Work, Power and Energy

DISCLAIMER

Work, Power and Energy

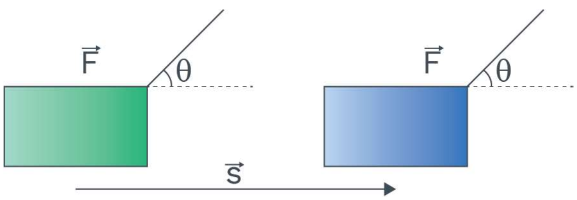
Work done by a constant force:

Assume $\vec{F}$ be a constant force acting on a body. If the body goes through a displacement $\vec{s}$, then the work done by the force $\vec{F}$ is given by W = Fs cos θ, where θ = angle between force vector $\vec{F}$ and displacement vector $\vec{s}$.

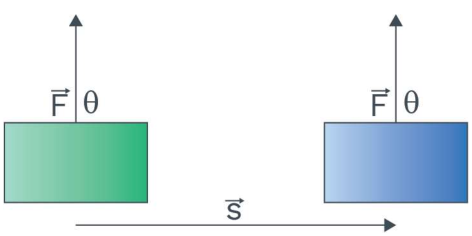
F = magnitude of force $\vec{F}$ and

s = magnitude of displacement $\vec{s}$.

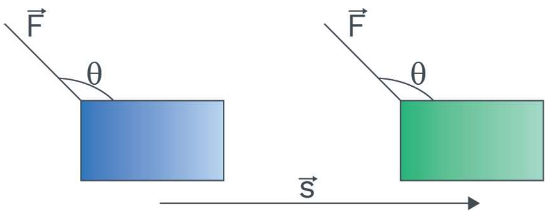
- If θ is acute (θ < 90°), work done (W) is positive (force tries to increase the speed of the body).



- If θ = 90° i.e., force is perpendicular to displacement, W = 0.



- If θ is obtuse (θ > 90°), work done (W) is negative (force tries to decrease the speed of the body).



Definitions

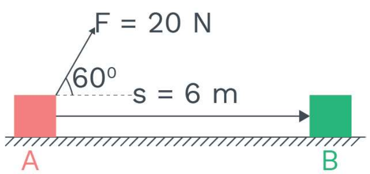
- The work done by the force $\vec{F}$ is given by W = Fs cos θ, where θ = angle between force vector $\vec{F}$ and displacement vector $\vec{s}$.

  F = magnitude of force $\vec{F}$ and s = magnitude of displacement $\vec{s}$.

Concept Reminder

Work is a scalar as it is dot product of two vectors having dimension $[ML^2T^{-2}]$ and SI unit N-m which is more specifically called Joule (J)

Ex. Find work done by force F from A to B ?



Sol. $W = Fs\cos\theta = 20 \times 6 \times \cos(60°) = 60$ J

- In vector from

$W = \vec{F} \cdot \vec{r}$

In $\vec{F} = \hat{i}F_x + \hat{j}F_y + \hat{k}F_z$ and $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$, the work done will be,

$$W = F_x \cdot x + F_y \cdot y + F_z \cdot z$$

Note: The force of gravity is the example of constant force, hence work done by it is the example of work done by a constant force.

Ex. A force $\vec{F} = (2\hat{i} - 3\hat{j} + 4\hat{k})$N displaces a particle from point A (0, 3, 5) m to a point B (4, 4, 2) m then find Work done by this force?

Sol. $\vec{s} = \vec{r}_f - \vec{r}_i = 4\hat{i} + \hat{j} - 3\hat{k}$

$$\vec{F} = 2\hat{i} - 3\hat{j} + 4\hat{k}$$

$$W = \vec{F} \cdot \vec{s} = 8 - 3 - 12$$

$$W = -7 \text{ J}$$

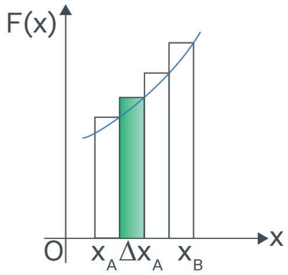
Work Done by a Variable Force:

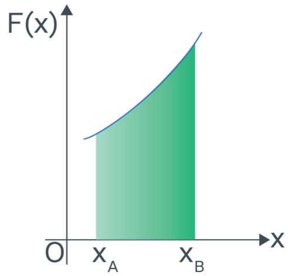
When the force (F) is an arbitrary function of position (x), we need the techniques of calculus to evaluate the work done by it. The figure shows F_x as some function of the position x. We begin by replacing the actual variation of the force by a series of n small steps. The area under every segment of the curve is approximately same to the area of a rectangle. The height of the rectangle is a constant value of force, and its width is a small displacement Δx. Thus, the step involves an amount of work $\Delta W_n = F_n \Delta x_n$. The total work

KEY POINTS

- Work done by constant force.
- Work done by variable force.

done is approximately by the sum of the areas of the rectangles:





$W \approx \Sigma F_n \Delta x_n$

As the size of the steps is reduced, the tops of the rectangle more closely trace the actual curve shown in figure. In the limit $\Delta x \to 0$, which is equivalent to letting the number of steps tend to infinity, the discrete sum is replaced by a continuous integral.

$$\lim_{\Delta x_n \to 0} \Sigma F_n \Delta x_n = \int F_x dx$$

Thus, the work done by a force F_x from an initial point A to final point B is

$$W_{A \to B} = \int_{x_A}^{x_B} F_x dx = \text{area under } F_x - x \text{ graph}$$

Units & Dimensions:

- **Units:**

SI Unit: joule, which is represented by 'J'.

Joule: One joule of work is known as to be done when a force of one newton displaces a body by one meter in the direction of force.

1 joule = 1 newton × 1 meter = 1 kg-m^2/s^2

CGS UNIT: erg

erg: One erg of work is known as to be done when a force of one dyne displaces a body by one centimeter in the direction of force.

1 erg = 1 dyne × 1 cm = 1 g-cm^2/s^2

Concept Reminder

For variable force

$W = \int_{S_1}^{S_2} \vec{F} \cdot d\vec{s}$ = Area under F-s graph

Definitions

Joule: One joule of work is said to be done when a force of one newton displaces a body by one meter in the direction of force.
1 joule = 1 newton × 1 meter = 1 kg-m^2/s^2

Other Units:

(a) 1 joule = 10^7 ergs
(b) 1 erg = 10^{-7} joules
(c) 1 eV = 1.6×10^{-19} joules
(d) 1 joule = 6.25×10^{18} eV
(e) 1 MeV = 1.6×10^{-13} J
(f) 1 J = 6.25×10^{12} MeV
(g) 1 kilowatt hour (kWh) = 3.6×10^6 joules

- **Dimensions:**

  [Work] = [Force] [Displacement]

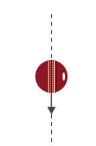
  $= [MLT^{-2}][L] = [ML^2T^{-2}]$

Angle dependency of work done/ nature of work:

Case-I:

- $W = Fs \cos \theta$

  If the angle 'θ' is acute ($\theta < 90°$) then the work is said to be positive. Positive work signifies that the external force applied the motion of the body.



Positive work done by gravitational force

When a body at rest starts falling freely under the motion of gravity ($\theta = 0°$), the work done by gravitational is positive (+ve).
When a spring is stretched, displacement and stretching force both are in the same direction ($\theta = 0°$). So, work done by stretching force is (+ve).

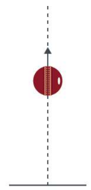
Case-II:

- $W = Fs \cos \theta$

  If the angle θ is obtuse angle ($\theta > 90°$).
  Then the work is known as to be negative (–ve). It shows that the direction of force is such that it opposes the motion of the object.

Rack your Brain

If a vector $2\hat{i} + 3\hat{j} + 8\hat{k}$ is perpendicular to the vector $4\hat{i} - 4\hat{j} + \alpha\hat{k}$ then find the value of α.

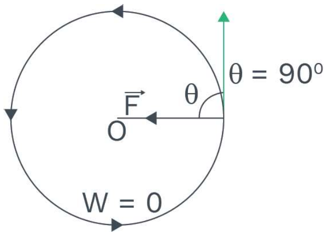


Negative work done
by gravitational force

Work done by frictional force is –ve when it opposes the motion and work done by braking force on the car is negative.

Case-III:

- $W = Fs \cos \theta$



Value of work done will be zero if ($F = 0$ or $s = 0$ or $\theta = 90°$)

- Body moving with uniform velocity. (Net W.D is zero)
- Net force on the particle is zero. (Net W.D is zero)
- We push the wall and it remains at rest.
- A pendulum is oscillating. (W.D by Tension)
- Electron is moving round the nucleus.
- Satellite is moving around the earth.
- Coolie “Sahayak” is carrying baggage on a horizontal platform (W.D. by gravitational force is zero)

Dependency on Frame of Reference:

- A force does not depend on frame of reference and assumes same value in all frames of references, but displacement depends on frame of reference and may assume different values

Concept Reminder

- When a person lifts a body from the ground, the work done by the lifting force is positive, but the work done by force of gravity is negative..

KEY POINTS

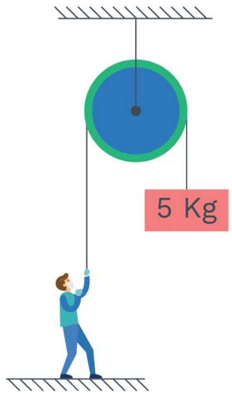
- Positive work.
- Negative work.
- Zero work.

