mg)^2 \times \ell}{A \times Y}$$

$$\Rightarrow \quad \text{E.P.E.} = \frac{m^2 g^2 \ell}{2\pi r^2 Y}$$

Ex. A block of mass 'm' is hanged with two massless wires P and Q as shown in figure. Then find ratio of elastic potential energy stored in wires P and Q.



Sol. E.P.E. $= \frac{1}{2} \times \frac{(\text{stress})^2}{Y} \times \text{volume} = \frac{m^2 g^2 \ell}{2\pi r^2 Y}$

$\Rightarrow$ E.P.E. $\propto \frac{\ell}{r^2 Y}$

$\Rightarrow$ $\frac{EPE_P}{EPE_Q} = \frac{\ell_P}{\ell_Q} \frac{r_Q^2}{r_P^2} \times \frac{Y_Q}{Y_P} = \frac{4}{1}$

Ex. A steel wire having 4.0 m in the length is stretched through 2.00 mm. The area of cross-sectional of the wire is 2.0 mm². If Young's modulus of the steel is 2.0 × 10^{11} N/m² determine:

(i) The energy density of the wire

(ii) The elastic potential energy stored in the wire.

Sol. (i) The energy density of stretched wire

$= \frac{1}{2} \times \text{stress} \times \text{strain} = \frac{1}{2} \times Y \times (\text{strain})^2$

$= \frac{1}{2} \times 2.0 \times 10^{11} \times \left[\frac{2 \times 10^{-3}}{4}\right]^2$

$= 2.5 \times 10^4$ J/m³

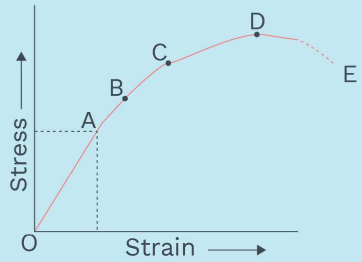
(ii) Elastic potential energy = energy density × volume

$= 2.5 \times 10^4 \times (2.0 \times 10^{-6}) \times 4.0$ J

$= 20 \times 10^{-2} = 0.20$ J

Concept Reminder

VARIATION OF STRAIN WITH STRESS:



OA → Limit of proportionality

OB → Elastic limit

C → Yield point

CD → Plastic behaviour

D → Ultimate point

DE → Fracture

Ex. The length of a wire is being increased by 0.04%. Then find out the energy stored in its per unit volume [Y = 2× 10^9 N/m^2].

Sol. $PED = \frac{1}{2} \times Y \times (strain)^2$

$= \frac{1}{2} \times 2 \times 10^9 \times (4 \times 10^{-4})^2 = 160 \text{ J/m}^3$

Ex. A wire of length 4m is hanged vertically from ceiling if it is stretched due to its own weight, then find out the energy stored per unit volume. [Y = 2 × 10^7 N/m^2 and r = 10^3 kg/m^3]

Sol. $PED = \frac{1}{2} \times Y \times (strain)^2$

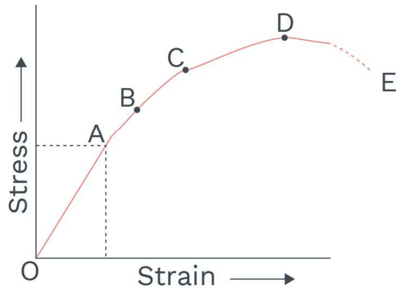
$= \frac{1}{2} \times Y \times \left(\frac{\rho g \ell}{2Y}\right)^2$ $\qquad \because \Delta\ell = \frac{\rho g \ell^2}{2Y}$

$\Rightarrow \quad strain = \frac{\Delta\ell}{\ell} = \frac{\rho g \ell}{2Y}$

$PED = \frac{\rho^2 g^2 \ell^2}{8Y} = 10 \text{ J / m}^3$

VARIATION OF STRAIN WITH STRESS

When a wire is being stretched by a load, it is seen that for small value of load, the extension produced in the wire is proportional to the load. On removing the load, the wire returns back to its original length. The wire regains its original dimensions only when the load applied is less or its equal to a certain limit. This limit is called elastic limit. Thus, the elastic limit is the maximum stress on whose removal, the bodies regain their original dimensions. In shown figure, the type of behaviour is represented by OB portion of the graph. Till A the stress is proportional to strain and from A to B if deforming forces are removed then the wire comes to its original length but here stress is not proportional to strain.



