Vector

DISCLAIMER

Vector

PHYSICAL QUANTITIES:
Based on directions, there are mainly two types of physical quantities.

(1) Scalar Quantities:
A physical quantity that can be described completely by its magnitudes only and doesn't require a direction is known as a scalar quantity.
Example: Distance, mass, time, speed, density, volume etc.

(2) Vector Quantities:
A physical quantity that has magnitude and direction and obeys all the laws of vector algebra is called a vector quantity.
Examples: Displacement, velocity, acceleration, force etc.

KEY POINTS

- Scalar
- Vector

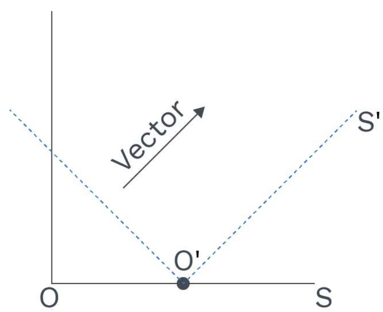
SCALAR QUANTITY	VECTOR QUANTITY
They are described by its magnitude only.	They are described by its magnitude and direction.
They can be changed by changing its magnitude only.	They can be changed by changing its • Magnitude • direction • Both magnitude and direction.
They follow ordinary rule of algebra.	They follow vector algebra.
Example :- Distance, Mass, Time, etc.	Example :- Displacement, Velocity, Acceleration, etc.

Notes:

- If vector is rotated through an angle other than multiple of 2π (or 360°) it changes. Because its direction gets changed but magnitude remains same.
- If the reference frame is translated or rotated, the vector doesn't change (though its components may change).

Concept Reminder

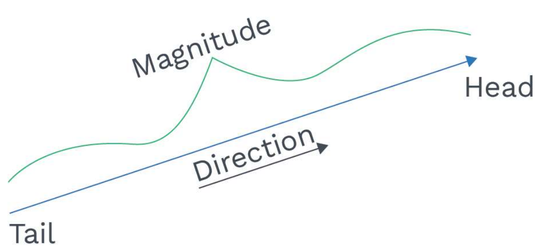
Vector quantities follow vector algebra not the ordinary law of algebra.



If physical quantity is a vector, it has a direction, but the converse may or may not be true, that means if a physical quantity has direction, it may or may not be a vector, example- time, pressure, surface tension or current, etc., have direction but are not vectors.

Representation of a vector:

- A vector is shown by a line headed with an arrow. Its length is proportional to its magnitude.



$\vec{A}$ is a vector.

$\vec{A} = \overrightarrow{PQ}$

Magnitude of $\vec{A} = |\vec{A}|$ or $A = |\overrightarrow{PQ}|$

Magnitude of a vector is always positive.

Angles between vectors:-

- The angle between the two vectors means smaller of the two angles between them when they are placed tail-to-tail or head-to-head by displacing either of the vectors parallel to itself (i.e., $0 \le \theta \le \pi$).

Definitions

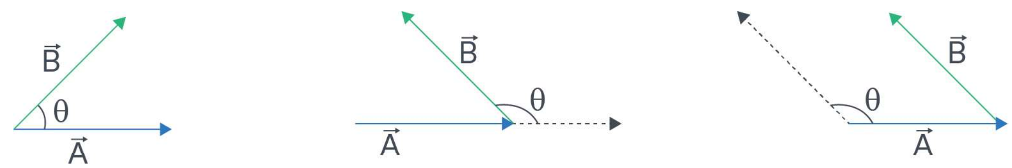
- **Scalar**:- A physical quantity that can be completely defined by its magnitude only.
- **Vector**:- Those physical quantities for which we need both magnitude and direction to define them.

Rack your Brain

Identify the vector quantity among the following.
(1) Distance
(2) Angular momentum
(3) Heat
(4) Energy

Concept Reminder

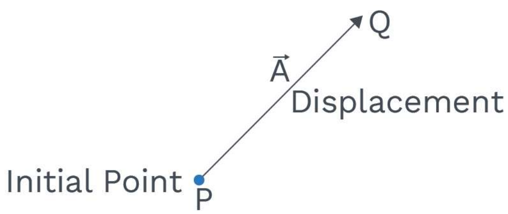
Magnitude of a vector is always positive. If a vector is negative, then –ve sign represents direction of that vector.



Polar Vector:

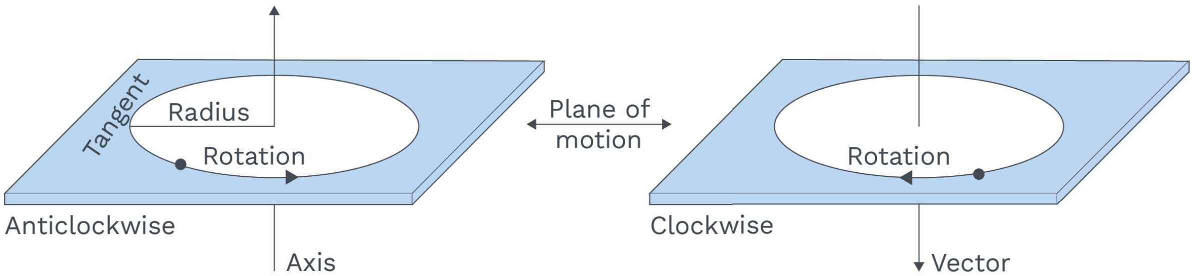
- Vector which has initial point, or a point of application are called polar vectors.

Examples: Displacement, force etc.



Axial Vector:

- Axial vectors are used in rotational motion to describe rotational effects.
- Direction of axial vectors is always along the axis of rotation in according to right hand screw rule or right-hand thumb rule.



Axial Vectors

Examples: Small angular displacement $(\overrightarrow{d\theta})$,

Angular velocity $(\vec{\omega})$, Angular momentum $(\vec{J})$,

Angular acceleration $(\vec{\alpha})$ and Torque $(\vec{\tau})$

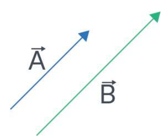
Definitions

Equal Vectors: -

Vectors which have equal magnitude and same direction.

Parallel Vectors:

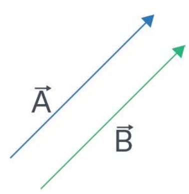
Those vectors which have same direction are called parallel vectors.



Angle between two parallel vectors is always zero.

Equal Vectors:

- Vectors which have same magnitude and same direction are called equal vectors.



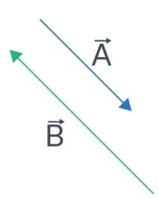
- $\vec{A} = \vec{B}$

NOTE:

- If a vector is moved parallel to itself, it represents a vector equal to itself, such a vector which can be freely moved parallel to themselves are called free vectors.

Anti-parallel Vectors:-

- Those vectors which have opposite direction are called anti-parallel vector.



Angle between two anti-parallel vectors is always 180°

KEY POINTS

- Axial Vector
- Parallel Vector
- Equal Vector

Concept Reminder

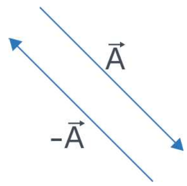
Angle between two

(i) Parallel vectors is always zero

(ii) Anti-parallel vectors is always 180°.

Opposite (or Negative) Vectors:-

- Vectors which have same magnitude, but opposite direction are called opposite vectors.



NOTE:

- If we multiply a vector by real positive number, then only its magnitude changes direction remains same.
- If we multiply a vector by real negative number, then its direction changes by 180°.
- If we multiply a vector by zero, then it becomes null vector.

Coplanar Vectors:

- Vectors placed in the same plane are called coplanar vectors.

Note: Two vectors are always coplanar.

Null or Zero Vector:

- Vector with zero magnitude is knwon as a null vector.

Its direction is arbitrary and is not specified.

Example: Sum of two vectors is always a vector.

Therefore $\overrightarrow{(A)} + \overrightarrow{(-A)} = \vec{0}$

Here $\vec{0}$ is a zero vector or null vector.

Unit Vector:

- A vector with unit (1) magnitude, is called unit vector.
- A unit vector is represented by $\hat{A}$

(A cap or A hat or A caret).

Unit Vector $\hat{A} = \dfrac{\text{Vector}}{\text{Magnitude of the vector}} = \dfrac{\vec{A}}{|\vec{A}|}$

KEY POINTS

- Anti-parallel vectors
- Opposite Vector
- Coplanar Vector
- Null Vector

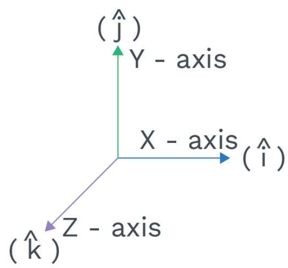
Concept Reminder

- Unit vector is used to represent a direction of vector
- $\hat{A} = \dfrac{\vec{A}}{|\vec{A}|}$

$\therefore$ $\boxed{\vec{A} = A\hat{A}}$ or $\vec{A} = |\vec{A}|\hat{A}$

Note: A unit vector is used to specify the direction of a vector.

Three Standard Unit Vectors:



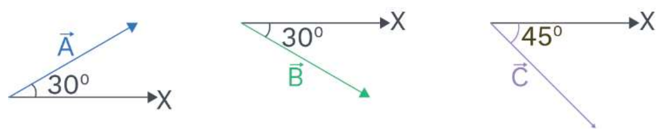
In x – y – z co-ordinate frame there are three-unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ which are used to indicate X, Y and Z axes, respectively. These three-unit vectors are mutually perpendicular i.e., $\hat{i} \perp \hat{j} \perp \hat{k}$

Ex. Three vectors $\vec{A}, \vec{B}, \vec{C}$ are shown in the figure. Find angle between

(i) $\vec{A}$ and $\vec{B}$

(ii) $\vec{B}$ and $\vec{C}$

(iii) $\vec{A}$ and $\vec{C}$



Sol. To measured the angle between two vectors we connect the tails of the two vectors. We can shift the vectors parallel to themselves such that tails of $\vec{A}$, $\vec{B}$ and $\vec{C}$ are connected as shown in figure.