Concept Reminder

- Work done in displacing a particle under the action of a number of forces is equal to work done by the resultant force.

relative to different reference frames. Therefore, work of a force depends on choice of reference frame.

Ex. If a man starts pulling the string with a rate of 4 m/s^2. Find work done by tension on block in 2 seconds.



Sol. By $F_{net} = Ma$

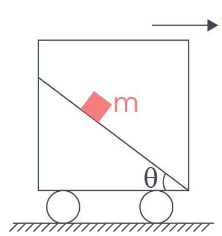
$\Rightarrow$ $T - 50 = 5 \times 4$ $\Rightarrow$ $T = 70$

$\Rightarrow$ $s = \frac{1}{2} \times 4 \times (2)^2 = 8\,m$

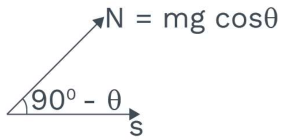
$\Rightarrow$ Work done = $FS \cos \theta$

$= 70 \times 8 \times 1 = 560$ J

Ex. A cart with a rough inclined base moves in horizontal direction with a speed of v m/s. If a block is placed on the inclined and given that θ is less than angle of repose find work done by normal reaction on block in time "t".



Sol. Block is horizontal and $s = v \times t$

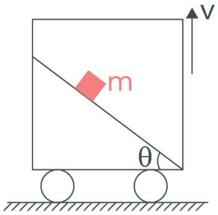


Work done = N . s cos θ

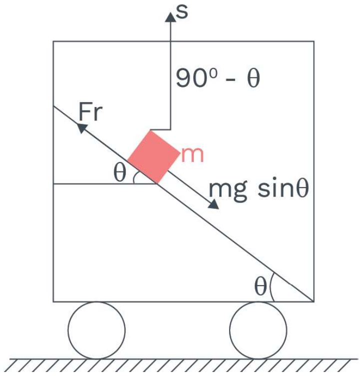
$= mg\cos\theta \times vt \cdot \cos(90 - \theta)$

$= \dfrac{mgvt\sin 2\theta}{2}$

Ex. If cart in previous question is moved vertically upward with speed v m/s. Find work done by friction in time "t".



Sol. As block will not slide $f_r = mg \sin\theta$



Work done = $F \cdot s \cdot \cos\theta$

$= mg \sin\theta \,(vt) \cos(90 - \theta)$

work done = $mgvt \sin^2\theta$

Ex. A force $\vec{F} = (3x^2\hat{i} - 4y\hat{j} + 3\hat{k})N$ displaces a particle from A(0, 1, 2)m to B(1, 2, 1)m. Then find W.

Sol. $\vec{F} = (3x^2\hat{i} - 4y\hat{j} + 3\hat{k})N$

$$\int dw = \int_0^1 3x^2dx + \int_1^2 -4ydy + 3\int_2^1 dz$$

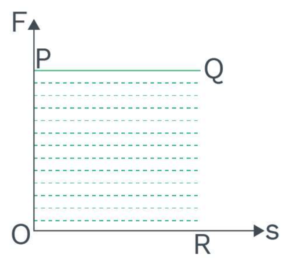
$$= |\, x^3 \,|_0^1 - |\, 2y^2 \,|_1^2 + |\, 3z \,|_2^1$$

$$= 1 - 6 - 3$$

$$W = -8 \text{ J}$$

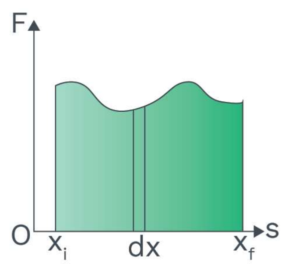
Graphical representation of work done:

- The area enclosed by the F-s graph and displacement axis gives the amount of work done by the force.



Work done = Fs = Area of PQRO

Calculate work done by variable force F(x).



- For a small displacement dx the value of work done will be the area of the strip of width dx

$$W = \int_{x_i}^{x_f} dW = \int_{x_i}^{x_f} F\,dx$$

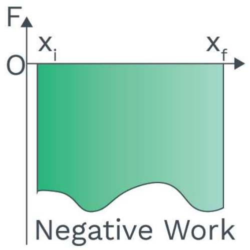
- If area enclosed above X-axis, work done is positive and if the area enclosed below X-axis, work done is negative.

Rack your Brain

A particle moves from a point $(-2\hat{i} + 5\hat{j})$ to $(4\hat{j} + 3\hat{k})$ when a force of $(4\hat{i} + 3\hat{j})$ N is applied. How much work has been done by the force?

Concept Reminder

If a person is pushing a box in a moving train, the work done in frame of train will be $\vec{F} \cdot \vec{s}$ while in frame of earth will be $\vec{F} \cdot (\vec{s} + \vec{s}_0)$, where $\vec{s}$ is a displacement of train relative to ground.

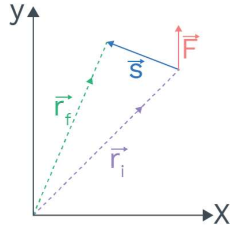


Applications on work:

- If a force is changing linearly from F_1 to F_2 over a displacement S, then work done is

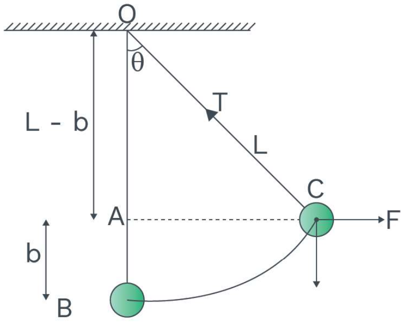
$$W = \left(\frac{F_1 + F_2}{2}\right)s$$

- If a force displaces the particle from its initial position $\vec{r}_i$ to final position $\vec{r}_f$ then displacement vector is $\vec{s} = \vec{r}_f - \vec{r}_i$.



$$W = \vec{F} \cdot \vec{s} = \vec{F} \cdot (\vec{r}_f - \vec{r}_i)$$

- Work done in pulling the bob of mass m of a simple pendulum of length L through an angle θ to vertical by means of a horizontal force F.



Rack your Brain

When a body moves with a constant speed along a circle choose right statement.
(a) No work is done on it
(b) No acceleration is produced in it
(c) Its velocity remains constant

$$\cos\theta = \frac{L-b}{L} = 1-\frac{b}{L}; \quad \frac{b}{L} = 1-\cos\theta$$

$$b = L(1-\cos\theta)$$

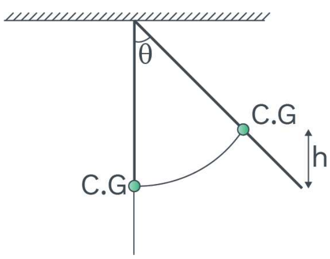
Work done by gravitational force

$$W = -mgb = -mgL(1-\cos\theta)$$

Work done by horizontal force F is W= FL sin θ

Work done by tension T in the string is zero.

Work done by gravitational force in pulling a uniform rod of mass m and length l through an angle θ is given by

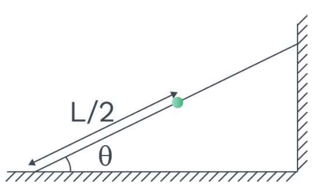


$$W = -mg\frac{l}{2}(1-\cos\theta)$$

Where l/2 is the distance of centre of mass from the support.

- A ladder of mass m and length L resting on a level floor is lifted and held against a wall at an angle θ with the floor

  Work done by gravitational force is



$$W_g = -mgh = -mg\left(\frac{L}{2}\right)\sin\theta$$

- A bucket fully fill of water of total mass M is pulled by using a uniform rope of mass m and length l. Work done by pulling force

$$W = Mgl + mg\frac{l}{2}$$

Concept Reminder

Work done by gravitational force in pulling a uniform rod of mass m and length l through an angle θ is

$$W = -mg\frac{l}{2}(1-\cos\theta)$$

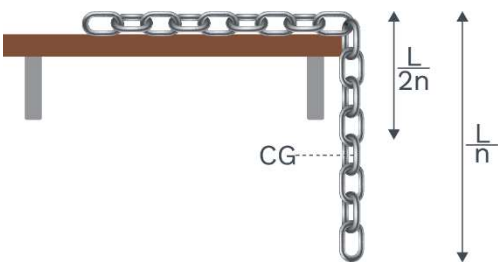
- A particle of mass 'm' is suspended vertically using (mass of rope of negligible). If the rope is used to lift the particle vertically up with uniform acceleration 'a', work done by tension in the rope is

  $W = m(g + a)h \quad (h = \text{height})$

  If block is lowered with acceleration 'a', then

  $W = -m(g - a)h$

  A uniform chain of mass M and length L is kept on smooth horizontal table such that $\frac{1}{n^{th}}$ of its length is hanging over the edge of the table. Value of work done by the pulling force to bring the hanging part onto the table is



$$W = \left(\frac{M}{n}\right)gh = \left(\frac{M}{n}\right)g\left(\frac{L}{2n}\right) = \frac{MgL}{2n^2}$$

Mass of hanging part is $\frac{M}{n}$

- A uniform chain of mass M and length L rests on a smooth horizontal table with $\frac{1}{n_1^{th}}$ part of its length is hanging from the side of the table. Work-done in pulling the chain partially such that $\frac{1}{n_2^{th}}$ part is hanging from the edge of the table is given by