OA → Limit of Proportionality
OB → Elastic limit
C → Yield Point
CD → Plastic behaviour
D → Ultimate point
DE → Fracture

As we go beyond the point B as shown above, then even for a very small increase in stress, the strain produced is very large. This type of behaviour is observed around point C and at this stage the wire begins to flow like a viscous fluid. The point C is called yield point. If the stress is then further increased, then the wire breaks off at a point D called the breaking point. The stress which is corresponding to this point is called breaking stress or tensile strength of the material of the wire. The material for which the plastic range CD is relatively high is called ductile material. These materials get permanently deformed before breaking. The materials for which the plastic range is relatively small are called brittle materials. These materials break as soon as the elastic limit is crossed.

Important points:

- Breaking stress = Breaking force/area of the cross section.
- The breaking stress is constant for a material.
- Breaking force depends upon the area of the section of the wire of a given material.
- The working stress is always kept lower than that of a breaking stress so that the safety factor = breaking stress/working stress may have a large value.
- Breaking strain = elongation or compression/ original dimension.
- Breaking strain is constant for material.

Elastic after effect:

We know that some material bodies take some time in order to regain their original configuration when the deforming force is being removed. The delay in the regaining of original configuration by the bodies on the removal of deforming force is called elastic after the effect. The elastic after the effect is negligibly small for quartz fibre and phosphor bronze. For this reason, the suspensions are made up of quartz and phosphor- bronze are used in galvanometers and electrometers.

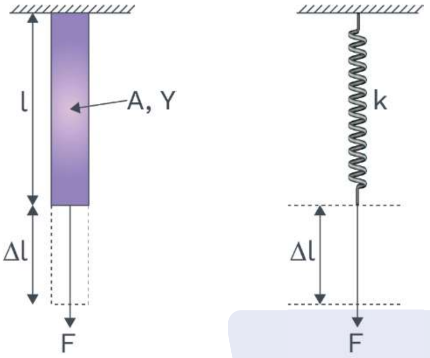
For glass fibre elastic after effect is very large. It takes hours for glass fibre to return to its original state on removal of deforming force.

Elastic Fatigue

The loss of the strength of the material due to repeated strains on the material is called elastic fatigue. That is why the bridges are declared unsafe after a long time of their use.

Analogy of Rod as a spring:

$Y = \frac{\text{stress}}{\text{strain}} \Rightarrow Y = \frac{F\ell}{A\Delta\ell}$

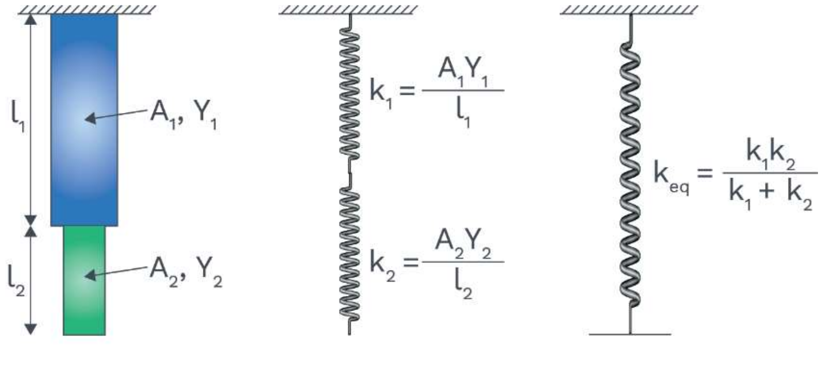


or $F = \frac{AY}{\ell}\Delta\ell$

$\frac{AY}{\ell}$ = constant, depends on the type of material and geometry of rod.

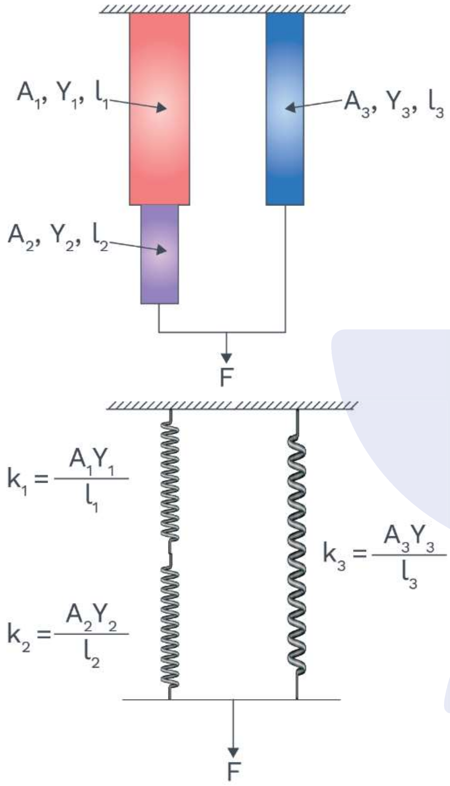
$F = k\Delta l$

Where $k = \frac{AY}{\ell}$ = equivalent spring constant.



for the system of the rods shown in figure (a), the replaced spring system is shown in figure (b) two spring in series. Figure (c) represents equivalent spring system.