Now we observe that angle between $\vec{A}$ and $\vec{B}$ is 60°, $\vec{B}$ and $\vec{C}$ 15°, between $\vec{A}$ and $\vec{C}$ is 75°

Rack your Brain

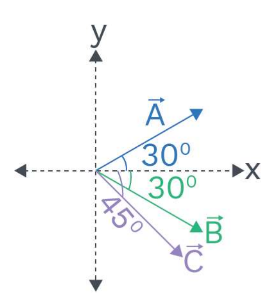
If a unit vector is represented by $0.5\hat{i} - 0.8\hat{j} + c\hat{k}$ then find value of c.

Definitions

Vector whose magnitude is zero is known as Null vector.

Concept Reminder

- $\hat{i}, \hat{j}$ and $\hat{k}$ are three standard unit vectors used to represent direction in x, y and z axis respectively.

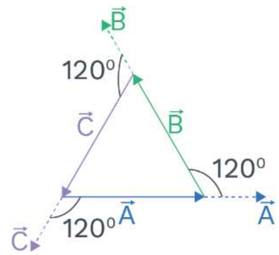


Ex. If $\vec{A}, \vec{B}, \vec{C}$ represents the three sides of an equilateral triangle taken in the same order then find the angle between

(i) $\vec{A}$ and $\vec{B}$

(ii) $\vec{B}$ and $\vec{C}$

(iii) $\vec{A}$ and $\vec{C}$



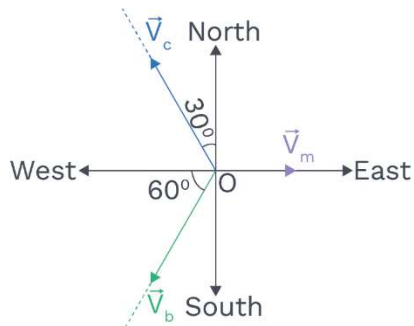
Sol. From the diagram the angle between vectors $\vec{A}$ and $\vec{B}$ is 120°, between $\vec{B}$ and $\vec{C}$ is 120° and between $\vec{A}$ and $\vec{C}$ is 120°

Ex. Vijay walks towards east with certain velocity. A Jeep is travelling along a road which is 30° west of north. While a bus is travelling in another road which is 60° south of west. Find the angle between velocity vector of

(i) Vijay and Jeep

(ii) Jeep and bus

(iii) bus and Jeep

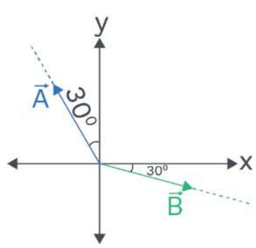


Sol. From the diagram the angle between velocity vector of man and car is 90° + 30° = 120°

The angle between velocity vector of car and bus is 60° + 60° = 120°

The angle between velocity vector of bus and man is 30° + 90° = 120°

Ex. A vector $\vec{A}$ makes an angle 30° with the y-axis in anticlockwise direction. Another vector $\vec{B}$ makes an angle 30° with the x-axis in clockwise direction. Find angle between vectors $\vec{A}$ and $\vec{B}$.



Sol. From the diagram the angle between $\vec{A}$ and $\vec{B}$ is 30° + 90° + 30° = 150°

Concept Reminder

- Angle between two vectors always lie within range
- $0° \leq \theta \leq 180°$

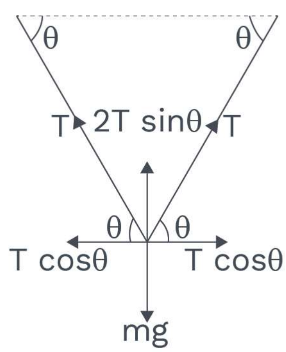
Ex. The components of a vector along the x and y directions are (n + 1) and 1 respectively. If the coordinate system is rotated by an angle 60° then the components change to n and 3. Find the value of n.

Sol. Length of the vector does not change on rotation.

$$\sqrt{(n+1)^2 + 1^2} = \sqrt{n^2 + 3^2} \Rightarrow n = \frac{7}{2} = 3.5$$

Ex. A weight 'mg' is suspended from the middle of a rope whose ends are rigidly clamped at the same level. The rope is no longer horizontal. What is the minimum tension required to completely straighten the rope?

Sol. From the diagram



$$2T \sin\theta = mg \Rightarrow T = \frac{mg}{2\sin\theta}$$

The rope will be straight when $\theta = 0°$

$$T = \frac{mg}{2\sin 0°} = \infty$$

The tension required to completely straighten the rope is infinity.

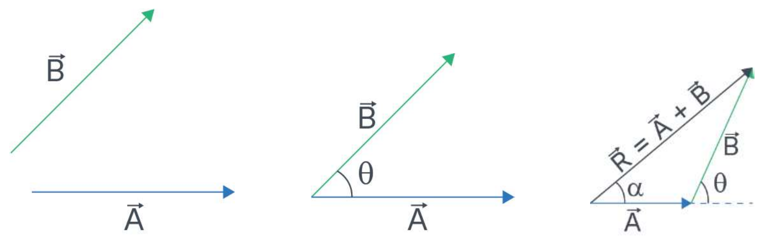
ADDITION OF TWO VECTORS:

- Resultant vector is defined as the sum of two vectors.

Rack your Brain

The magnitude of vectors $\vec{A}, \vec{B}$ and $\vec{C}$ are 3, 4 and 5 units respectively. If $\vec{A} + \vec{B} = \vec{C}$, Find the angle between $\vec{A}$ and $\vec{B}$.

Triangle Law of Addition of Two Vectors:

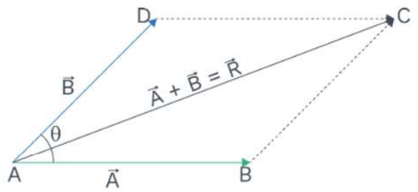


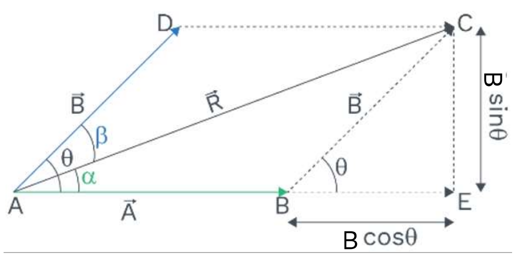
If two vectors are shown by two sides of a triangle in same order, then their sum or **'resultant vector'** is define by the 3rd side of the triangle taken in opposite order of the first two vectors.

Definitions

- Sum of two vectors is known as resultant of those vectors.

Parallelogram Law of Addition of Two Vectors (Alternate Method):





- If two vectors are shown by two adjacent sides of a parallelogram which are directed away from their common point, then their sum (i.e., resultant vector) is given by the diagonal of the parallelogram passing away through that common point.

  Here $\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AC} = \vec{R}$

  or $\vec{R} = \vec{A} + \vec{B}$

  Using ΔACE

KEY POINTS

- Triangular law of Addition
- Parallelogram law of Addition
- Resultant vector

$\Rightarrow \ (AC)^2 = (AE)^2 + (CE)^2$

$R^2 = (A + B\cos\theta)^2 + (B\sin\theta)^2$

The magnitude of resultant is.

$R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$,

Using ΔACE

The angle between $\vec{A}$ and $\vec{R}$ is given by.

$$\tan\alpha = \frac{CE}{AE} = \frac{B\sin\theta}{A + B\cos\theta}$$

Similarly angle between $\vec{B}$ and $\vec{R}$ is given by.

$$\tan\beta = \frac{A\sin\theta}{B + A\cos\theta}$$

Important points:

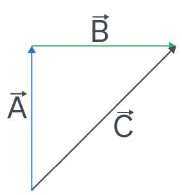
1. Vector addition is commutative, i.e.

$\vec{A} + \vec{B} = \vec{B} + \vec{A}$

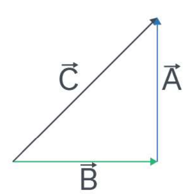
2. Vector addition is associative, i.e.

$\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$

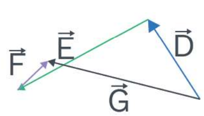
$\vec{A} + \vec{B} = \vec{C}$



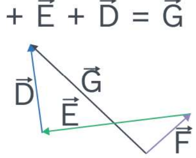
$\vec{B} + \vec{A} = \vec{C}$



$\vec{D} + \vec{E} + \vec{F} = \vec{G}$



$\vec{F} + \vec{E} + \vec{D} = \vec{G}$



3. Resultant of the two vectors will be maximum when they are parallel that means angle between them is zero.

$$R_{max} = \sqrt{A^2 + B^2 + 2AB\cos 0^\circ}$$

Concept Reminder

- Triangular law of Addition and parallelogram law of Addition both give same result. They are just two ways to find same resultant.

Rack your Brain

If $|\vec{A} + \vec{B}| = |\vec{A}| + |\vec{B}|$ then find angle between A and B.

or $\quad R_{max} = \sqrt{(A+B)^2} \qquad (\because \ \cos 0° = 1)$

or $\quad R_{max} = A + B$

4. Resultant of the two vectors will be minimum if they are antiparallel that means angle between them is 180°.

$$R_{min} = \sqrt{A^2 + B^2 + 2AB\cos 180°}$$

or $\quad R_{min} = \sqrt{(A-B)^2} \qquad (\because \ \cos 180° = -1)$

or $\quad R_{min} = A - B$

(Bigger – smaller)

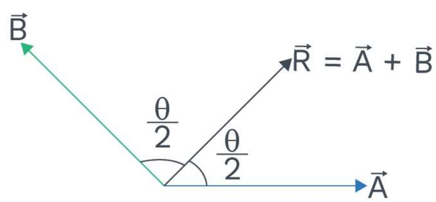
5. Resultant of the two vectors of unequal magnitude will never be zero.