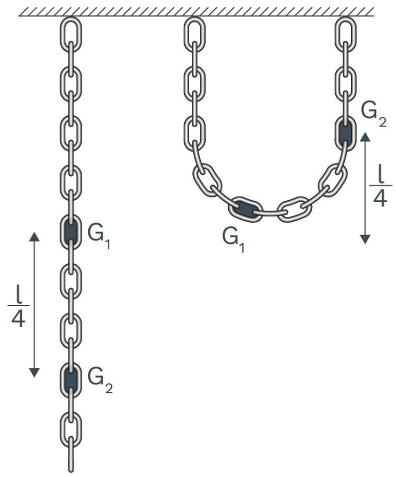
$$W = \frac{MgL}{2}\left[\frac{1}{n_1^2} - \frac{1}{n_2^2}\right]$$

Concept Reminder

The work done by pulling force to bring the hanging part onto the table is

$$W = \frac{MgL}{2n^2}$$

- A uniform massive chain of mass 'M' and length L is suspended vertically as shown in figure. The lower end of the massive chain is lifted upto point of suspension

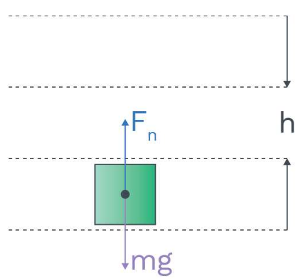


$h = \frac{L}{4} + \frac{L}{4} = \frac{L}{2}$ = raise in centre of mass of lower half of the chain.

Work done by gravitational force is

$$W_g = -\frac{M}{2} g \frac{L}{2} = -\frac{MgL}{4}$$

The Work-done in lifting a body of mass 'm' having density d_1 inside a liquid of density d_2 through a height h is



$$W = mgh\left[1 - \frac{d_2}{d_1}\right]$$

- A block of mass 'm' is placed on a frictionless horizontal surface. A force F acts on the block

Concept Reminder

In a conservative field work is path independent e.g., if we shift a body in equilibrium from A to C in a gravitational field via AC or ABC, the work done will be same.

Concept Reminder

Work done in lifting a body of mass m having density d_1 inside of density d_2 through height h is

$$W = mgh\left[1 - \frac{d_2}{d_1}\right]$$

parallel to the surface such that it moves with an acceleration 'a', through a displacement 's'. The work done by the force is

$W = Fs = mas$ ($\because \theta = 0°$)

- A block of mass 'm' is placed on a rough horizontal surface of coefficient of friction μ. A force F acts on the block parallel to the surface such that it moves with an acceleration 'a', through a displacement 's'. The work-done by the frictional force is

$f = \mu mg \cos\theta$; but $\theta = 0°$

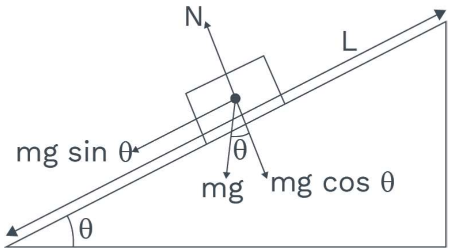
$\therefore \quad f = \mu mg \cos 0° = \mu mg \Rightarrow W_f = \mu mgs$

$W_{net} = (f + mg)s = (\mu mg + ma)s = m(\mu g + a)s$

If the body moves with uniform velocity, then

$W = fs = \mu mgs$

- A body of mass m is sliding down on a smooth inclined plane of inclination θ. If L is length of inclined plane, then work done by gravitational force is



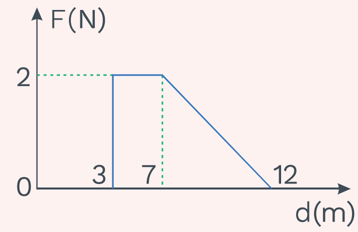
$W_g = Fs = mg \sin\theta L$

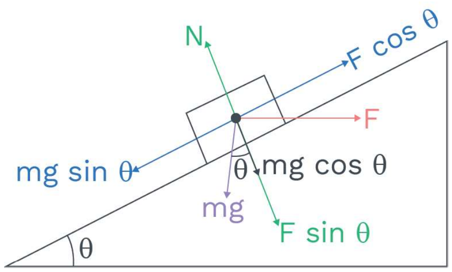
- A body of mass 'm' is moved up the smooth inclined plane of inclination θ and length L by a constant horizontal force F then work done by the resultant force is

$W = (F\cos\theta - mg\sin\theta)L$

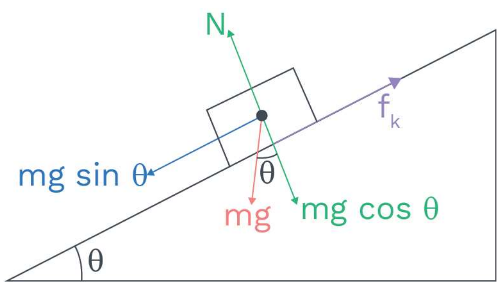
Rack your Brain

Force F on a particle moving in a straight line varies with distance d as shown. Find work done on particle during its displacement for 12 m.





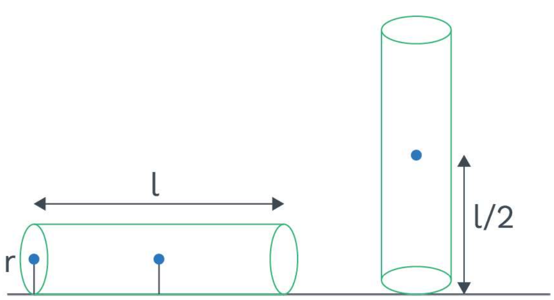
- A object of mass 'm' is sliding down on rough inclined plane of inclination θ. f L is the length of incline and μ_k is the coefficient of kinetic friction then work done by the resultant force on the object is



$$W = (mg\sin\theta - f_k)L = (mg\sin\theta - \mu_k mg\cos\theta)L$$

$$= mgL(\sin\theta - \mu_k \cos\theta)$$

- A uniform solid cylinder of mass m, length l and radius r is lying on ground with curved surface in contact with ground. If it is turned such that its circular face is in contact with ground, then work done by applied force is



Concept Reminder

Work done by friction can be positive also. For example, in two block system if force is applied on lower body and they both moving together then work done by friction on upper body is positive.

Concept Reminder

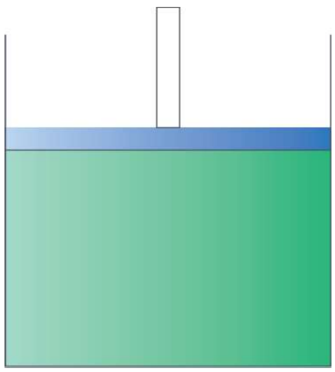
Work done by the gas during change in its volume from V_1 to V_2 is

$$W = \int_{V_1}^{V_2} P\,dV$$

Work is positive if $V_2 > V_1$ i.e., gas expands and negative if $V_2 < V_1$ i.e., gas compressed.

$$W = mgh = mg\left(\frac{l}{2} - r\right) \quad \left(\because\ h = \frac{l}{2} - r\right)$$

A gas at a pressure P is enclosed in a cylinder with a movable piston. Work done by the gas in producing small displacement dx of the piston is



$$dW = Fdx = PAdx = PdV$$

Total work-done by the gas during the change in its volume from V_1 to V_2 is

$$W = \int_{V_1}^{V_2} PdV$$

- Two blocks of masses m_1 and m_2 $(m_1 > m_2)$ connected by an inextensible string are passing over a smooth, massless pulley. The two blocks are released from the same level. At any instant 't', if 'x' is the displacement of each block then

Work done by gravity on block m_1,

$$W_1 = +\,m_1gx$$

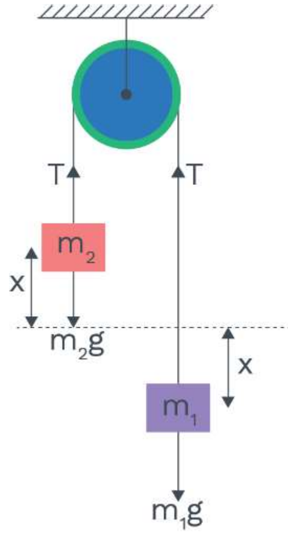
Work done by gravity on block m_2,

$$W_2 = -\,m_2gx$$

Work done by gravitational force on the system,

$$W_g = m_1gx - m_2gx$$

$$W_g = (m_1 - m_2)gx = (m_1 - m_2)g\left(\frac{1}{2}at^2\right)$$



$[\because v^2 - u^2 = 2as]$

$$W_g = \frac{(m_1 - m_2)^2 g^2 t^2}{2(m_1 + m_2)} \left[\because a = \frac{(m_1 - m_2)g}{m_1 + m_2}\right]$$

Note: In this case work done on the two blocks by tension is zero.

$W = T(x) + T(-x) = 0$

Ex. A body is displaced from $\vec{r}_A = (2\hat{i} + 4\hat{j} - 6\hat{k})$ to $\vec{r}_B = (6\hat{i} - 4\hat{j} + 2\hat{k})$ under a constant force $\vec{F} = (2\hat{i} + 3\hat{j} - \hat{k})$. Find the work done.