Given figure represents another combination of rods and their replaced spring system.



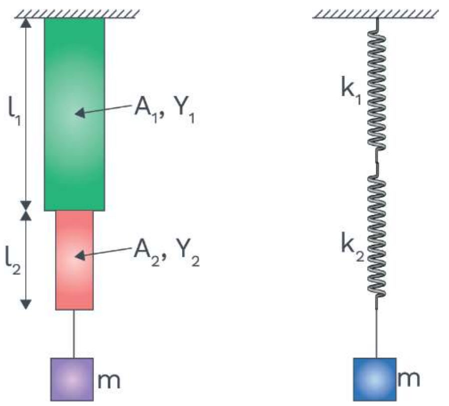
Concept Reminder

Strain energy stored in equivalent spring

$$U = \frac{1}{2}kx^2$$

$$U = \frac{1}{2}\frac{F^2\ell}{AY}$$

Ex. A mass m is attached with the rods as shown in the figure. This mass is slightly stretched & released whether the motion of mass is S.H.M., if yes then find out the time period.



Sol. $k_{eq} = \frac{k_1k_2}{k_1 + k_2}$

$$T = 2\pi\sqrt{\frac{m}{k_{eq}}} = 2\pi\sqrt{\frac{m(k_1 + k_2)}{k_1k_2}}$$

where $k_1 = \frac{A_1Y_1}{\ell_1}$ and $k_2 = \frac{A_2Y_2}{\ell_2}$

> **Concept Reminder**
>
> **Energy density:**
>
> - $U = \frac{1}{2}\frac{(\text{stress})^2}{Y}$
> - $U = \frac{1}{2}\text{stress} \times \text{strain}$
> - $U = \frac{1}{2}Y(\text{strain})^2$

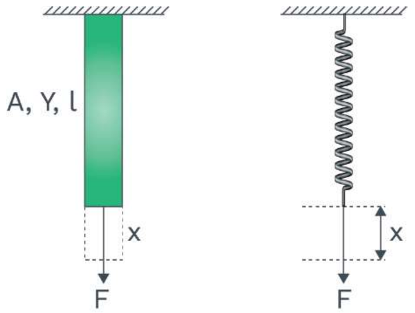
ELASTIC POTENTIAL ENERGY STORED IN THE STRETCHED WIRE OR IN A ROD:

Strain energy stored in the equivalent spring

$$U = \frac{1}{2}kx^2$$

where $x = \frac{F\ell}{AY}$, $k = \frac{AY}{\ell}$

$$U = \frac{1}{2}\frac{AY}{\ell}\frac{F^2\ell^2}{A^2Y^2} = \frac{1}{2}\frac{F^2\ell}{AY}$$



Equation can be re-arranged,

$$U = \frac{1}{2}\frac{F^2}{A^2} \times \frac{\ell A}{Y}$$

[ℓA = volume of rod, F/A = stress]

$$U = \frac{1}{2}\frac{(\text{stress})^2}{Y} \times \text{volume}$$

Again,

$$U = \frac{1}{2}\frac{F}{A} \times \frac{F}{AY} \times A\ell \qquad \left[\text{strain} = \frac{F}{AY}\right]$$

$$U = \frac{1}{2}\text{stress} \times \text{strain} \times \text{volume}$$

Again,

$$U = \frac{1}{2}\frac{F^2}{A^2Y^2}A\ell Y$$

$$\Rightarrow \quad U = \frac{1}{2}Y(\text{strain})^2 \times \text{volume}$$

Strain energy density

$$= \frac{\text{strain energy}}{\text{volume}} = \frac{1}{2}\frac{(\text{stress})^2}{Y}$$

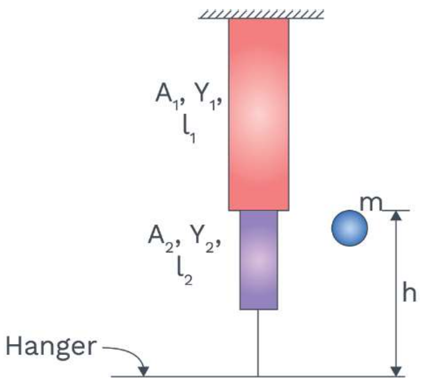
$$= \frac{1}{2}Y(\text{strain})^2 = \frac{1}{2}\text{stress} \times \text{strain}$$

Rack your Brain

The work done per unit volume to stretch the length of area of cross-section 2 mm^2 by 2% will be: ($Y = 8 \times 10^{10}$ N/m^2)