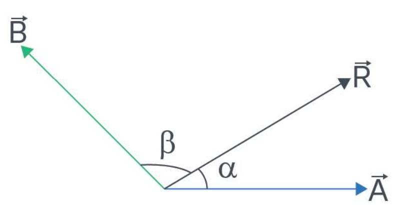
6. If the vectors are of unequal magnitude, then minimum three coplanar vector are needed for zero resultant.

7. Resultant of the two vectors of equal magnitude will be at their bisector.

If $|\vec{A}| = |\vec{B}|$, then $\alpha = \beta = \frac{\theta}{2}$

But if $|\vec{A}| > |\vec{B}|$, then angle $\alpha < \beta$





$\therefore \quad \vec{R}$ will incline more towards vector of bigger magnitude.

8. If the two vectors have equal magnitude i.e.

Concept Reminder

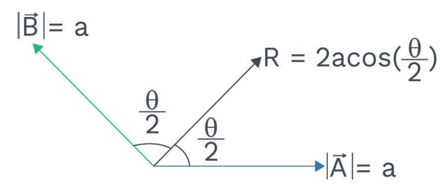
- Range of resultant is $A - B \le R \le A + B$

Concept Reminder

- If $|\vec{R}| = |\vec{A}| = |\vec{B}|$ then angle between $\vec{A}$ and $\vec{B}$ is 120°.

$|\vec{A}| = |\vec{B}| = a$ and angle between them is θ then resultant will be at bisector of $\vec{A}$ and $\vec{B}$ and its magnitude is equal to $2a\cos\left[\frac{\theta}{2}\right]$.

$$|\vec{R}| = |\vec{A} + \vec{B}| = 2a\cos\left[\frac{\theta}{2}\right]$$



Special Case: If $\theta = 120°$

then $R = 2a\cos\left[\frac{120°}{2}\right] = a$

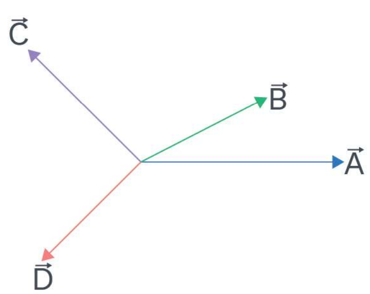
i.e., If $\theta = 120°$ then $|\vec{R}| = |\vec{A} + \vec{B}| = |\vec{A}| = |\vec{B}| = a$

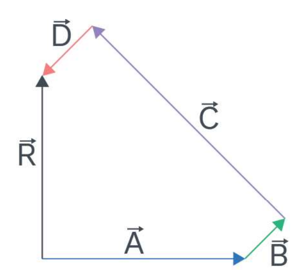
9. If resultant of the two unit vectors is another unit vector then the angle between them $(\theta) = 120°$.

OR

If the angle between two-unit vectors $(\theta) = 120°$, then their resultant is another unit vector.

Addition of More Than Two Vectors (Law of Polygon):-





If some vectors are from by sides of a polygon in same order, then their resultant vector will be represent by the closing side of polygon in the opposite order.

Here $\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D}$

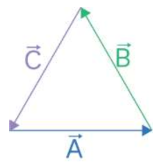
KEY POINTS

- Polygon's law of Addition
- Subtraction of vectors.

IMPORTANT POINTS

1. In a polygon if all vectors are in same order, then their resultant is null vector.

$\vec{R} = \vec{A} + \vec{B} + \vec{C}$



2. If 'n' vectors of equal magnitude are arranged at equal angles of separation, then their resultant is always zero.

Ex. Two force of magnitudes of 3N and 4N respectively are acting on an object. Find the resultant force if the angle between them is-

(i) 0°

(ii) 180°

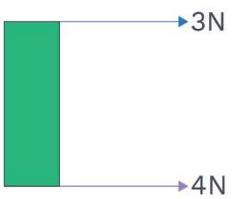
(iii) 90°

Sol. (i) If $\theta = 0°$ i.e., both the forces are parallel then R = A + B

∴ Net force or resultant force

(R) = (3 + 4)N = 7N

Direction of resultant is along both the forces.

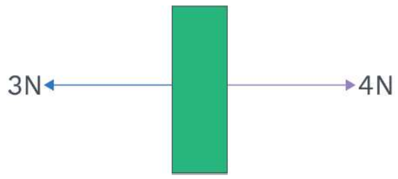


(ii) If $\theta = 180°$

i.e., both the forces are antiparallel, then R = A – B

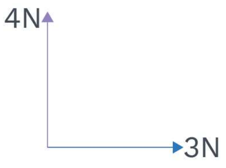
∴ Net force or resultant force = (4 – 3) N = 1N

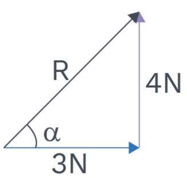
Direction of net force is along the bigger force means along 4N.



(iii) If $\theta = 90°$ i.e., both the forces are perpendicular then

$R = \sqrt{A^2 + B^2 + 2AB\cos 90°}$





Concept Reminder

- Direction of resultant of two vectors is always closer to vector having larger magnitude.

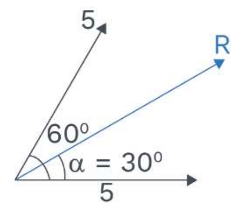
$\therefore \quad = \sqrt{A^2 + B^2} = \sqrt{3^2 + 4^2} = 5N$

$\tan\alpha = \frac{4}{3}$ or $\alpha = \tan^{-1}\left(\frac{4}{3}\right) = 53°$

i.e., magnitude of resultant is 5N which is acting at an angle of 53° from 3N force.

Ex. Two vectors having equal magnitude of 5 units, have an angle of 60° between them. Find the magnitude of their resultant vector and its angle from one of the vectors.

Sol. $R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$



If $|\vec{A}| = |\vec{B}| = a$ then $R = 2a\cos\frac{\theta}{2}$

Here a = 5 unit and $\theta = 60°$

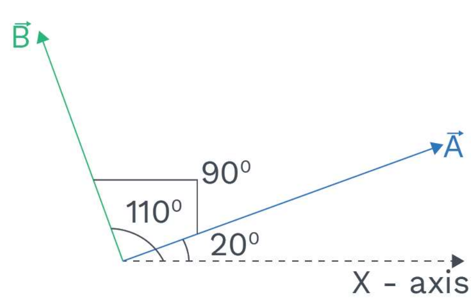
$\therefore \quad R = 2\times 5\times\cos\left(\frac{60°}{2}\right) = 2\times 5\times\frac{\sqrt{3}}{2} = 5\sqrt{3}$ unit

Since both the vectors have equal magnitude therefore resultant $(\vec{R})$ will be at their bisector.

$\therefore$ Angle of $\vec{R}$ from each given vector = 30°

Ex. A vector $\vec{A}$ and $\vec{B}$ are at an angles of 20° and 110° respectively with the horizontal axis. The magnitudes of these vectors are 5m and 12m, respectively. Calculate their resultant vector.

Sol. According to figure, angle between the vectors $\vec{A}$ and $\vec{B}$ = 110° – 20° = 90°



$R = \sqrt{A^2 + B^2 + 2AB\cos 90°} = \sqrt{5^2 + 12^2} = 13\,m$

Let angle of $\vec{R}$ from $\vec{A}$ is α then.

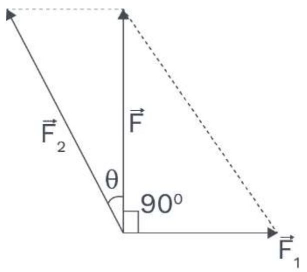
$\tan\alpha = \frac{B\sin\theta}{A + B\cos\theta} = \frac{12\sin 90°}{5 + 12\cos 90°}$

$= \frac{12\times 1}{5 + 12\times 0} = \frac{12}{5}$

or $\alpha = \tan^{-1}\left(\frac{12}{5}\right)$ with vector $\vec{A}$

or $(\alpha + 20°)$ with x-axis

Ex. The sum of magnitudes of the two forces acting at a point is 16N. If their resultant is normal to the smaller force and has a magnitude of 8N. Then the forces are



Sol. Let $\vec{F}$ be the resultant of two forces $\vec{F_1}$ and $\vec{F_2}$ as shown in figure with $F_2 > F_1$

$F_2 \sin\theta = F_1$ .....(i)

$F_2 \cos\theta = F = 8$ .....(ii)

Squaring and adding Equations (i) and (ii), we get

$F_2^2 = F_1^2 + 64$ .....(iii)

Given $F_1 + F_2 = 16$ .....(iv)

Solving Equations (iii) and (iv), we get

$F_1 = 6N$ and $F_2 = 10N$.