Sol. Work done

$$W = \vec{F} \cdot \vec{s}; \quad W = \vec{F} \cdot (\vec{r}_B - \vec{r}_A)$$

$$W = (2\hat{i} + 3\hat{j} - \hat{k})\left[(6\hat{i} - 4\hat{j} + 2\hat{k}) - (2\hat{i} + 4\hat{j} - 6\hat{k})\right]$$

$$W = (2\hat{i} + 3\hat{j} - \hat{k}) \cdot (4\hat{i} - 8\hat{j} + 8\hat{k})$$

$$W = 8 - 24 - 8 = -24 \text{ units}$$

Ex. A force $\vec{F} = 2x\hat{i} + 2\hat{j} + 3z^2\hat{k}$ N is acting on a particle. Find out the work done by the force in displacing the body from (1, 2, 3)m to (3, 6, 1)m.

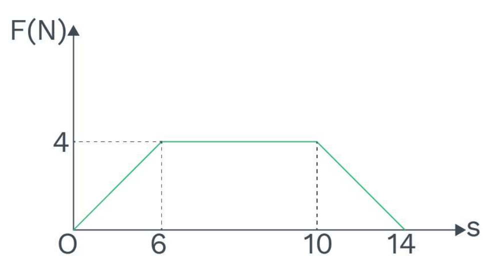
Sol. Work done

$$W = \int_{x_1}^{x_2} F_x dx + \int_{y_1}^{y_2} F_y dy + \int_{z_1}^{z_2} F_x dz$$

$$W = \int_1^3 2x dx + \int_2^6 2\, dy + \int_3^1 3z^2 dz$$

$$W = 2\left[\frac{x^2}{2}\right]_1^3 + 2[y]_2^6 + 3\left[\frac{z^3}{3}\right]_3^1 = -10\text{ J}$$

Ex. The force acting on an object varies with the distance travelled by the object as shown in the figure. Find out the work done by the force in moving the object from x = 0 m to x = 14 m.



Sol. Work done = Area under F - s curve.

$$W = \left(\frac{1}{2} \times 6 \times 4\right) + (4 \times 4) + \left(\frac{1}{2} \times 4 \times 4\right) = 36\,J$$

Ex. When a rubber band is stretch by a distance 'x', it exerts a restoring force of magnitude $F = ax + bx^2$, where a and b are constants. Find the work done in stretching the unstretched rubber band by 'L'.

Sol. The restoring force exerted by the rubber band when it is stretched by a distance 'x 'is

$F = ax + bx^2$

The small amount of work done on the rubber band in stretching through a small distance 'dx' is

$dW = Fdx = (ax + bx^2)dx$

The total work done in stretching the unstretched rubber band by 'L' is

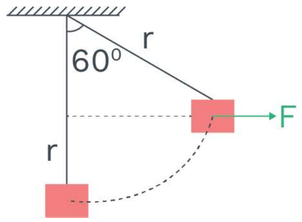
$$W = \int_0^L Fdx = \int_0^L (ax + bx^2)dx = \int_0^L ax\,dx + \int_0^L bx^2 dx$$

$$W = a\left[\frac{x^2}{2}\right]_0^L + b\left[\frac{x^3}{3}\right]_0^L = \frac{aL^2}{2} + \frac{bL^3}{3}$$

Ex. A 10 kg block is pulled along a frictionless surface in the form of an arc of a circle of radius 10 m. The applied force is 200 N. Find the work done by-

(a) Applied force and

(b) Gravitational force in displacing through an angle 60°.



Sol. Work done by applied force $W = Fr \sin \theta$

$W = 200 \times 10 \times \sin 60°$

$= 200 \times 10 \times \frac{\sqrt{3}}{2} = 1732\,J$

Work done by gravitational force

$$W = -mgr(1-\cos\theta)$$

$$W = -10 \times 9.8 \times 10(1-\cos 60^\circ)$$

$$W = -98 \times 10\left(1-\frac{1}{2}\right) = -490\,J$$

Ex. A uniform chain of length 2 metre is kept on a table such that a length of 60 cm hangs freely from the side of the table. The total mass of chain is 4 kilogram. Find out the work done in pulling the entire chain back onto the table?

Sol. M = 4 kg, L = 2 m, l = 0.6 m, g = 10 m/s^2

Work done

$$W = mg\frac{l}{2} = \left(\frac{M}{L}\right)lg\frac{l}{2}$$

$$W = \left(\frac{4}{2}\right) \times 0.6 \times 10 \times \frac{0.6}{2} = 3.6\ J$$

Ex. Find the work done in lifting a body of mass 20 kg and specific gravity 3.2 to a height of 8 m in water? (g = 10 m/s^2)

Sol. Given specific gravity

$$\frac{\rho_b}{\rho_w} = 3.2$$

$$\rho_b = 3.2 \times \rho_w = 3.2 \times 1000 = 3200$$

Work done

$$W = mgh\left(1-\frac{\rho_w}{\rho_b}\right) = 20 \times 10 \times 8\left(1-\frac{1000}{3200}\right)$$

$$W = 20 \times 10 \times 8\left(\frac{2200}{3200}\right) = 1100\,J$$

Ex. A body of mass 'm' is lowered with the help of a rope of negligible mass through a distance 'd' with an acceleration of g/3. Find out the work done by the rope on the body?

Sol. During lowering a body, tension in rope is

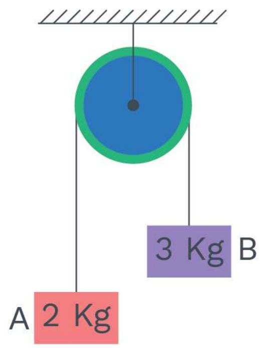
$T = m(g - a)$ and distance $s = d$

Work-done

$W = -\ m(g - a)d$

$$W = -m\left(g - \frac{g}{3}\right)d = -\frac{2mgd}{3}$$

Ex. If the system shown is released from rest. Find the net work done by tension in first one second (g = 10m/s²)



Sol. $a = \left(\frac{m_2 - m_1}{m_1 + m_2}\right)g = \left(\frac{3-2}{2+3}\right)10 = 2m/s^2$

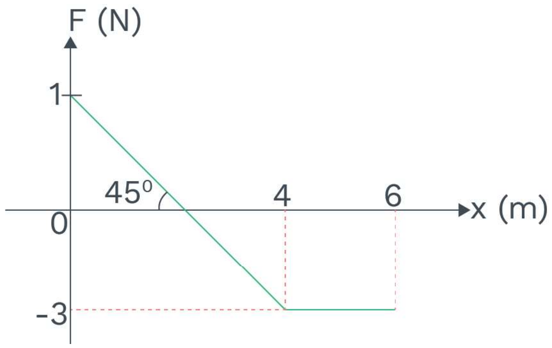
$$T = \frac{2m_1m_2g}{m_1 + m_2} = \frac{2 \times 2 \times 3 \times 10}{2+3} = 24\,N$$

For each block

$$s = \frac{1}{2}at^2 = \frac{1}{2} \times 2 \times 1 = 1\,m$$

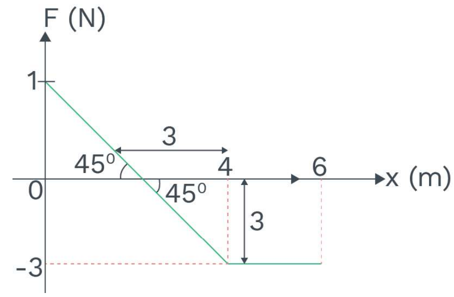
$$\therefore \quad W_{net} = W_1 + W_2 = Ts - Ts = 0$$

Ex. A force applied on a body and displaces it from x = 0 to x = 6 m. Find work done by the force in the process.



Sol. W.D = Area under F-x curve

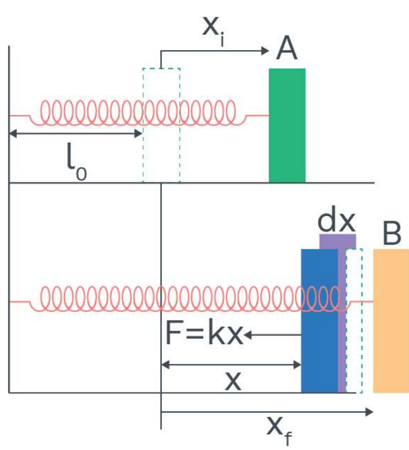
$$= \frac{1}{2} + \left[\frac{-9}{2}\right] + [-6] = -10\text{ J}$$



Work done by spring force (W_S):

$$dW = -kx \cdot dx$$

$$W = \int dW = \int_{x_i}^{x_f} -kx \cdot dx$$



Concept Reminder

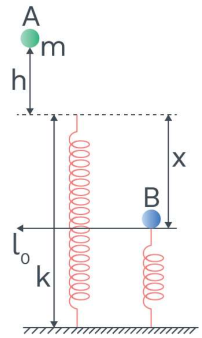
Work done by spring force when spring is displaced from x_i to x_f is

$$W = \frac{1}{2}K(x_i^2 - x_f^2)$$

$$W = -k\left|\frac{x^2}{2}\right|_{x_i}^{x_f}$$

$$W = \frac{1}{2}k(x_i^2 - x_f^2)$$

Ex. From A to B, find
(i) W. D by spring force
(ii) W. D by gravity



Sol. From A to B, find

(i) $W_S = \frac{1}{2}k(0^2 - x^2) = -\frac{1}{2}kx^2$

(ii) W_g = mg(h + x)

Ex. A force F = (2 + x) acts on a particle in x-direction where 'F' is in newton and 'x' in meter. Find out the work-done by this force during a displacement from x = 1.0 m to x = 2.0 m.