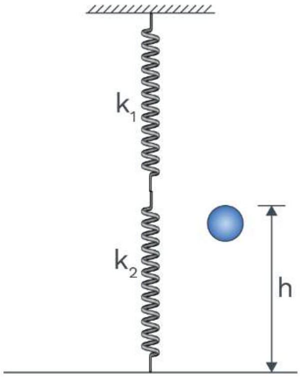
(1) 40 MJ/m^3
(2) 16 MJ/m^3
(3) 64 MJ/m^3
(4) 32 MJ/m^3

Ex. Hanger is massless. A ball of mass m drops from a height h, which sticks to the hanger after striking. Neglect over turning, find out the maximum extension in the rod. (Assuming rod is massless and hanger are massless).



Sol. Applying energy conservation

$$mg\,(h + x) = \frac{1}{2}\frac{k_1k_2}{k_1 + k_2}x^2$$



where, $k_1 = \dfrac{A_1Y_1}{\ell_1}$, $k_2 = \dfrac{A_2Y_2}{\ell_2}$

and $k_{eq} = \dfrac{A_1A_2Y_1Y_2}{A_1Y_1\ell_2 + A_2Y_2\ell_1}$

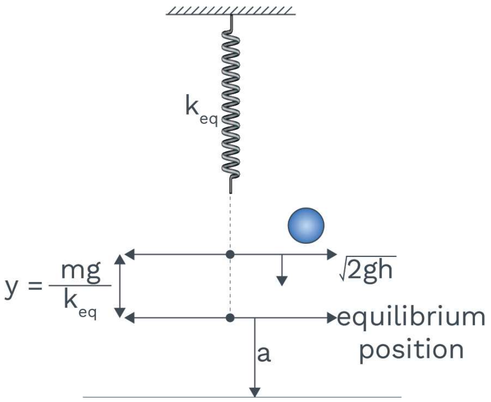
$$k_{eq}x^2 - 2mgx - 2mgh = 0$$

$$x = \frac{2mg \pm \sqrt{4m^2g^2 + 8\,mghk_{eq}}}{2\,k_{eq}}$$

$$x_{max} = \frac{mg}{k_{eq}} + \sqrt{\frac{m^2g^2}{k_{eq}^2} + \frac{2\,mgh}{k_{eq}}}$$

Alternate by S.H.M.

$$\omega = \sqrt{\frac{k_{eq}}{m}},\quad v = \omega\sqrt{a^2 - y^2}$$



Rack your Brain

A uniform cubical block is subjected to volumetric compression, which decreases its each side by 2%. The Bulk strain produced in it is:

(1) 0.03 (2) 0.02

(3) 0.06 (4) 0.12

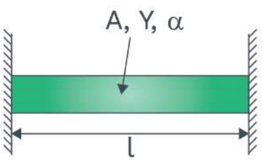
$$\sqrt{2gh} = \sqrt{\frac{k_{eq}}{m}}\sqrt{a^2 - y^2}$$

$$\Rightarrow \quad \sqrt{\frac{2mgh}{k_{eq}} + \frac{m^2g^2}{k_{eq}^2}} = a$$

maximum extension,

$$a + y = \frac{mg}{k_{eq}} + \sqrt{\frac{m^2g^2}{k_{eq}} + \frac{2mgh}{k_{eq}}}$$

Thermal Stress:



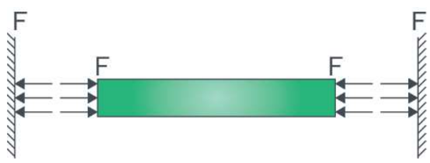
If temperature of rod is increased by ΔT, then change in length

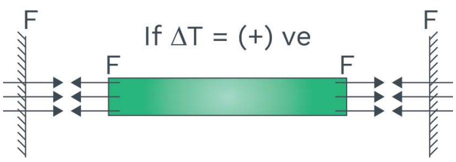
$$\Delta\ell = \ell\,\alpha\,\Delta T$$

$$\text{strain} = \frac{\Delta\ell}{\ell} = \alpha\Delta T$$

But due to the rigid support, there is no strain. Supports provide force on stresses in order to keep the length of rod same.

$$Y = \frac{\text{stress}}{\text{strain}}$$





thermal stress = Y strain = YαΔT.