Ex. If vector $\vec{A}$ and $\vec{B}$ are $3\hat{i} - 4\hat{j} + 5\hat{k}$ and $2\hat{i} + 3\hat{j} - 4\hat{k}$ respectively then find the unit vector parallel to $\vec{A} + \vec{B}$

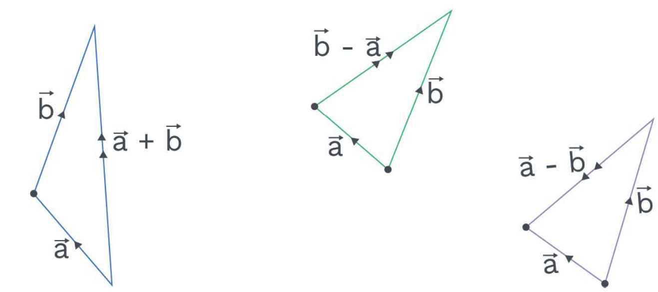
Sol. $\hat{n} = \frac{\vec{A} + \vec{B}}{|\vec{A} + \vec{B}|} = \frac{(3\hat{i} - 4\hat{j} + 5\hat{k}) + (2\hat{i} + 3\hat{j} - 4\hat{k})}{|5\hat{i} - \hat{j} + \hat{k}|}$

$$= \frac{(5\hat{i} - \hat{j} + \hat{k})}{\sqrt{27}}$$

Concept Reminder

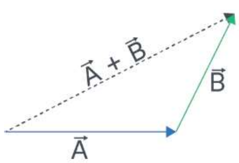
- Magnitude of vector $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ is given as $|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$

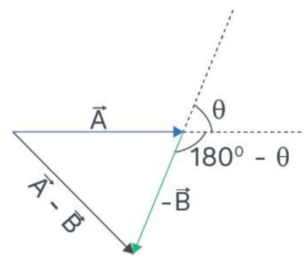
Addition and Subtraction of Vectors



Subtraction of vectors:

Subtraction of a vector from another vector is the addition of negative vector, i.e., $\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$





For subtraction of a vector from another vector

$$R = \sqrt{A^2 + B^2 + 2AB\cos(180° - \theta)}$$

$$R = \sqrt{A^2 + B^2 - 2AB\cos\theta}$$

IMPORTANT POINTS

1. The vector subtraction does not follow commutative law i.e.,

$\vec{A} - \vec{B} \neq \vec{B} - \vec{A}$

2. The vector subtraction does not follow associative law i.e.,

$\vec{A} - (\vec{B} - \vec{C}) \neq (\vec{A} - \vec{B}) - \vec{C}$

3. If two vectors have same magnitude, i.e., $|\vec{A}| = |\vec{B}| = a$ and θ is the angle between them, then.

$$|\vec{A} - \vec{B}| = \sqrt{a^2 + a^2 - 2a^2\cos\theta} = 2a\sin\left[\frac{\theta}{2}\right]$$

KEY POINTS

- Commutative law
- Associative law

Special case: If $\theta = 60°$ then $2a\sin\left(\frac{\theta}{2}\right) = a$

i.e., $|\vec{A} - \vec{B}| = |\vec{A}| + |\vec{B}| = a$ at $\theta = 60°$

4. If difference of two-unit vectors is another unit vector then the angle between them will be 60° or If two unit vectors are at angle of 60°, then their difference will also be a unit vector.
5. In physics whenever we want to find the change in a vector quantity, we have to use vector subtraction. For example, change in velocity $(\Delta\vec{V}) = \vec{V}_2 - \vec{V}_1$ or $\vec{V}_{final} - \vec{V}_{initial}$
6. If the two vectors are such that their sum and their difference vectors have equal magnitude then angle between the given vectors $(\theta) = 90°$.

$$|\vec{A} + \vec{B}| = |\vec{A} - \vec{B}|$$

or $\quad A^2 + B^2 + 2AB\cos\theta = A^2 + B^2 - 2AB\cos\theta$

or $\quad \theta = 90°$

7. If $\vec{A} + \vec{B} = \vec{A} - \vec{B}$ then $\vec{B} = \vec{0}$ (a null vector)

Concept Reminder

- Subtraction of two vectors is simply a addition of one positive and other negative vector.

Resolution of vectors into rectangular components and vector product

- When a vector is split into components which are at right angle to each other than the components are called **rectangular or orthogonal components** of that vector.

In ΔOAB, $\frac{OB}{OA} = \cos\theta$ and $\frac{AB}{OA} = \sin\theta$

or $\boxed{a_x = a\cos\theta}$ and $\frac{AB}{OA} = \sin\theta$

or $\quad AB = OA\sin\theta = OC$

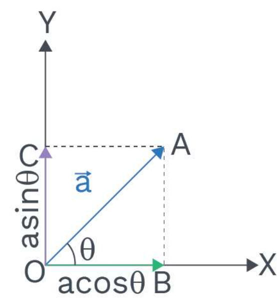
or $\boxed{a_y = a\sin\theta}$

Now according to rule of vector addition

$\overrightarrow{OA} = \overrightarrow{OB} + \overrightarrow{OC}$ or $\boxed{\vec{a} = a_x\hat{i} + a_y\hat{j}}$

From Pythagoras theorem $\boxed{a = \sqrt{a_x^2 + a_y^2}}$

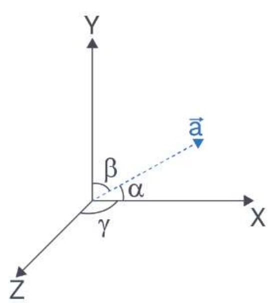
Angle of $\vec{a}$ from x-axis is given by $\boxed{\tan\theta = \dfrac{a_y}{a_x}}$



RECTANGULAR COMPONENTS OF A VECTOR IN THREE DIMENSIONS(3-D):-

- In terms of x, y and z-components $\vec{a}$ is given as

$$\boxed{\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}} \quad .....(1)$$



Magnitude of $\vec{a}$ is given as

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2} \quad .....(2)$$

- As shown in the figure, if $\vec{a}$ make angles α, β and γ from x, y and z-axis respectively, then

$$a_x = a\cos\alpha \Rightarrow \cos\alpha = \frac{a_x}{a}$$

$$a_y = a\cos\beta \Rightarrow \cos\beta = \frac{a_y}{a}$$

and $a_z = a\cos\gamma \Rightarrow \cos\gamma = \dfrac{a_z}{a}$

here $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called **directional cosines** of the vector.

Putting the value of a_x, a_y and a_z in eq. (2) we get

$$a^2 = a^2\cos^2\alpha + a^2\cos^2\beta + a^2\cos^2\gamma$$

or $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

KEY POINTS

- Resolution of Vector
- Rectangular Component
- Directional cosines

Concept Reminder

- $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$
- $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

or $(1-\sin^2\alpha)+\left(1-\sin^2\beta\right)+\left(1-\sin^2\gamma\right)=1$

or $3-(\sin^2\alpha+\sin^2\beta+\sin^2\gamma)=1$

or $\sin^2\alpha+\sin^2\beta+\sin^2\gamma=2$

Note: *"It means that the sum of the squares of the direction cosines of a vector is always one and the and sum of the squares of the direction sines of a vector is always two."*

1. Any vector can be resolved into maximum infinite number of components.

   For example, $10\hat{i}=\hat{i}+\hat{i}+\hat{i}$ ....10 times

   or $=\frac{\hat{i}}{2}+\frac{\hat{i}}{2}+\frac{\hat{i}}{2}$ ....20 times

2. Maximum number of rectangular components of a vector in a plane (2-dimensions) is two. But maximum number of rectangular components in space (3-dimensions) is three which are along 'X', 'Y' and 'Z' axis.
3. A vector doesn't depend on the orientation of axis, but the components of that vector depend upon the orientation of axis.
4. The components of a vector along its perpendicular direction are always zero.

Concept Reminder

- We can resolve a vector in any two rectangular components in perpendicular direction.

Ex. If $\vec{P}=3\hat{i}+4\hat{j}+12\hat{k}$ then find

(i) $\left|\vec{P}\right|$ and

(ii) the direction cosines of the $\vec{P}$.

Sol. (i)

$$\left|\vec{P}\right|=P=\sqrt{P_x^2+P_y^2+P_z^2}=\sqrt{3^2+4^2+12^2}=13$$

(ii) $\cos\alpha=\frac{P_x}{P}=\frac{3}{13}$,

$$\cos\beta=\frac{P_y}{P}=\frac{4}{13},\ \cos\gamma=\frac{P_z}{P}=\frac{12}{13}$$

Ex. Find out the angle made by $(\hat{i}+\hat{j})$ vector from X and Y axis, respectively.

Sol. $\cos\alpha = \frac{a_x}{a}$ and $\cos\beta = \frac{a_y}{a}$

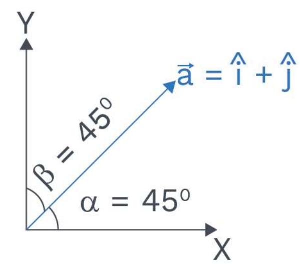
here $a = \sqrt{a_x^2 + a_y^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\cos\alpha = \frac{1}{\sqrt{2}}$

or $\alpha = 45°$

or $\beta = \frac{1}{\sqrt{2}}$

or $\beta = 45°$ i.e., $\hat{i}+\hat{j}$ is at bisector of X and Y axis.



Ex. Find out the angle made by $\vec{A} = \hat{i}+\hat{j}+\hat{k}$ vector from X, Y and Z axis respectively.

Sol. $\cos\alpha = \frac{A_x}{A}$, $\cos\beta = \frac{A_y}{A}$, $\cos\gamma = \frac{A_z}{A}$

here $A_x = A_y = A_z = 1$

and $A = \sqrt{A_x^2 + A_y^2 + A_z^2} = \sqrt{1+1+1} = \sqrt{3}$

$\therefore \quad \cos\alpha = \frac{1}{\sqrt{3}}$ or $\alpha = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\cos\beta = \frac{1}{\sqrt{3}}$ or $\beta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\cos\gamma = \frac{1}{\sqrt{3}}$ or $\gamma = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

Ex. A bird moves with velocity 20 m/s in a direction making an angle of 60° with the eastern line and 60° with the vertical upward. Represent the velocity vector in rectangular form.

Sol. Let eastern line be taken as x-axis, northern as y-axis and vertical upward as z-axis.

Concept Reminder

Minimum number of unequal Magnitude vectors whose sum can be zero is.

- 3 for co-planar vectors.
- 4 for non-coplanar vectors.

Let the velocity makes angle and

with x, y and z axis respectively.