Sol. As the force is depends on x, we shall find the work-done in a small displacement from x to x + dx and then integrate it to calculate the total work. The work-done in this small displacement is

$$dW = Fdx = (2 + x)dx$$

Thus,

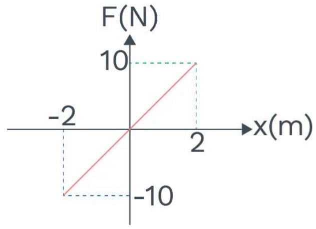
$$W = \int_{1.0}^{2.0} dW = \int_{1.0}^{2.0} (2 + x)dx$$

$$= \left[2x + \frac{x^2}{2}\right]_{1.0}^{2.0} = 3.5 \text{ J}$$

Ex. A force $F = -\frac{k}{x^2} (x \neq 0)$ acts on a object in x-direction. Find the work-done by this force in displacing the object from x = a to x = +2a. Here k is a positive constant.

Sol. $W = \int F\,dx = \int_{+a}^{+2a} \left(\frac{-k}{x^2}\right)dx = \left[\frac{k}{x}\right]_{+a}^{+2a} = -\frac{k}{2a}$

Ex. A force F acting on a body varies with the position x as shown in figure. Find the work done by this force in displacing the body from
(a) x = – 2 m to x = 0 (b) x = 0 to x = 2 m



Sol. (a) From x = – 2 m to x = 0,
Displacement of the body is along positive x-direction while force acting on the particle is along negative x-direction. Therefore, work done is negative and given by the area under F-x graph

$\therefore$ $W = -\frac{1}{2}(2)(10) = -10\ J$

(b) From x = 0 and x = 2m,
Displacement of particle and force acting on the particle both are along positive x-direction. Therefore, work done is positive and given by the area under F-x graph,

or $W = \frac{1}{2}(2)(10) = 10\ J$

ENERGY

- Energy is the capacity or ability to do work. Greater the value of energy possessed by the particle, greater the work-done it will be able to do.
- Energy is cause for doing work and work is effect of energy.
- Energy is a scalar. Energy and work have same units and dimensions.
- The different types of energy are Mechanical energy, Heat energy, light energy, Sound energy, Electrical energy, Nuclear energy etc.
- Energy of system always remain constant it can neither be created nor it can be destroyed however it may be converted from one form to another .

Definitions

Energy is the ability or capacity to do work. Greater the amount of energy possessed by the body, greater the work it will be able to do.

- Examples are as following.

Electric energy	$\xrightarrow{\text{Motor}}$	Mechanical energy
Mechanical energy	$\xrightarrow{\text{Generator}}$	Electrical energy
Light energy	$\xrightarrow{\text{Photocell}}$	Electrical energy
Electrical energy	$\xrightarrow{\text{Heater}}$	Heat energy
Electrical energy	$\xrightarrow{\text{Radio}}$	Sound energy
Nuclear energy	$\xrightarrow{\text{Nuclear Reactor}}$	Electrical energy
Chemical energy	$\xrightarrow{\text{Cell}}$	Electrical energy
Electrical energy	$\xrightarrow{\text{Secondary Cell}}$	Chemical energy
Heat energy	$\xrightarrow{\text{Incendence nt lamp}}$	Light

Energy is a scalar quantity.
Unit : Its unit is same as that of work or torque.
In MKS : Joule, watt sec
In CGS : erg

Note: 1 eV = 1.6×10^{-19} joule
1 kWh = 36×10^{5} joule
10^{7} erg = 1 joule

- Dimension $[M^1L^2T^{-2}]$
According to Einstein's mass energy equivalence principle mass and energy are inter convertible i.e. they can be changed into each other
Energy equivalent of mass m is, $E = mc^2$

Concept Reminder

A conservation law states that value of some quantity such as energy, linear momentum, angular momentum or charge does not change with time.

Where, m: mass of the particle [in Kg]
c : velocity of light
E : equivalent energy corresponding to mass m.

- **Mechanical energy is of two types**:
  (a) Potential Energy
  (b) Kinetic Energy

Potential Energy (U):

Potential energy of a particle is the energy possessed by a particle by virtue of its position or configuration in the field.

- Potential energy is explained only for conservative forces. It does not exist for non-conservative forces. In case of conservative forces.

$$F = -\left(\frac{dU}{dr}\right)$$

$$\therefore \quad dU = -\vec{F}\cdot\overrightarrow{dr}$$

$$\Rightarrow \int_{U_1}^{U_2} dU = -\int_{r_1}^{r_2} \vec{F}\cdot\overrightarrow{dr}$$

$$U_2 - U_1 = -\int_{r_1}^{r_2} \vec{F}\cdot\overrightarrow{dr} = -W$$

If $r_1 = \infty$, $U_1 = 0$

$$\therefore \quad U = \int_{\infty}^{r} \vec{F}\cdot\overrightarrow{dr} = -W$$

- Potential energy is a relative quantity.
- Potential energy of a particle at any position in a conservative force field is defined as the external work done against the action of conservative force in order to shift it from a certain reference point (PE = 0) to the present position.
- At a certain reference position, the potential energy of the particle is assumed to be zero.
- Relationship between conservative force field and potential energy:

KEY POINTS

- Energy
- Potential energy
- Kinetic energy

Definitions

Potential energy of a body is the energy possessed by a body by virtue of its position or configuration in the field.

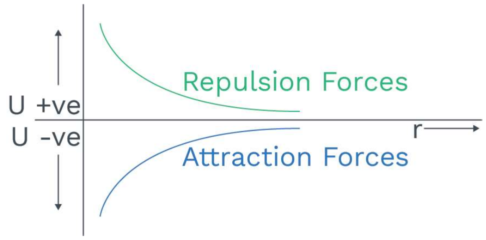
Concept Reminder

Potential energy is related to conservative force by formula

$$\vec{F} = -\nabla U = \text{grad}(U) = -\frac{\partial U}{\partial x}\hat{i} - \frac{\partial U}{\partial y}\hat{j} - \frac{\partial U}{\partial z}\hat{k}$$

$$\vec{F} = -\nabla U = \text{grad}(U) = -\frac{\partial U}{\partial x}\hat{i} - \frac{\partial U}{\partial y}\hat{j} - \frac{\partial U}{\partial z}\hat{k}$$

- Potential energy may be positive or negative or even zero



(i) Potential energy is positive, if force field is repulsive in nature.

(ii) Potential energy is negative, if force field is attractive in nature.

- If r ↑ (separation between particle and force centre), U ↑, force field is attractive or vice-versa.
- If r ↓, U ↓, force field is repulsive in nature.

Rack your Brain

The potential energy of a system increases if work is done:

(1) Upon the system by conservative force

(2) By the system against conservative force

(3) Upon system by non-conservative force

(4) By system against non-conservative force

Ex. Find force at (1,2) if potential energy for particle is $U = x^3 - x^2 + 2y + y^2$.

Sol. $\vec{F} = -\left[\left(3x^2 - 2x\right)\hat{i} + (2 + 2y)\hat{j}\right]$

$$\vec{F} = -\left[\left(3x^2 - 2x\right)\hat{i} + (2 + 2y)\hat{j}\right]$$

$$\vec{F}_{(1,2)} = -\left[\hat{i} + 6\hat{j}\right]$$

Ex. Potential energy of a body is given equation $U = 20 + (x - 1)^2$ the find. (Given mechanical energy = 45 J)

(a) Minimum potential energy.

(b) Maximum kinetic energy

Sol. (a) $U_{min} = 20$ J [as min. value of square is zero]

(b) $K + U = 45$

$K_{max} + U_{min} = 45$

$K_{max} = 25$ as $[U_{min} = 20]$

Ex. Force between the diatomic molecule has its origin from the interaction between the electrons and the nuclei present in each atom. This force is conservative and associated potential energy U(r) is, to a good approximation, represented by the Lennard-Jones potential function.

$$U(r) = U_0\left\{\left(\frac{a}{r}\right)^{12} - \left(\frac{a}{r}\right)^{6}\right\}$$

Here r is the distance between the two atoms and U_o and a are positive constants. Develop expression for the associated force and find the equilibrium separation between the atoms.

Sol. Using equation $F = -\frac{dU}{dr}$, we obtain the expression for the force

$$F = \frac{6U_0}{a}\left\{2\left(\frac{a}{r}\right)^{13} - \left(\frac{a}{r}\right)^{7}\right\}$$

At equilibrium, force must be zero. Therefore, the equilibrium separation is $r_0 = 2^{1/6}a$.

Ex. Potential energy of a particle of mass 2kg depends on position x as, $U = x^2 - 10x + 20$ J. If kinetic energy of particle at x = 2 m is 7 J, then find:

(i) Total mechanical energy of particle

(ii) Find minimum potential energy of particle

(iii) Find maximum K.E of particle

(iv) Find maximum speed of particle during motion

Sol. (i) At x = 2 m, K.E = 7 J

at x = 2 m, $U = (2)^2 - 10(2) + 20 = 4$ J

So, at x = 2 m, M.E. = K.E + U

M.E. = 7 + 4 = 11 J

As particle is moving under the effect of only conservative force, so, M.E is constant.