If ΔT = (positive)

Concept Reminder

Four important Relations between Y, B, η and σ

(i) $\eta = \frac{Y}{2(1+\sigma)}$

(ii) $\frac{9}{Y} = \frac{3}{\eta} + \frac{1}{B}$

(iii) $\sigma = \frac{3B - 2\eta}{6B + 2\eta}$

(iv) $B = \frac{Y}{3(1-2\sigma)}$

$$\frac{F}{A} = Y\alpha\Delta T \quad \Rightarrow \quad F = AY\alpha\Delta T$$

APPLICATIONS OF ELASTICITY

Some of the important applications of elasticity of the materials are discussed below:

- The material used in the bridges lose its elastic strength with time bridges are declared unsafe after long use.
- To determine the maximum height of a mountain:

The pressure at the base of mountain = $h\rho g$ = stress. The elastic limit of the typical rock is 3×10^8 Nm^{-2}.

The stress must be kept less than the elastic limits, otherwise the rock will begin to flow.

$$h < \frac{3 \times 10^8}{\rho g} < \frac{3 \times 10^8}{3 \times 10^3 \times 10} < 10^4 m$$

($\because$ $\rho = 3 \times 10^3$ kg m^{-3}; g = 10 ms^{-2})

or h = 10 km

It may be noted that height of Mount Everest is nearly 9 km.

Torsion constant of a wire:

$$C = \frac{\pi\eta r^4}{2\ell}$$

Where η is modulus of rigidity r and ℓ are radius and length of wire respectively.

(a) Toque required for twisting by angle θ, $\tau = C\theta$.

(b) Work done in twisting by angle θ,

$$W = \frac{1}{2}C\theta^2$$

Concept Reminder

The stress must be less than the elastic limits, otherwise the rock begins to flow.

$$h < \frac{3 \times 10^8}{\rho g} < \frac{3 \times 10^8}{3 \times 10^3 \times 10} < 10^4 m$$

($\because$ $\rho = 3 \times 10^3$ kg m^{-3};
g = 10 ms^{-2})

or h = 10 km

It may be noted that the height of Mount Everest is nearly 9 km.

Concept Reminder

Torsion constant of a wire:

$$C = \frac{\pi\eta r^4}{2\ell}$$

Where η is modulus of rigidity r and ℓ are radius and length of wire respectively.

EXAMPLES

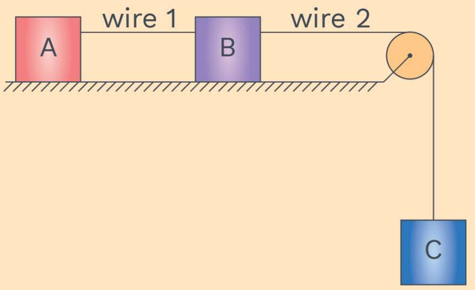
Q1 **If a compressive force of 3.0×10^4 N is exerted on the end of 20 cm long bone of cross-sectional area $3.6\ cm^2$**
(a) will the bone break &
(b) if not, then how much will it shorten ?
Given, the compressive strength of bone = $7.7 \times 10^8\ Nm^{-2}$ and Young's modulus of bone = $1.5 \times 10^{10}\ Nm^{-2}$

Sol: (a) $\sigma = \frac{F}{A} = \frac{3 \times 10^4}{3.6 \times 10^{-4}} = \frac{30}{36} \times 10^8 = \frac{5}{6} \times 10^8 < 7.7 \times 10^8\ N/m^2$

Hence it will not break.

(b) $\Delta\ell = \frac{FL}{AY} = \frac{3 \times 10^4 \times 20 \times 10^{-2}}{3.6 \times 10^{-4} \times 1.5 \times 10^{10}} = \frac{40 \times 10^{-4}}{3.6} = \frac{10}{9} \times 10^{-3}\ m$

Q2 **Three blocks A, B and C each having mass 4 kg are attached as shown in the figure. Both the wires have equal cross sectional area $5 \times 10^{-7}\ m^2$. The surface is smooth. Find the longitudinal strain in each of the wire if Young modulus of both the wires is $2 \times 10^{11}\ N/m^2$**
(take g = 10 m/s²)



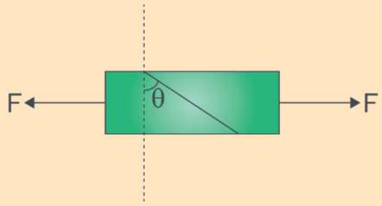
Sol: $a = \frac{40}{12} = \frac{10}{3}\ m/s^2$

$T_1 = \frac{40}{3} N$ and $T_2 = \frac{80}{3} N$

Strain in wire 1 $= \frac{40}{3 \times 5 \times 10^{-7} \times 2 \times 10^{11}} = \frac{4}{3} \times 10^{-4}$

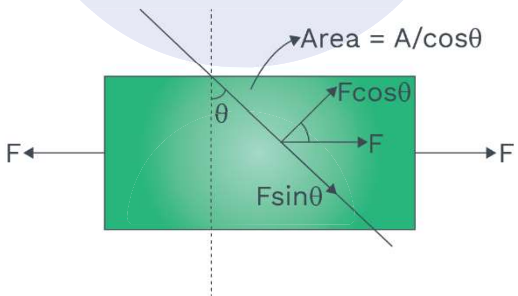
Strain in wire 2 $= \frac{80}{3 \times 5 \times 10^{-7} \times 2 \times 10^{11}} = \frac{8}{3} \times 10^{-4}$.