(= 60°, = 60°)

so

Ex. A force of 4N is inclined at angle of 60° from the vertical. Calculate out its components along horizontal and vertical directions.

Sol. Vertical Component = 4 cos 60° = 2N

Horizontal component = 4 sin 60° =

Ex. Calculate the resultant of the vectors shown in figure.

Sol.

Concept Reminder

- If $R = R_x$ then $R_y = 0$ and if $R = R_y$ then $R_x = 0$.
  i.e., component of a vector perpendicular to itself is always zero.

$\vec{R} = 4\hat{i} + 3\hat{j} + 3\hat{i} + 4\hat{j} = 7\hat{i} + 7\hat{j}$

$\therefore$ $R = 7\sqrt{2}$ cm and $\alpha = 45°$ with horizontal.

Ex. A force is inclined at angle of 60° from the x-axis. If the x-axis component of the force is 40N, then find the y-axis component.

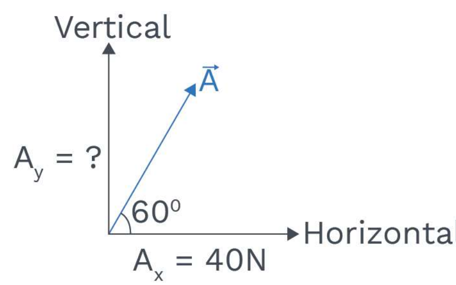
Sol. $A_x = 40N$, $A_y = ?$, $\theta = 60°$

As $A_x = A\cos\theta$

$\therefore$ $40 = A\cos 60°$ or $40 = \frac{A}{2}$ or $A = 80N$

Now $A_y = A\sin 60° = \frac{A\sqrt{3}}{2}$

$= \frac{80\sqrt{3}}{2} = 40\sqrt{3}\,N$



Ex. Find the vector for which when added to the resultant of $\vec{P} = 2\hat{i} + 7\hat{j} - 10\hat{k}$ and $\vec{Q} = \hat{i} + 2\hat{j} + 3\hat{k}$ gives a unit vector along X-axis.

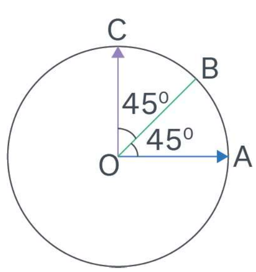
Sol. Resultant $\vec{R} = \vec{P} + \vec{Q}$

$= (2\hat{i} + 7\hat{j} - 10\hat{k}) + (\hat{i} + 2\hat{j} + 3\hat{k}) = 3\hat{i} + 9\hat{j} - 7\hat{k}$

Required vector

$= \hat{i} - \vec{R} = \hat{i} - (3\hat{i} + 9\hat{j} - 7\hat{k}) = -2\hat{i} - 9\hat{j} + 7\hat{k}$

Ex. Calculate the resultant of the vectors $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ as shown in figure. The radius of the circle is r.



Rack your Brain

If the magnitude of sum of two vectors is equal to the magnitude of difference of the two vectors, Find the angle between these vectors.

Sol. $\vec{R} = \overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$

$\vec{R} = r\hat{i} + r\cos 45°\hat{i} + r\sin 45\hat{j} + r\hat{j}$

$\vec{R} = \left(r + \frac{r}{\sqrt{2}}\right)\hat{i} + \left(r + \frac{r}{\sqrt{2}}\right)\hat{j}$

$|\vec{R}| = \left(\sqrt{2}r + r\right)$ along $\overrightarrow{OB}$.

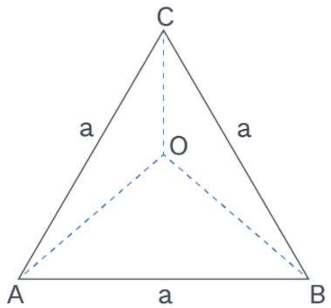
Ex. ABC is an equilateral triangle. Length of each side is 'a' and centroid is point O. Find.

(i) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = ?$

(ii) $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = ?$

(iii) If $\left|\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{AC}\right| = na$ then n = ?

(iv) If $\overrightarrow{AB} + \overrightarrow{AC} = n\ \overrightarrow{AO}$ then n =?



Sol. (i) $\overrightarrow{AB}$, $\overrightarrow{BC}$ & $\overrightarrow{CA}$ form a closed triangle in the same order. Therefore, their resultant is zero.

$\therefore$ $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$

(ii) $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ are three vectors of equal magnitude which are separated by equal angles of 120°. Therefore, their resultant is zero.

$\therefore$ $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \vec{O}$

(iii) Here $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{AC}$

$\therefore$ $\left(\overrightarrow{AB} + \overrightarrow{BC}\right) + \overrightarrow{AC} = \overrightarrow{AC} + \overrightarrow{AC} = 2\overrightarrow{AC}$

$\therefore$ $\left|\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{AC}\right| = \left|2\overrightarrow{AC}\right| = 2\left|\overrightarrow{AC}\right| = 2a$

$\therefore$ n = 2

(iv) $\overrightarrow{AB} + \overrightarrow{AC} = ?$

here $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

and $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$

$\therefore$ $\overrightarrow{AB} + \overrightarrow{AC} = 2\overrightarrow{AO} + \overrightarrow{OB} + \overrightarrow{OC}$ .....(1)

but $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \vec{O}$

[from (ii) part of question]

Concept Reminder

- A vector can be a null vector only if all its component are zero.

.....(2)

Using equation (1) and (2)

.....(3)

So, n = 3

Ex. Add vectors , and which have equal magnitude of 50 unit and are inclined at angles of 45°, 135° and 315° respectively from x-axis.

Sol. Angle between and = 315° – 135° = 180°

They balance each other.

sum = = 50 unit at 45° from X-axis

Ex. Vector is 2 cm long and is 60° above the x-axis in the first quadrant, vector is 2 cm long and is 60° below the x-axis in the fourth quadrant. (As shown in figure) Find .

Sol.

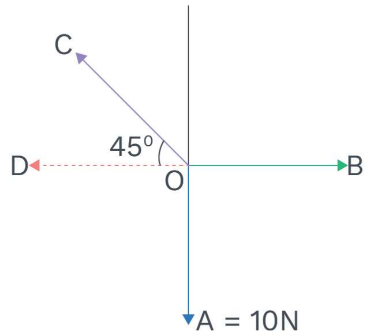
⇒

⇒

R = 2 cm, along x-axis

Ex. The sum of three vectors is shown in below figure, is zero.

(i) What is the magnitude of vectors $\overrightarrow{OB}$?



(ii) What is the magnitude of vector $\overrightarrow{OC}$?

Sol. Resolve $\overrightarrow{OC}$ into two components.

OE = OC sin 45° and OD = OC cos 45°

For zero resultant OE = OA

or $\quad OC \times \dfrac{1}{\sqrt{2}} = 10\,N$

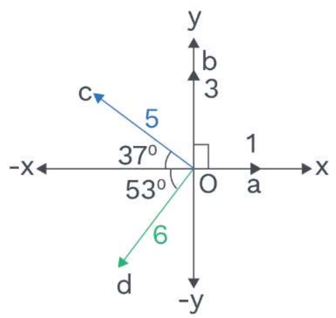
or $\quad OC = 10\sqrt{2}\,N$

and $\quad$ OD = OB or OC cos 45° = OB

or $\quad 10\sqrt{2} \times \dfrac{1}{\sqrt{2}} = OB$ or $OB = 10\,N$

$\therefore \quad OC = 10\sqrt{2}\,N$ and $OB = 10\,N$

25. Find the resultant of the vectors shown in figure by the component method.



Sol. $\overrightarrow{R_x} = 1\hat{i} - 5\cos 37°\hat{i} - 6\cos 53°\hat{i}$

$\overrightarrow{R_x} = 1\hat{i} - 4\hat{i} - 3.6\hat{i} \quad \therefore \overrightarrow{R_x} = -6.6\hat{i}$

$\overrightarrow{R_y} = 3\hat{j} + 5\sin 37°\hat{j} - 6\sin 53°\hat{j}$

$\overrightarrow{R_y} = 3\hat{j} + 3\hat{j} - 4.8\hat{j} \quad \therefore \overrightarrow{R_y} = 1.2\hat{j}$

$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(-6.6)^2 + (1.2)^2} = 6.7$

MULTIPLICATION OF VECTORS:-

- Vectors of different type can be multiplied to create new physical quantities which can be a scalar or a vector. If, in multiplication of the two vectors, the generated physical quantity is a scalar, then their product is known as **scalar or dot product** and if it is a vector, then their product is called **vector or cross product**.