(ii) $U = x^2 - 10x + 20$ J

at maximum or minimum-

$$\frac{dU}{dx} = 0 \quad \Rightarrow 2x - 10 = 0$$

so, x = 5 m

So, U will be minimum at x = 5 m $\left[\frac{d^2U}{dx^2} > 0\right]$

and $U_{min} = (5)^2 - 10(5) + 20$; $U_{min} = -5$ J

(iii) $M.E = (K.E)_{max} + U_{min}$

$11\ J = (K.E)_{max} + (-5)\ 16\ J$

$\Rightarrow \quad (K.E)_{max} = 91\ J$

(iv) $(KE)_{max} = \frac{1}{2}mv^2_{max} = 16\ J$

$\Rightarrow \quad \frac{1}{2}mv^2_{max} = 16\ J \quad \Rightarrow v_{max} = 4\ m/sec$

Kinetic Energy:

- Kinetic energy (K.E.) is the energy possessed by an object by virtue of its motion.
- Kinetic energy of a body of mass 'm' moving with a velocity 'v', $KE = \frac{1}{2}mv^2$.
- Kinetic energy is a scalar quantity, which has only magnitude.
- The kinetic energy of a body is a measure of the work a body can do by the virtue of its motion.
  Examples for bodies having K.E
  (1) A vehicle in motion.
  (2) Water flowing in a river.
  (3) A bullet fired from a gun.
- Kinetic energy depends on frame of reference. Ex: kinetic energy of a person of mass m sitting in a train moving with speed v is zero in the frame of train but $\frac{1}{2}mv^2$ in the frame of earth.

Definitions

- Kinetic energy is the energy possessed by a body by virtue of its motion.
- Kinetic energy of a body of mass 'm' moving with a velocity 'v', $KE = \frac{1}{2}mv^2$.

Relation between K.E. and linear momentum:

- $KE = \frac{1}{2}mv^2 = \frac{P^2}{2m} = \frac{1}{2}Pv \quad (\because P = mv)$
- If two different masses bodies have same momentum, then lighter body will have greater KE $\left(\because KE \propto \frac{1}{m}\right)$.
- When a bullet is shoot from a gun the linear momentum of the bullet and gun are equal and opposite.

i.e. $\frac{KE_{bullet}}{KE_{gun}} = \frac{M_{gun}}{M_{bullet}}$

Concept Reminder

- $KE = \frac{1}{2}mv^2 = \frac{P^2}{2m}$
- $P = \sqrt{2mKE}$

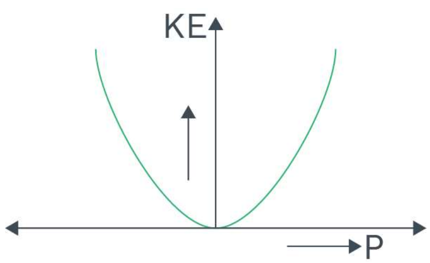
Therefore, the KE of the bullet is greater than that of the gun.

- A body can have energy without momentum. But it cannot have momentum without energy.
- A m mass bullet moving with velocity v stops in wooden block after penetrating through a distance 'x'. If resistance force F is offered by the block to the bullet. (Assuming F is constant inside the block)

$$\frac{1}{2}mv^2 = Fx; \quad F = \frac{mv^2}{2x}, \quad \therefore v^2 \propto x$$

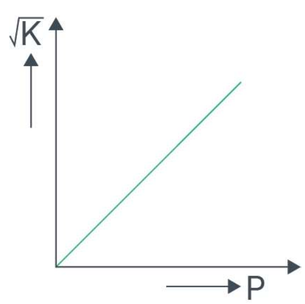
For a given body:

(a) The graph between KE and P is a parabola.



(b) The graph between $\sqrt{KE}$ and P is a straight line passing through the origin.

Its slope = $\frac{1}{\sqrt{2m}}$



(c) The graph between $\sqrt{KE}$ and $\frac{1}{P}$ is a rectangular hyperbola.

Concept Reminder

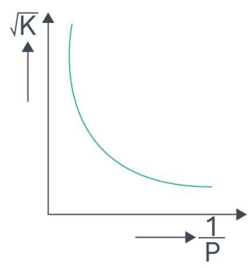
- Kinetic energy is a scalar property that is associated with state of motion of an object.
- An aeroplane in straight and flight has translational KE and a rotating wheel has rotational KE.

Rack your Brain

The kinetic energy acquired by a mass m in travelling distance d, starting from rest, under the action of constant force is directly proportional to:

(1) m (2) m^0

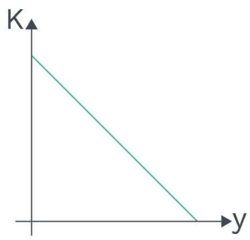
(3) $\sqrt{m}$ (4) $1/\sqrt{m}$



- A particle is projected up from a point at an angle ‘θ’ with the horizontal. At any time ‘t’ if ‘P’ is linear momentum, ‘y’ is vertical displacement and ‘x’ is horizontal displacement, then nature of the curves drawn for KE of the particle (K) against these parameters are

(i) K-y graph:

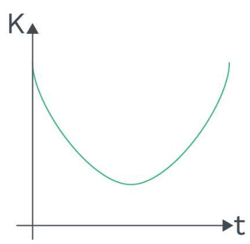
$K = K_i - mgy$; It is a straight line



(ii) K-t graph:

$$K = K_i - mg\left(u_y t - \frac{1}{2}gt^2\right)$$

$\because \quad y = u_y t - \frac{1}{2}gt^2$; It is a parabola



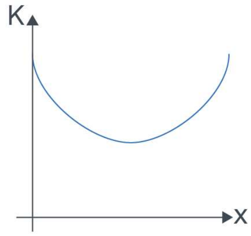
(iii) K-x graph:

$$K = K_i - mg\left(x\tan\theta - \frac{gx^2}{2u_x^2}\right)$$

Concept Reminder

The kinetic energy depends on frame of reference, e.g., KE of a person of mass m, sitting in a train moving with speed u is zero in the frame of train but $\frac{1}{2}mu^2$ in frame of earth.

$\because \quad y = (\tan\theta)x - \left(\frac{g}{2u_x^2}\right)x^2$; It is also a parabola



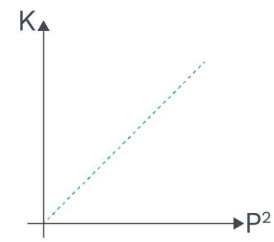
(iv) K- P^2 graph:

It is a straight line curve, which is passing through origin and

slope = $\frac{1}{2m}$

$P^2 = 2\ mK$

$P^2 \propto K$



CONSERVATIVE AND NON-CONSERVATIVE FORCES:

While analysing the problems utilise the principle of energy conservation, it is necessary to distinguish between two types of forces:

- **Conservative Forces:-**

There are two path in which we can explained a conservative force.

A force is conservative if the net work-done against the force in moving a mass between two points depends only on the location of two points and not depend on the path length followed.

OR

The net work-done against the force in moving a mass through any closed path is zero. These two criteria are equivalent. A conservative force follows both properties.

Examples of conservative forces are Gravitational force, electrostatic forces, ideal spring forces. We can always explained potential energy for

Rack your Brain

A force is conservative if the net work done against the force in moving a mass between two points depends only on the location of two points and not on the path followed.

every conservative force. Corresponding to these conservative forces we have elastic potential energy, gravitational potential energy and electrostatic potential energy.

- **Non-Conservative Forces:**
  Those forces which do not satisfy the above-mentioned criteria are not conservative in nature. Viscous forces and friction are the most common examples of non-conservative forces.

KEY POINTS

- Conservative force
- Non-conservative force
- Potential energy of spring

DIFFERENCE BETWEEN CONSERVATIVE FORCES & NON-CONSERVATIVE FORCES	
CONSERVATIVE FORCES	**NON-CONSERVATIVE FORCES**
Work done does not depend upon path.	Work done depends upon path
Work done in a round trip is zero e.g. gravitational force	Work done in a round trip is not zero e.g. friction
Central forces, spring forces etc. are conservative forces	Forces which are velocity–dependent in nature e.g. dragging force, viscous force, etc
When a conservative force acts within a system, the kinetic energy and potential energy can change. However, their sum, the mechanical energy of the system, does not change.	Work done against a non–conservative force may be dissipated as heat energy.
Work done is completely recoverable	Work done is not completely recoverable

Potential energy stored in a spring:

- The potential energy change of a system corresponding to a conservative internal force is

$$dU = -\int_0^x \vec{F}\cdot\overrightarrow{dx},$$

dU = – (work-done by the spring force)

$$dU = -\left(\frac{-Kx^2}{2}\right);\ U_f - U_i = \frac{1}{2}Kx^2$$

Since U_i is zero when spring is at its natural length

$$\therefore\quad U_f = \frac{1}{2}Kx^2$$

Ex. Two spheres balls whose radii are in the ratio 1 : 2 are moving with speeds in the ratio 3 : 4. If their densities ratio 3 : 2, then find out the ratio of their kinetic energies.

Sol. $\frac{r_1}{r_2} = \frac{1}{2}, \frac{v_1}{v_2} = \frac{3}{4}, \frac{\rho_1}{\rho_2} = \frac{3}{2}$

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}(\rho V)v^2 = \frac{1}{2}\left(\frac{4}{3}\pi r^3 \rho\right)v^2$$

$$\frac{KE_1}{KE_2} = \frac{\rho_1}{\rho_2} \times \left(\frac{r_1}{r_2}\right)^3 \times \left(\frac{v_1}{v_2}\right)^2 = \frac{3}{2} \times \left(\frac{1}{2}\right)^3 \times \left(\frac{3}{4}\right)^2$$

$$\frac{KE_1}{KE_2} = \frac{3}{2} \times \frac{1}{8} \times \frac{9}{16} = \frac{27}{256}$$

Ex. A particle is projected at 60° to the horizontal with a kinetic energy 'K'. Find the kinetic energy at the highest point?