Q3 **A bar of cross-section A is subjected to an equal & opposite tensile forces F at its ends. Consider a plane through the bar which is making an angle θ with a plane at right angles to the bar.**



(a) What is the tensile stress at the plane in terms of F, A and θ ?
(b) What is the shearing stress at this plane, in terms of F, A and θ ?
(c) For what value of 'θ' is the tensile stress be maximum ?
(d) For what value of 'θ' is the shearing stress be maximum ?

Sol:



(a) tensile stress = $\dfrac{F\cos\theta}{A/\cos\theta} = \dfrac{F\cos^2\theta}{A}$

(b) shearing stress = $\dfrac{F\sin\theta}{A/\cos\theta} = \dfrac{F}{A}\sin\theta\cos\theta$

(c) for maximum tensile stress
$\theta = 0°$

(d) for maximum shearing stress
$\theta = 45°$

Q4 **Calculate the increase in energy of a brass bar of length 0.2 m and cross-sectional area 1 cm² when compressed with a load of 5 kg weight along its length.**
(Young's modulus of brass = 1.0 × 10¹¹ N/m² and g = 9.8 m/s²).

Sol:

$$K = \frac{AY}{\ell} = \frac{\left(1\times 10^{-4}\times 1\times 10^{11}\right)}{0.2} = 5\times 10^{7}$$

$$U = \frac{F^2}{2K} = \frac{\left(5\times 9.8\right)^2}{2\times 5\times 10^{7}} = \frac{49\times 49}{10^{8}} = 2.40\times 10^{-5}\,J.$$

Q5 **When the load on a wire is increased slowly from 2 kg wt. to 4 kg wt., then the elongation increases from 0.6 mm to 1.00 mm. How much work is done during the extension of the wire? [g = 9.8 m/s²]**

Sol:

$$W = \frac{1}{2}\left(F_2x_2 - F_1x_1\right)$$

$$= \frac{1}{2}\left(4\times 9.8\times 10^{-3} - 2\times 9.8\times 0.6\times 10^{-3}\right)$$

$$= 2\times 9.8\times 10^{-3}\times 0.7$$

$$= 13.72\times 10^{-3}\,J.$$

Q6 **A spherical ball of radius 3.0 × 10⁻⁴ m and density 10⁴ kg/m³ falls freely under gravity through a distance h before entering a tank of water. If after entering the water the velocity of the ball does not change, find h. Viscosity of water is 9.8 × 10⁻⁶ N-s/m². [g = 9.8 m/s²]**

Sol:

$$v = \frac{2}{9\eta}r^2.\left(\rho_0 - \rho_w\right)g$$

$$= 180\ m/sec.$$

$$h = \frac{v^2}{2g} = \frac{32400}{2\times 9.8} = \frac{81}{49}\times 10^{3}\,m.$$

Q7

During the Searle's experiment, zero of the Vernier scale lies between 3.20 × 10⁻² m and 3.25 × 10⁻² m of the main scale. The 20th division of the Vernier scale exactly coincides with one of the division on the main scale. When the additional load of 2 kg is being applied to the wire, the zero of the Vernier scale still lies between 3.20 × 10⁻² m and 3.25 × 10⁻² m of the main scale but now the 45th division of the Vernier scale coincides with one of the main scale divisions. The length of the thin metallic wire is 2 m & its cross-sectional area is 8 × 10⁻⁷ m². The least count of Vernier scale is 1.0 × 10⁻⁵ m. The maximum percentage error in Young's modulus of the wire is

Sol: **Observation - 1**

Let weight used is W_1, extension ℓ_1

$$Y = \frac{W_1 / A}{\ell_1 / L} \Rightarrow W_1 = \frac{YA\ell_1}{L} \Rightarrow \ell_1 = 3.2 \times 10^{-2} + 20 \times 10^{-5}$$