KEY POINTS

- Multiplication of vectors
- Scalar or dot product
- Vector or Cross product

TYPE OF VECTOR PRODUCT

- **SCALAR PRODUCT OF TWO VECTORS:-**

Definition:- The scalar product or dot product of the two vectors is defined as the product of their magnitudes with cosine of angle between them.
Thus, If there are the two vectors $\vec{A}$ and $\vec{B}$ having angle θ between them then their scalar product is written as $\vec{A} \bullet \vec{B} = AB\cos\theta$

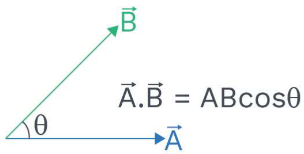
PROPERTIES:

1. It is always a scalar, which will be positive if angle between vectors is acute (i.e., θ > 90°) and will be negative if angle between them is obtuse (i.e., 90° < θ < 180°)

2. it is commutative i.e.,
$\vec{A} \bullet \vec{B} = \vec{B} \bullet \vec{A}$

3. It is distributive i.e.
$\vec{A} \bullet (\vec{B} + \vec{C}) = \vec{A} \bullet \vec{B} + \vec{A} \bullet \vec{C}$

4. According to definition
$\vec{A} \bullet \vec{B} = AB\cos\theta$



The angle between the vectors

$$\theta = \cos^{-1}\left[\frac{\vec{A}\bullet\vec{B}}{AB}\right]$$

5. Scalar product of the two vectors will be maximum when $\cos\theta$ = 1, i.e., $\theta = 0°$, i.e., vectors are parallel. $(\vec{A}\bullet\vec{B})_{max}$ = AB

6. Scalar product of the two vectors will be zero when $\cos\theta = 0$, i.e., $\theta = 90°$ therefore $(\vec{A}\bullet\vec{B}) = 0$

i.e., *if the scalar product of two nonzero vectors is zero then vectors are orthogonal or perpendicular to each other.*

7. In case of orthogonal unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$

$$\hat{i}\bullet\hat{j} = \hat{j}\bullet\hat{k} = \hat{k}\bullet\hat{i} = 1\times1\times\cos 90° = 0$$

8. The scalar product of a vector by itself is termed as self dot product and is given by

9. In case of unit vector $\hat{n}$

$$\hat{n}\bullet\hat{n} = 1\times1\times1\cos 0° = 1$$

So $\quad \hat{n}\bullet\hat{n} = \hat{i}\bullet\hat{i} = \hat{j}\bullet\hat{j} = \hat{k}\bullet\hat{k} = 1$

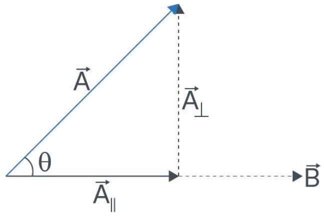
10. In terms of components

$$\vec{A}\bullet\vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\bullet(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

or $\quad \vec{A}\bullet\vec{B} = (A_xB_x + A_yB_y + A_zB_z)$

So $\quad \hat{n}\bullet\hat{n} = \hat{i}\bullet\hat{i} = \hat{j}\bullet\hat{j} = \hat{k}\bullet\hat{k} = 1$

11. Projection of vector: -



Concept Reminder

- In vector algebra, there is no law of division i.e., $\frac{A}{\vec{B}}$ or $\frac{\vec{A}}{\vec{B}}$ are not defined.

Rack your Brain

If a vector $2\hat{i} + 3\hat{j} + 8\hat{k}$ is perpendicular to the vector $4\hat{j} - 4\hat{i} + \alpha\hat{k}$, then find the value of α.

Concept Reminder

- If scalar product between two non-zero vector vanishes then the vectors are orthogonal.

- Component (or projection) of vector $\vec{A}$ along vector $\vec{B}$ is given by.

$$\vec{A}_{||} = (A\cos\theta)\hat{B}$$

$$\Rightarrow \frac{\vec{A}\cdot\vec{B}}{|\vec{B}|}\cdot\frac{\vec{B}}{|\vec{B}|} = \left[\frac{\vec{A}\cdot\vec{B}}{|\vec{B}|^2}\right]\vec{B}$$

- Component of vector $\vec{A}$ perpendicular to vector $\vec{B}$ is given by.

$$\vec{A}_{\perp} = \vec{A} - \vec{A}_{||} = \vec{A} - (\vec{A}\cdot\hat{B})\hat{B}$$

Ex. Find the angle between two vectors $\vec{A} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{B} = \hat{i} - \hat{k}$

Sol.

$$\cos\theta = \frac{\vec{A}.\vec{B}}{|\vec{A}||\vec{B}|} = \frac{2+0+1}{\sqrt{6}\sqrt{2}} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2} \Rightarrow \theta = 30°$$

Ex. If $\vec{a}_1$ and $\vec{a}_2$ are two non collinear unit vector inclined at 60° to each other then the value of $(\vec{a}_1 - \vec{a}_2).(2\vec{a}_1 + \vec{a}_2)$ is

Sol. $a_1 = a_2 = 1$

Now,

$$(\vec{a}_1 - \vec{a}_2).(2\vec{a}_1 + \vec{a}_2) = 2a_1^2 - a_2^2 - a_1a_2\cos\theta$$

$$= 2 - 1 - \frac{1}{2} = \frac{1}{2}$$

Ex. If $\vec{a}$ and $\vec{b}$ are two unit vectors such that $\vec{a} + 2\vec{b}$ and $5\vec{a} - 4\vec{b}$ are perpendicular to each other, then the angle between $\vec{a}$ and $\vec{b}$ is

Sol. $a = b = 1$

$$(\vec{a} + 2\vec{b}).(5\vec{a} - 4\vec{b}) = 0$$

$$5a^2 - 4ab\cos\theta + 10ab\cos\theta - 8b^2 = 0$$

$$5 - 4\cos\theta + 10\cos\theta - 8 = 0$$

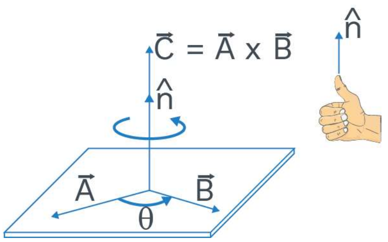
$$-3 + 6\cos\theta = 0 \Rightarrow \cos\theta = 1/2 \Rightarrow \theta = 60°$$

Rack your Brain

Find the angle between two vectors $\vec{A} = 3\hat{i} + 4\hat{j} + 5\hat{k}$ and $\vec{B} = 3\hat{i} + 4\hat{j} - 5\hat{k}$.

- **VECTOR PRODUCT OF TWO VECTORS:-**

Definition: The vector product or cross product of the two vectors is defined as a vector having magnitude equal to the product of their magnitudes with the sine of angle between them, and its direction will be perpendicular to the plane containing both the vectors according to right hand screw rule or right hand thumb rule.

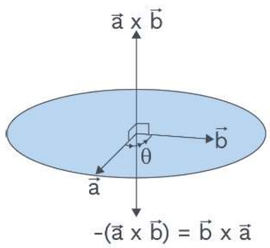


Thus, if $\vec{A}$ and $\vec{B}$ are two vectors, then their vector product i.e. $\vec{A}\times\vec{B}$ is a vector $\vec{C}$ defined by

$\vec{C} = \vec{A}\times\vec{B} = AB\sin\theta\ \hat{n}$

Concept Reminder

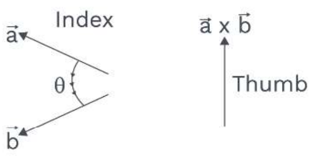
- The result of dot product is always scalar therefore it is known as scalar product. Similarly, the result of cross product is always vector therefore it is known as vector product.



$\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta$

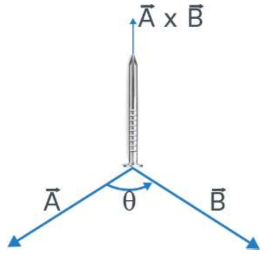
$0^\circ \le \theta \le 180^\circ$

Use 'Right Hand Rule'



Right Hand Screw Rule:

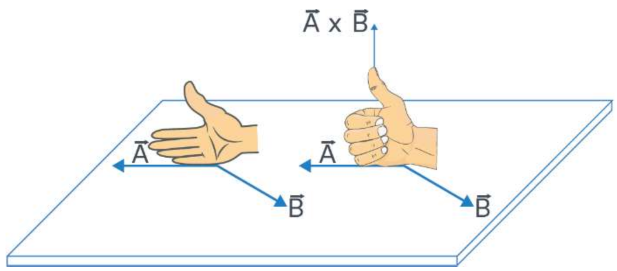
The direction of $\vec{A}\times\vec{B}$, i.e., $\vec{C}$ is perpendicular to the plane containing vectors $\vec{A}$ and $\vec{B}$ and towards the advancement of a right handed screw rotated from $\vec{A}$ (first vector) to $\vec{B}$ (second vector) through the smaller angle between them. Thus if, a right handed screw whose axis is perpendicular to the plane formed by $\vec{A}$ to $\vec{B}$



through the lesser angle between them then the direction of advancement of the screw will give the direction $\vec{A} \times \vec{B}$.

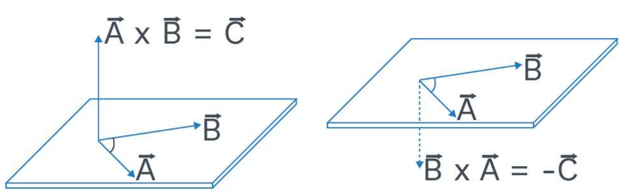
Right hand Thumb Rule:

Place the vector $\vec{A}$ and $\vec{B}$ tail to tail. Now put stretched fingers and thumb of right hand perpendicular to the plane of $\vec{A}$ and $\vec{B}$ such that fingers are along the vector $\vec{A}$. If the fingers are now closed through smaller angle so as to go towards $\vec{B}$, then the thumb gives the direction of $\vec{A} \times \vec{B}$ i.e., $\vec{C}$.



1. Vector product of the two vectors will always will be a vector perpendicular to the plane containing the two vectors, that means orthogonal (perpendicular) to both the vectors $\vec{A}$ and $\vec{B}$.

2. Vector product of two vectors is not commutative i.e., $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$ But $\left|\vec{A} \times \vec{B}\right| = \left|\vec{B} \times \vec{A}\right| = AB\sin\theta$



Note: $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$ i.e., in case of vectors $(\vec{A} \times \vec{B})$ and $(\vec{B} \times \vec{A})$, magnitudes are equal, but

Concept Reminder

- $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$ But $\vec{A} \times \vec{B} = -\left(\vec{B} \times \vec{A}\right)$

  because in cross-product direction of result is involved.

directions are opposite.