Sol. Initial kinetic energy is $K = \frac{1}{2}mu^2$

The velocity at highest point $v_x = u \cos \theta$

Kinetic energy of a particle at highest point

$$K_H = \frac{1}{2}mv_x^2 = \frac{1}{2}mu^2 \cos^2 \theta = K \cos^2 60° = \frac{K}{4}$$

Ex. An athlete in the Olympic games covers a distance of 100 m in 10 s. His kinetic energy can be estimated to be in the range.

Sol. Approximate mass of the athlete = 60 kg

Average velocity = 10 m/s

Approximate

$$KE = \frac{1}{2}mv^2 = \frac{1}{2} \times 60 \times 10^2 = 3000\,J$$

Range of KE = 2000 J to 5000 J

Ex. Kinetic energy of an object moving along a circle of radius 'r' depends on the distance as KE = cs^2, (c is constant, s is displacement). Find the force acting on the object.

Sol. $\frac{1}{2}mv^2 = cs^2 \quad \Rightarrow v = \left(\sqrt{\frac{2c}{m}}\right)s$

$$a_t = \frac{dv}{dt} = \sqrt{\frac{2c}{m}} \times \frac{ds}{dt} = v\sqrt{\frac{2c}{m}}$$

$$F_t = ma_t = mv\sqrt{\frac{2c}{m}} = \left[m\sqrt{\frac{2c}{m}}s\right]\sqrt{\frac{2c}{m}} = 2cs$$

Total force

$$F = \sqrt{F_t^2 + F_c^2} = \sqrt{(2cs)^2 + \left(\frac{mv^2}{r}\right)^2}$$

$$F = 2cs\sqrt{1 + \frac{s^2}{r^2}}$$

Work-Energy Theorem:

- Work-done by all forces acting on a body is equal to change in its kinetic energy.

  i.e., $W = K_f - K_i = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$

  Where K_i and K_f are the initial and final kinetic energies of the body.
- Work-energy theorem is applicable not only for a single body but also for a system of particles.
- This theorem is applied to a system of two or more bodies change in kinetic energy of the system is equal to work done on the system by the external as well as internal forces.
- Work energy theorem can also be applied to a system under the action of variable forces, pseudo forces, conservative as well as non-conservative forces.

Concept Reminder

According to work-energy theorem

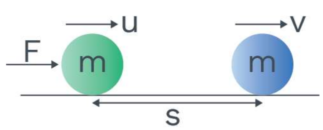
$$W = k_f - k_i = \frac{1}{2}m(v^2 - u^2)$$

Concept Reminder

In uniform circular motion work done by centripetal force is always zero. So, kinetic energy and speed remains constant.

Proof:

For constant force:



$v^2 = u^2 + 2as$

$$\Rightarrow \quad v^2 - u^2 = 2\left(\frac{F}{m}\right)s$$

$$\Rightarrow \quad s = \frac{(v^2 - u^2)m}{2F} = \frac{\Delta KE}{F}$$

$\Rightarrow$ Fs = ΔKE $\qquad \Rightarrow$ W = ΔKE

For variable force:

$$W = \int F dx = \int ma\, dx = \int m\left(\frac{v\,dv}{dx}\right)dx$$

$$= \int_u^v mv\, dv = m\left[\frac{v^2}{2}\right]_u^v = \frac{m}{2}[v^2 - u^2]$$

$\Rightarrow$ W = ΔKE

How to use work-energy theorem:

- Identify the initial (1) and final (2) positions of a system or a particle then write expressions for initial and final kinetic energy.
- Write work done by all the forces acting on the body during it goes from initial to final position.
- Equate total work-done to change in kinetic energy

  W.D. = $KE_2 - KE_1$

Applications of work-energy theorem:

- A body of mass 'm' starting from rest acquire a velocity 'v' due to constant force F. Neglecting air resistance.

  Work done = change in kinetic energy = $\frac{1}{2}mv^2$

- A body of mass m is thrown vertically up with a speed u. Neglecting the air friction, the work-done by gravitational force, as body reaches maximum height is

$$W_g = \Delta K = K_f - K_i$$

$$W_g = \frac{1}{2}m(0)^2 - \frac{1}{2}m \times u^2 = -\frac{1}{2}mu^2$$

Rack your Brain

A body of mass 3 kg is under a constant force which causes a displacement s in meters in it, given by relation $s = \frac{1}{3}t^2$, where t is in seconds. Find work done by the force in 2 s.

KEY POINTS

- Work-Energy theorem
- Applications of work-energy theorem

- A particle of mass m falls freely from a height h in air medium onto the ground. If v is the velocity with which it reaches the ground, the work done by air friction is W_f and work done by gravitational force W_g then,

$$W_g + W_f = \frac{1}{2}mv^2 - 0 = \frac{1}{2}mv^2$$

- A block of mass m slides down a frictionless incline of inclination 'θ' to the horizontal. If 'h' is the height of incline, the velocity with which body reaches the bottom of incline is

$$W_g = \Delta K; \quad mgh = \frac{1}{2}mv^2 - 0$$

$$mgh = \frac{1}{2}mv^2; \quad v = \sqrt{2gh}$$

- A block of mass m starts from rest (initial speed = 0) from the top of a rough inclined plane of inclination θ and length l. The velocity v with which it reaches the bottom of incline if μ_k is the coefficient of kinetic friction is

$$W_g + W_f = \Delta k$$

$$(mg\sin\theta)l + (-\mu_k mg\cos\theta)l = \frac{1}{2}mv^2 - 0$$

$$v = \sqrt{2gl(\sin\theta - \mu_k\cos\theta)}$$

- A bob of mass m suspended from a string of length l is given a speed u at its lowest position then the speed of the bob v when it makes an angle θ with the vertical is

$$W_g + W_r = \Delta K$$

$$\Rightarrow \quad -mgl(1-\cos\theta) + 0 = \frac{1}{2}m(v^2 - u^2)$$

$$v = \sqrt{u^2 - 2gl(1-\cos\theta)}$$

- A m mass bullet moving with velocity v stops in a wooden block after penetrating through a distance x. If f is the resistance offered by the block to the bullet.

Concept Reminder

Work-energy theorem is applicable for all types of forces i.e. external, internal, conservative, non-conservative, pseudo forces.

Rack your Brain

Consider a drop of rain water having mass 1 g falling from a height of 1 km. It hits the ground with speed of 50 ms^{-1}. The work done by the:

(a) Gravitational force

(b) Resistive force of air

$$W_f = K_f - K_i; \quad -fx = 0 - KE_i$$

i.e., stopping distance

$$x = \frac{KE_i}{f} = \frac{mv^2}{2f} = \frac{P^2}{2mf}$$

- A block of mass 'm' attached to a spring of spring constant 'K' oscillates on a smooth horizontal table. The other end of the spring is fixed to a wall. It has a speed 'v' when the spring is at natural length. The distance it moves on a table before it comes to rest is calculated as below

$$W_{SF} + W_g + W_N = \Delta K \quad \text{(SF = spring force)}$$

Let the mass be oscillating with amplitude 'x'.
On compressing the spring

$$W_{SF} = -\frac{1}{2}Kx^2, \quad W_g = FS\cos 90°;$$

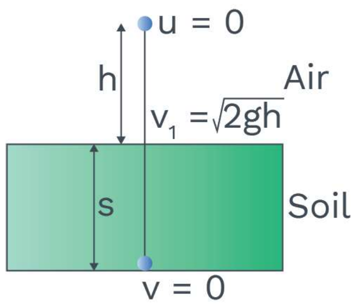
$$W_N = NS\cos 90° = 0$$

$$W_{SF} = K_f - K_i$$

$$\Rightarrow \quad -\frac{1}{2}Kx^2 = 0 - \frac{1}{2}mv^2 \quad \Rightarrow x = v\sqrt{\frac{m}{K}}$$

$$mg(h+s) + (-Rs) = 0 \quad \Rightarrow R = mg\left(1 + \frac{h}{s}\right)$$

R = resistive force of soil in below figure

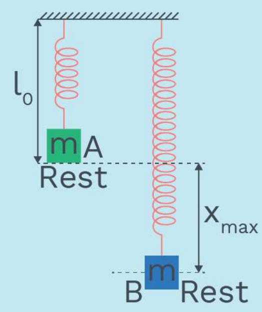


In this time of penetration is given by impulse

$$(R - mg)t = 0 - m\sqrt{2gh} \Rightarrow t = \frac{m\sqrt{2gh}}{mg - R}$$

Concept Reminder

Maximum extension in a spring hanged vertically is $x_{max} = \frac{2mg}{K}$



- A body of mass m is initially at rest. By the use of a constant force, its speed changes to v_0 in time to the kinetic energy of the body at time t is

$$W = \Delta K = K_f - K_i = K_f - 0$$

$$K_f = W = mas = ma\left(\frac{1}{2}at^2\right) = \frac{1}{2}ma^2t^2$$

Since, $a = \frac{v_0}{t_0}$; $K_f = \frac{1}{2}m\left(\frac{v_0}{t_0}\right)^2 t^2$

Ex. A uniform chain of length 'l' and mass 'M' is on a smooth horizontal table, with $\frac{1}{n^{th}}$ part of its length hanging from the edge of the table. Find out the kinetic energy of the chain as it completely slips off the table.