Observation - 2

Let weight used is W_2 extension ℓ_2

$$Y = \frac{W_2 / A}{\ell_2 / L} \Rightarrow W_1 = \frac{YA\ell_2}{L} \Rightarrow \ell_1 = 3.2 \times 10^{-2} + 45 \times 10^{-5}$$

$$W_2 - W_1 = \frac{YA}{L}(\ell_2 - \ell_1) \Rightarrow Y = \frac{(W_2 - W_1)L}{A(\ell_2 - \ell_1)}$$

$$\left(\frac{\Delta Y}{Y}\right)_{max} = \frac{\Delta\ell_2 + \Delta\ell_1}{\ell_2 - \ell_1} = \frac{2 \times 10^{-5}}{25 \times 10^{-5}}$$

$$\left(\frac{\Delta Y}{Y}\right)_{max} \times 100\% = \frac{2}{25} \times 100\% = 8\% .$$

Q8 **Two exactly similar wires of steel & copper are stretched by equal force. If the difference in the elongation both the wires is 0.5 cm, find out that by how much each wire has elongated. (Given Young's modulus for the steel = 2×10^{12} dyne cm^{-2} and for copper 12×10^{11} dyne cm^{-2}).**

Sol:

$$\Delta\ell_S = \frac{F\ell}{Y_S A}$$

$$\Delta\ell_C = \frac{F\ell}{Y_C A}$$

$$\Delta\ell_S - \Delta\ell_C = 0.5$$

$$\frac{F\ell}{A}\left(\frac{1}{Y_S} - \frac{1}{Y_C}\right) = 0.5$$

$$\frac{F\ell}{A} = \frac{Y_C Y_S \times 0.5}{(Y_C - Y_S)}$$

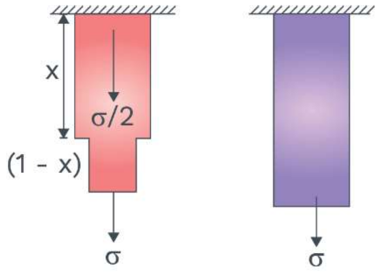
$$\frac{F\ell}{A} = 15 \times 10^{11}$$

$$\Delta\ell_S = \left(\frac{F\ell}{A}\right)\frac{1}{Y_S} = \frac{15 \times 10^{11}}{2 \times 10^{12}} = 0.75 \text{ cm}$$

$$\Delta\ell_C = \left(\frac{F\ell}{A}\right) \times \frac{1}{Y_C} = \frac{15 \times 10^{11}}{12 \times 10^{11}} = 1.25 \text{ cm}.$$

Q9 **A rod 1 m long is 10 cm^2 in area for a portion of its length & 5 cm^2 in area for the remaining. The strain energy of this stepped bar is 40 % of that a bar of 10 cm^2 in area & 1 m long under the same maximum stress. What is the length of the portion which is 10 cm^2 in area.**

Sol:



$$\frac{\left(\frac{\sigma}{2}\right)^2}{2Y}10x + \frac{\sigma^2}{2Y}(1-x)5 = \frac{40}{100}\left[\frac{\sigma^2}{2Y}10 \times 1\right]$$

2.5x + 5 − 5x = 4

2.5x = 1

x = 40 cm.

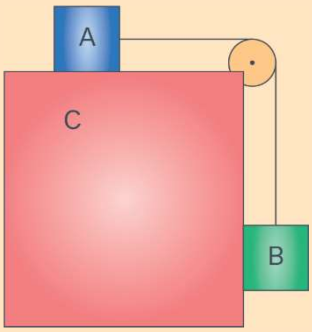
Q10 **The cross-section of a bar is given by $\left[1+\frac{x^2}{100}\right]$ cm², where 'x' is the distance from one end. Find the extension under a load of '20 k N' on a length of 10 cm.**
$Y = 2 \times 10^5$ N/mm².

Sol:

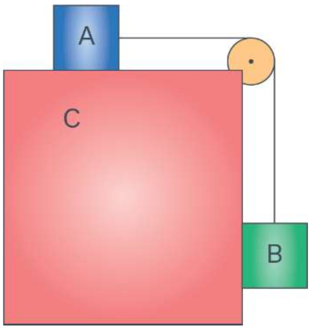
$$\Delta\ell = \int \frac{Fdx}{AY}$$

$$= \int_0^{10} \frac{20 \times 10^3 dx}{\left(1+\frac{x^2}{100}\right) \times 2 \times 10^7} = 0.008 \text{ cm.}$$

Q11 **Two block A & B are connected to each other by a string, passing over a frictionless pulley as shown in the figure. The block A slides over the horizontal top surface of a stationary block C & block B slide along the vertical side of C both with uniform speed. The coefficient of the friction between the surface of blocks is 0.2. String stiffness is 1960 N/m. If the mass of block B is 2 kg. Calculate the mass of block A and the energy stored in the string.**



Sol:



$Kx = \mu M_A g$ ...(i)