3. The vector product is distributive when the order of the vectors is strictly maintained, i.e., $\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B}+\vec{A}\times\vec{C}$

4. According to definition of vector product of two vectors $\vec{A}\times\vec{B} = AB\sin\theta\ \hat{n}$

   So, $|\vec{A}\times\vec{B}| = AB\sin\theta$

   i.e., $\theta = \sin^{-1}\left[\dfrac{|\vec{A}\times\vec{B}|}{|\vec{A}||\vec{B}|}\right]$

5. The vector product of two vectors will be maximum when $\sin\theta = 1$ (max.) i.e., $\theta = 90^\circ$

   $|\vec{A}\times\vec{B}|_{max} = AB\sin 90^\circ = AB$

   i.e., vector product is maximum if the vectors are orthogonal (perpendicular).

6. The vector product of the two non-zero vectors will be zero vector when sin $\theta = 0^\circ$,

   i.e., if, $\theta = 0^\circ$ or 180° then $\vec{A}\times\vec{B} = \vec{0}$

   Therefore, if the vector product of the two non-zero vectors is zero vector, then the vectors are collinear.

7. The self cross product, i.e., cross product of a vector by itself is a zero vector or a null vector.

   i.e., $\vec{A}\times\vec{A} = AA\sin 0\ \hat{n} = \vec{0}$

8. In case of unit vector $\hat{n}$

   $\hat{n}\times\hat{n} = 1\times 1\times\sin 0^\circ\ \hat{n} = \vec{0}$

   so that $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \vec{0}$

9. In case of orthogonal unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$; according to right hand thumb rule

   $\hat{i}\times\hat{j} = \hat{k},\ \hat{j}\times\hat{k} = \hat{i},\ \hat{k}\times\hat{i} = \hat{j}$

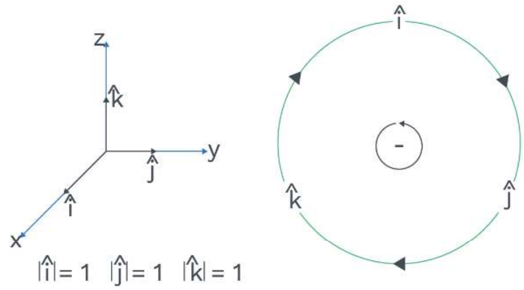
   and $\hat{j}\times\hat{i} = -\hat{k},\ \hat{k}\times\hat{j} = -\hat{i},\ \hat{i}\times\hat{k} = -\hat{j}$

Concept Reminder

- Self cross product is a cross product of a vector by itself and it is always equal to zero.

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$$

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$$



$$|\hat{i}| = 1 \quad |\hat{j}| = 1 \quad |\hat{k}| = 1$$

$$\hat{i} \times \hat{j} = \hat{k} \qquad \hat{j} \times \hat{i} = -\hat{k}$$

$$\hat{j} \times \hat{k} = \hat{i} \qquad \hat{k} \times \hat{j} = -\hat{i}$$

$$\hat{k} \times \hat{i} = \hat{j} \qquad \hat{i} \times \hat{k} = -\hat{j}$$

10. In terms of components

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \hat{i}(A_yB_z - A_zB_y) - \hat{j}(A_xB_z - A_zB_x) + \hat{k}(A_xB_y - A_yB_x)$$

11. A unit vector $(\hat{n})$ perpendicular to $\vec{A}$ as well as $\vec{B}$ is given by $\hat{n} = \dfrac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|}$.

12. If $\vec{A}$, $\vec{B}$ and $\vec{C}$ are coplanar, then $\vec{A}.(\vec{B} \times \vec{C}) = 0$.

13. Angle between $(\vec{A} + \vec{B})$ and $(\vec{A} \times \vec{B})$ is 90°.

Example of Vector product:

(i) Torque $(\vec{\tau}) = \vec{r} \times \vec{F}$

(ii) Angular momentum $(\vec{J}) = \vec{r} \times \vec{p}$

(iii) Velocity $(\vec{v}) = \vec{\omega} \times \vec{r}$

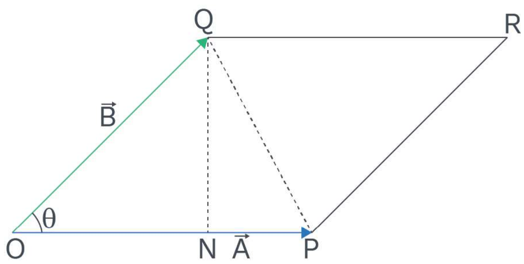
(iv) Acceleration $(\vec{a}) = \vec{\alpha} \times \vec{r}$

Here $\vec{r}$ is position vector or radius vector and $\vec{F}, \vec{p}, \vec{\omega}$ and $\vec{\alpha}$ are force, linear momentum, angular velocity, and angular acceleration respectively.

Concept Reminder

- Torque, Angular momentum, Angular vector etc. are an example of the Axial vectors.

GEOMETRICAL MEANING OF VECTOR PRODUCT OF TWO VECTORS



Area of $\Delta POQ = \dfrac{\text{base} \times \text{height}}{2}$

$$= \frac{(OP)(NQ)}{2} = \frac{A \times B \sin\theta}{2} = \frac{1}{2}\left|\vec{A} \times \vec{B}\right|$$

$\therefore$ Area of parallelogram OPRQ = 2 [Area of ΔOPQ] = $\left|\vec{A} \times \vec{B}\right|$

Formulae to Find Area:

1. If $\vec{A}$ & $\vec{B}$ are two sides of a triangle, then its area $= \dfrac{1}{2}\left|\vec{A} \times \vec{B}\right|$

2. If $\vec{A}$ & $\vec{B}$ are two adjacent sides of a parallelogram then its area $= \left|\vec{A} \times \vec{B}\right|$

3. If $\vec{A}$ and $\vec{B}$ are diagonals of a parallelogram then its area $= \dfrac{1}{2}\left|\vec{A} \times \vec{B}\right|$

Ex. Find the unit vector perpendicular to $\vec{A} = 3\hat{i} + 2\hat{j} - \hat{k}$ and $B = \hat{i} - \hat{j} + \hat{k}$.

Sol. $\vec{A} \times \vec{B} - = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & -1 \\ 1 & -1 & 1 \end{vmatrix}$

$$= \hat{i}(2-1) - \hat{j}(3+1) + \hat{k}(-3-2)$$

$$= \hat{i} - 4\hat{j} - 5\hat{k}$$

$\therefore$ $\hat{n} = \dfrac{\vec{A} \times \vec{B}}{\left|\vec{A} \times \vec{B}\right|} = \dfrac{\hat{i} - 4\hat{j} - 5\hat{k}}{\sqrt{1 + 16 + 25}} = \dfrac{\hat{i} - 4\hat{j} - 5\hat{k}}{\sqrt{42}}$

Ex. Can scalar product be ever negative?

Sol. Yes, Scalar product will be negative if $\theta > 90°$.

$\because$ $\vec{P} \bullet \vec{Q} = PQ\cos\theta$

$\therefore$ When $\theta > 90°$ then $\cos\theta$ is negative and $\vec{P}.\vec{Q}$ will be negative.

Ex. If $|\vec{A}+\vec{B}| = |\vec{A}-\vec{B}|$, then find the angle between $\vec{A}$ and $\vec{B}$.

Sol. $|\vec{A}+\vec{B}| = |\vec{A}-\vec{B}|$

$\therefore \quad \sqrt{A^2+B^2+2AB\cos\theta} = \sqrt{A^2+B^2-2AB\cos\theta}$

or $\quad A^2+B^2+2AB\cos\theta = A^2+B^2-2AB\cos\theta$

or $\quad 2AB\cos\theta = -2AB\cos\theta$

or $\quad 4AB\cos\theta = 0$

or $\quad \cos\theta = 0$

$\therefore \quad \theta = 90°$

Ex. If $\vec{A} = 4\hat{i}+n\hat{j}-2\hat{k}$ and $\vec{B} = 2\hat{i}+3\hat{j}+\hat{k}$, then find the value of n so that $\vec{A} \perp \vec{B}$.

Sol. Dot product of two mutually perpendicular vectors is zero.

i.e. $\quad \vec{A}.\vec{B} = 0$

or $\quad (4\hat{i}+n\hat{j}-2\hat{k}).(2\hat{i}+3\hat{j}+\hat{k}) = 0$

or $\quad (4 \times 2) + (n \times 3) + (-2 \times 1) = 0$

or $\quad 3n = -6$

or $\quad n = -2$

Ex. If $\vec{F} = (4\hat{i}-10\hat{j})$ and $\vec{r} = (5\hat{i}-3\hat{j})$, then calculate torque.

Sol. Here $\vec{r} = 5\hat{i}-3\hat{j}+0\hat{k}$ and $\vec{F} = 4\hat{i}-10\hat{j}+0\hat{k}$

$$\therefore \quad \vec{\tau} = \vec{r}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & -3 & 0 \\ 4 & -10 & 0 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(-50+12) = -38\hat{k}$$

Ex. Calculate a unit vector perpendicular to both the vectors $(2\hat{i}+3\hat{j}+\hat{k})$ and $(\hat{i}-\hat{j}+2\hat{k})$

.

Sol. Let $\vec{A} = 2\hat{i}+3\hat{j}+\hat{k}$ and $\vec{B} = \hat{i}-\hat{j}+2\hat{k}$

A unit vector perpendicular to both $\vec{A}$ and $\vec{B}$ is given as $\hat{n} = \dfrac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|}$

Concept Reminder

- To find a unit vector perpendicular to given two vectors, we should always use cross product.

  $\vec{A}\times\vec{B} = AB\sin\theta\,\hat{n}$

  $\therefore \hat{n} = \dfrac{\vec{A}\times\vec{B}}{AB\sin\theta}$

But $\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 1 \\ 1 & -1 & 2 \end{vmatrix}$

$= \hat{i}(6+1) - \hat{j}(4-1) + \hat{k}(-2-3) = 7\hat{i} - 3\hat{j} - 5\hat{k}$

$\therefore \quad |\vec{A}\times\vec{B}| = \sqrt{7^2 + (-3)^2 + (-5)^2} = \sqrt{83}$ unit

$\therefore \quad \hat{n} = \dfrac{7\hat{i} - 3\hat{j} - 5\hat{k}}{\sqrt{83}}$

This is a unit vector perpendicular to both the given vectors.

Rack your Brain

$\vec{A}$ and $\vec{B}$ are two vectors and θ is the angle between them, if $|\vec{A}\times\vec{B}| = \sqrt{3}(\vec{A}.\vec{B})$, then find the value of θ.

Ex. The diagonals of a parallelogram are vectors $\vec{A}$ and $\vec{B}$. If $\vec{A} = 5\hat{i} - 4\hat{j} + 3\hat{k}$ and $\vec{B} = 3\hat{i} - 2\hat{j} - \hat{k}$. Calculate the magnitude of area of this parallelogram.

Sol. When $\vec{A}$ and $\vec{B}$ are the diagonals of a parallelogram, then its

Area $= \dfrac{1}{2}|\vec{A}\times\vec{B}| = \dfrac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & -4 & 3 \\ 3 & -2 & -1 \end{vmatrix}$

$= \dfrac{1}{2}\left|\hat{i}(4+6) - \hat{j}(-5-9) + \hat{k}(-10+12)\right|$

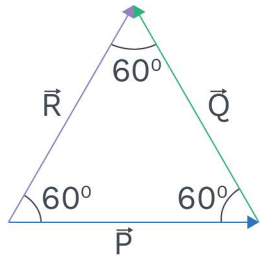
$= \dfrac{1}{2}\left|10\hat{i} + 14\hat{j} + 2\hat{k}\right| = \dfrac{1}{2}\sqrt{10^2 + 14^2 + 2^2}$

$= \dfrac{1}{2}\sqrt{300} = \dfrac{10}{2}\sqrt{3} = 5\sqrt{3}$ unit

Ex. Given that P = Q = R. If $\vec{P} + \vec{Q} = \vec{R}$ then the angle between $\vec{P}$ & $\vec{R}$ is θ_1. If $\vec{P} + \vec{Q} + \vec{R} = \vec{0}$ then the angle between $\vec{P}$ & $\vec{R}$ is θ_2. When is the relation between θ_1 and θ_2:

Sol. Since P = Q = R

Therefore, triangle formed by vectors $\vec{P}$, $\vec{Q}$ and $\vec{R}$ will be equilateral triangle.



Now if $\vec{P}+\vec{Q}=\vec{R}$, then $\vec{P}$ & $\vec{Q}$ will be in same order and $\vec{R}$ will be in their reverse order.

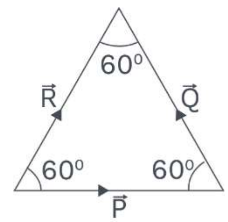
From the diagram, angle between $\vec{P}$ & $\vec{R} = \theta = 60°$

But if $\vec{P}+\vec{Q}+\vec{R}=\vec{0}$ then $\vec{P}$, $\vec{Q}$ & $\vec{R}$ will be in same order

From the diagram, angle between

$\vec{P}$ & $\vec{R} = \theta_2 = 180° - 60° = 120°$

$$\frac{\theta_1}{\theta_2} = \frac{60°}{120°} = \frac{1}{2} \Rightarrow \boxed{\theta_1 = \frac{\theta_2}{2}}$$



Ex . Given that $\vec{A}+\vec{B}+\vec{C}=\vec{0}$. From these three vectors two are equal in magnitude and the magnitude of the 3rd vector is $\sqrt{2}$ times as that of either of two having equal magnitude. Then the angles between vectors are given by:

(1) 30°, 60°, 90° (2) 45°, 45°, 90°

(3) 45°, 60°, 90° (4) 90°, 135°, 135°

Sol. Given A = B and $C = \sqrt{2}A$

$\because \quad \vec{A}+\vec{B}+\vec{C}=\vec{0}$

$\therefore \quad -\vec{C} = \vec{A}+\vec{B}$

$\therefore \quad |-\vec{C}| = \sqrt{A^2 + B^2 + 2AB\cos\theta}$

$\Rightarrow \quad C = \sqrt{A^2 + A^2 + 2A.A\cos\theta}$ $(\because B = A)$

$\Rightarrow \quad \sqrt{2}\,A = \sqrt{2A^2 + 2A^2\cos\theta}$ $(\because C = \sqrt{2}\,A)$

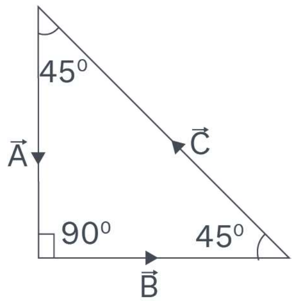
Squaring both sides

$2A^2 = 2A^2 + 2A^2\cos\theta$

$\Rightarrow \quad 2A^2\cos\theta = 0 \quad \Rightarrow \quad \cos\theta = 0$

$\Rightarrow \quad \theta = 90°$

Since two sides are equal and $\theta = 90°$, therefore the triangle formed by $\vec{A}$, $\vec{B}$ & $\vec{C}$ will be an isosceles right angled triangle.



From diagram

Angle between $\vec{A}$ & $\vec{B} = 90°$

Angle between $\vec{B}$ & $\vec{C} = 180° - 45° = 135°$

Angle between $\vec{C}$ & $\vec{A} = 180° - 45° = 135°$

Ex. The resultant of two vectors $\vec{P}$ and $\vec{Q}$ is $\vec{R}$. If $\vec{Q}$ is doubled then the new resultant vector is perpendicular to '$\vec{P}$'. Then R is equal to:

(1) $\left(\dfrac{P^2 - Q^2}{2PQ}\right)$ (2) Q

(3) $\dfrac{P}{Q}$ (4) $\dfrac{P+Q}{P-Q}$

Sol. When Q is doubled then $\vec{Q}$ becomes $2\vec{Q}$.

Suppose that the new resultant becomes $\vec{R}$,

$\therefore$ $\vec{R}' = \vec{P} + 2\vec{Q}$ & $\vec{R}' \perp \vec{P}$

From $\tan\alpha = \dfrac{B\sin\theta}{A + B\cos\theta}$

$$\Rightarrow \tan 90° = \frac{2Q\sin\theta}{P + 2Q\cos\theta} \Rightarrow \frac{1}{0} = \frac{2Q\sin\theta}{P + 2Q\cos\theta}$$

$$\Rightarrow P + 2Q\cos\theta = 0 \Rightarrow 2Q\cos\theta = -P$$

$$\therefore \quad \cos\theta = \frac{-P}{2Q} \quad \text{...(i)}$$

Now, according to question

$$\vec{R} = \vec{P} + \vec{Q}$$

$$\therefore \quad R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$$

Putting value of $\cos\theta$ from equation number (i)

$$R = \sqrt{P^2 + Q^2 + 2PQ\left(-\frac{P}{2Q}\right)}$$

$\Rightarrow \quad R = \sqrt{P^2 + Q^2 - P^2} \Rightarrow R = \sqrt{Q^2}$

$\therefore \quad R = Q$

Ex. If $\vec{F} = \hat{i} + 2\hat{j} - 3\hat{k}$

and $\vec{r} = 2\hat{i} - \hat{j} + \hat{k}$ find $\vec{r} \times \vec{F}$ (Torque)

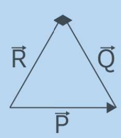
Sol. $\vec{r} \times \vec{F} = \begin{vmatrix} i & j & k \\ 2 & -1 & 1 \\ 1 & 2 & -3 \end{vmatrix}$

$= (3 - 2)\hat{i} - (-6 - 1)\hat{j} + (4 + 1)\hat{k} = \hat{i} + 7\hat{j} + 5\hat{k}$

Mind Map

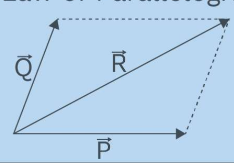
VECTORS

- **Vector Types**
  - Zero or Null vector — It has zero magnitude and arbitrary direction
  - Unit vector — It has magnitude as one or unity; $\hat{A} = \frac{\vec{A}}{|\vec{A}|}$
  - Equal vector — Vectors having same direction and magnitude
  - Negative vector — Vectors having same magnitude but opposite direction. Ex. $\vec{A}$ is a negative of $\vec{B}$ if, $\vec{A} = -\vec{B}$
  - Co-initial vector — Vectors having common starting point.
- **Addition of Vectors**
  - Law of Triangle



  $\vec{R} = \vec{P} + \vec{Q}$

  - Law of Parallelogram



- **Subtraction of two vectors** — $\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$
- **Multiplication of a vectors by a real number λ** — $\lambda \times \vec{A} = \lambda\vec{A}$
- **Position of displacement vector**
  - Position vector, $\vec{r} = x\hat{i} + y\hat{j}$
  - Displacement vector, $\Delta\vec{r} = \Delta x\hat{i} + \Delta y\hat{j}$
- **Velocity vector**
  - $\vec{v} = v_x\hat{i} + v_y\hat{j}$,
  - magnitude $|\vec{v}| = \sqrt{v_x^2 + v_y^2}$
  - Relative velocity
    - $\vec{V}_{AB} = \vec{V}_A - \vec{V}_B$
    - $\vec{V}_{BA} = \vec{V}_B - \vec{V}_A$
    - $\vec{V}_{AB} = -\vec{V}_{BA}$
    - $|\vec{V}_{AB}| = |-\vec{V}_{BA}|$
- **Acceleration vector**
  - $\vec{a} = a_x\hat{i} + a_y\hat{j}$; $a_x = dv_x/dt$ & $a_y = dv_y/dt$
  - $|\vec{a}| = \sqrt{a_x^2 + a_y^2}$