$kx = M_B g$ ...(ii)

$$\Rightarrow \quad x = \frac{M_B g}{K} = \frac{1}{100}$$

From equation (i) & (ii)

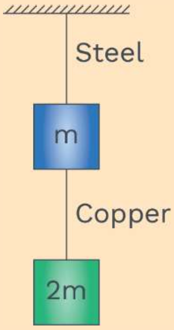
$0.2\, M_A = M_B$

$M_A = 10$ kg

$$E = \frac{1}{2}Kx^2 = \frac{1}{2} \times 1960 \times \left(\frac{1}{100}\right)^2$$

$E = 9.8 \times 10^{-2}$ J

Q12 **If the ratio of lengths, radii and Young's moduli of steel & brass wires in the figure are a, b and c respectively. Then the corresponding ratio of the increase in their lengths would be.**



Sol: $\frac{r_1}{r_2} = b$

$\frac{\ell_1}{\ell_2} = a$

$\frac{Y_1}{Y_2} = c$

$$\Delta\ell_1 = \frac{(3mg)\ell_1}{A_1Y_1}$$

$$\Delta\ell_2 = \frac{(2mg)\ell_2}{A_2Y_2}$$

$$\frac{\Delta\ell_1}{\Delta\ell_2} = \frac{3\ell_1}{2\ell_2A_1Y_1} \times A_2Y_2 = \frac{3}{2}\frac{a}{b^2c} = \frac{3a}{2b^2c}.$$

Q13 **Two rods identical in geometry but of different materials having co-efficient of thermal expansion α_1 and α_2 and Young's moduli Y_1 and Y_2 respectively are fixed between the two rigid massive walls. The rods are heated in such a way that they undergo the same increase in the temperature. There is no bending of the rods. If $\alpha_1 : \alpha_2 = 2 : 6$ the thermal stresses developed in two rods are equal provided $Y_1 : Y_2$ is equal to**

Sol: $\frac{\alpha_1}{\alpha_2} = \frac{2}{6}$

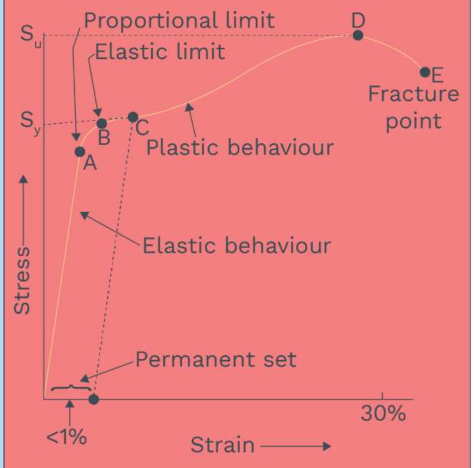
$\because \frac{F}{A} = Y\alpha\,\Delta\theta$ $\quad\quad\because$ $\Delta\theta$ is same for both

$$\frac{\frac{F_1}{A_1}}{\frac{F_2}{A_2}} = \frac{Y_1\alpha_1}{Y_2\alpha_2}$$

$$\frac{Y_1}{Y_2} = \frac{\alpha_2}{\alpha_1} = 3 : 1.$$

Mind Map

Stress-Strain Curve



Young's Modulus

$$Y = \frac{F_{\perp} l}{A \Delta l}$$

Shear Modulus

$$\eta = \frac{F_{\parallel}}{A\theta}$$

Bulk Modulus

$$B = \frac{-\Delta P}{\Delta V/V}$$

Hooke's Law

Stress ∝ Strain

Stress = E strain

E = Elastic modulus

Elasticity

Property of materials by virtue of which it regains its original confrigration after removal of deforming force

Stress

Restoring force per unit area i.e, $\frac{F}{A}$

Strain

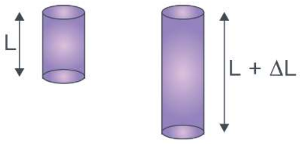
$\frac{\text{Change in configuration}}{\text{Original configuration}}$

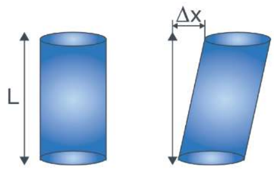
Types of Stress

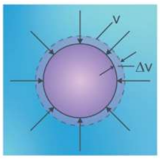
(i) Longitudinal stress = $\frac{F_{\perp}}{A}$

(ii) Shearing stress = $\frac{F_{\parallel}}{A}$

(iii) Volumetric stress = $\frac{F_v}{A}$







Types of Strain

(i) Longitudinal strain = $\frac{\Delta l}{l_o}$

(ii) Shearing strain = $\theta = \frac{\Delta x}{L}$

(iii) Volumetric strain = $\frac{\Delta V}{V_o}$