Sol. Work done

$$\Delta W = U_i - U_f = K_f - K_i$$

$$\frac{Mgl}{2} - \frac{Mgl}{2n^2} = \frac{1}{2}Mv^2;$$

$$v = \sqrt{gl\left[1 - \frac{1}{n^2}\right]}$$

Ex. A 2kg block slides on a horizontal floor with a speed of 4 m/s. It strikes an uncompressed spring and compresses it till the block is motionless. The kinetic frictional force is 15 N and spring constant is 10,000 Nm^{-1}. Find the compression in the spring?

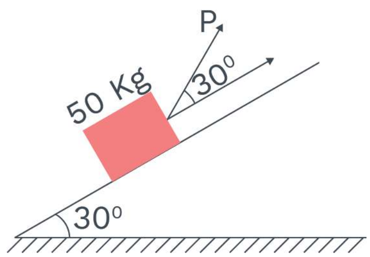
Sol. $KE = \frac{1}{2}mv^2 = W_{friction} + \frac{1}{2}Kx^2$

$\Rightarrow \quad \frac{1}{2} \times 2 \times 4^2 = 15x + \frac{1}{2} \times 10000 \times x^2$

$\Rightarrow \quad 5000x^2 + 15x - 16 = 0$

$\Rightarrow \quad x = 0.055$ m or $x = 5.5$ cm

Ex. In the below figure, what constant force 'P' is required to bring the 50 kg body, which starts from rest to a velocity of 10 m/s in moving 7 m along the plane? (Neglect friction)



Sol. Work done by force P in displacing the block by 7 m,

$$W_1 = (F\cos\theta)(s)$$

$$W_1 = (P\cos 30°)7 = \frac{7\sqrt{3}}{2}PJ$$

$$W_2 = -mgh = -50 \times 9.8 \times 7\sin 30° = -1715 \text{ J}$$

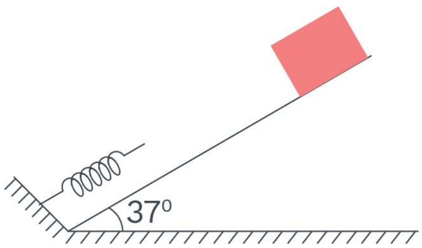
According to work energy theorem

$$W_1 + W_2 = \frac{1}{2}m(v_2^2 - v_1^2)$$

$$\frac{7\sqrt{3}}{2}P - 1715 = \frac{1}{2} \times 50 \times (10^2 - 0^2)$$

$\Rightarrow \quad P = 607$ N

Ex. Figure shown a spring fixed at the bottom end of an incline of inclination 37°. A small body of mass 2 kg starts slipping down the incline from a point 4.8 m away from the spring. The body compresses the spring by 20 cm, stops momentarily and then rebounds through a distance 1 m up the incline. Find (i) the friction coefficient between the plane and the body and (ii) the spring constant of the spring. ($g = 10$ ms^{-2})



Sol. Applying work energy theorem for downward motion of the body

$W = \Delta KE$

$$mg\sin\theta(x+d) - f\times l_1 - \frac{1}{2}Kx^2 = \Delta KE$$

$$20\sin 37°(5) - \mu\times 20\cos 37°\times 5 - \frac{1}{2}K(0.2)^2 = 0$$

$$80\mu + 0.02K = 60 \quad \text{...(i)}$$

For the upward motion of the body

$$-mg\sin\theta l_2 + (f\times l_2) + \frac{1}{2}Kx^2 = \Delta KE$$

$$-2\times 10\sin 37°\times 1 - \mu\times 20\cos 37°\times 1 + \frac{1}{2}K(0.2)^2 = 0$$

$$16\mu - 0.02K = -12 \quad \text{...(ii)}$$

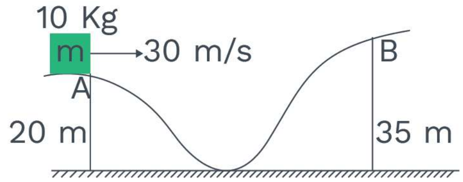
Adding equations (i) and (ii), we get

$$96\mu = 48 \quad \Rightarrow \mu = 0.5$$

Now, use the value of μ in equation (i), we get

K = 1000 N/m.

Ex. A block has given a velocity 30 m/s on a smooth track. Find velocity when it reaches point B.



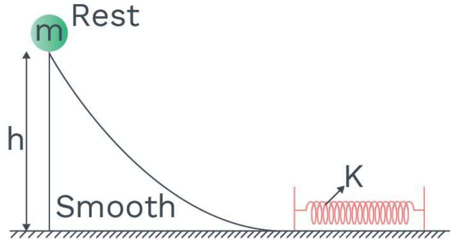
Sol. $WD = \Delta K$

$WD_{mg} = \Delta K$

$$-10\times 10\times[35-20] = \frac{1}{2}\times 10[v_f^2 - (30)^2]$$

$$V_f = \sqrt{600}\ m/s$$

Ex. Find maximum compression of spring during subsequent motion ?



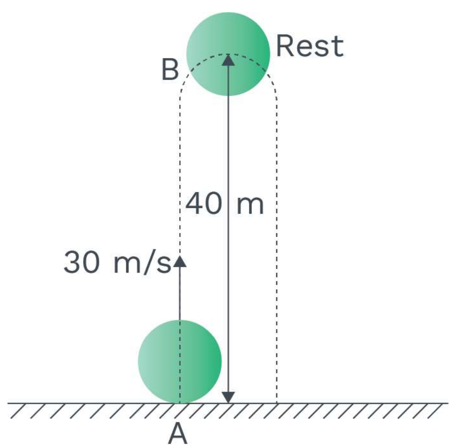
Sol. $W_N + W_g + W_s = \frac{1}{2}m[v_f^2 - v_i^2]$

$0 + mgh + \frac{1}{2}k[O - x_{max}^2] = 0 - 0$

$mgh = \frac{1}{2}kx_{max}^2 \quad \Rightarrow x_{max} = \sqrt{\frac{2mgh}{k}}$

Ex. A particle of mass 2kg is projected upwards with velocity 30 m/sec. If it attains a maximum height of 40 m, then find work done by air resistance.

Sol. From A to B

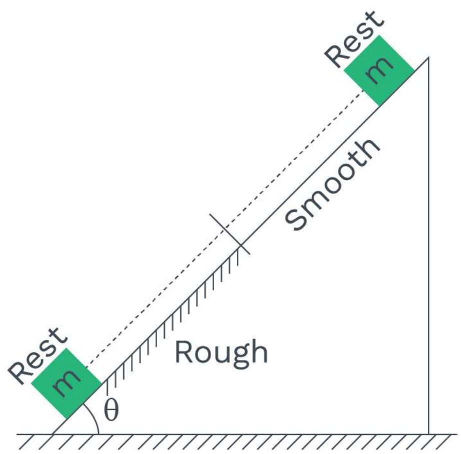


$W_g + W_{air} = \frac{1}{2}m[v_f^2 - v_i^2] \;\Rightarrow\; -2 \times 10 \times 40 + W_{air} = \frac{1}{2} \times 2[0^2 - 30^2]$

$-800 + W_{air} = -900$

$W_{air} = -900 + 800 = -100$ J

Ex. If first half part of inclined plane is smooth and next half part is rough and block start from rest and again comes to rest. Just before reaching at bottom of inclined plane then find μ of rough part?

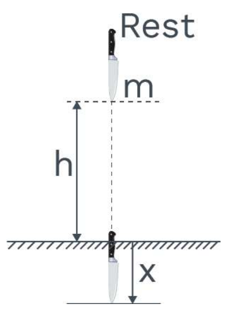


Sol. $W_N + W_g + W_f = m(v_f^2 - v_i^2)$

$0 + mg\sin\theta \times 2L - \mu mg\cos\theta \times L = 0 - 0$

$\Rightarrow$ $\mu = 2\tan\theta$

Ex. Knife is released from rest from a height h and it again comes to rest after entering the ground to a depth x. Then find resistance force offered by ground to knife?

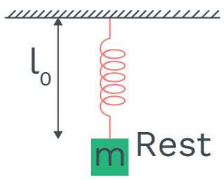


Sol. $W_g + W_F = \frac{1}{2}m(v_f^2 - v_i^2)$

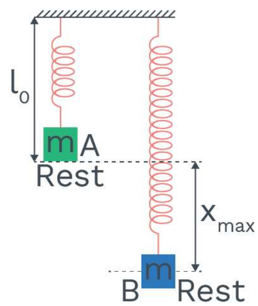
$mg(h + x) - Fx = 0 - 0$

$mg\left(\frac{h}{x} + 1\right) = F$

Ex. Block is released from rest when spring is at natural length l_0. Then find maximum elongation of spring during subsequent motion?



Sol. Apply W.E. theorem from position A to B:



$$W_g + W_s = \frac{1}{2}m[v_f^2 - v_i^2]$$

$$mg \times x_{max} = \frac{1}{2}kx_{max}^2$$

$$\frac{2mg}{k} = x_{max}$$

Ex. Velocity of a particle of mass 0.5 kg depends on position x as, $v = 5\, x^{3/2}$ m/sec. Find total work done on particle from x = 0 to x = 2 m ?

Sol. m = 0.5 kg

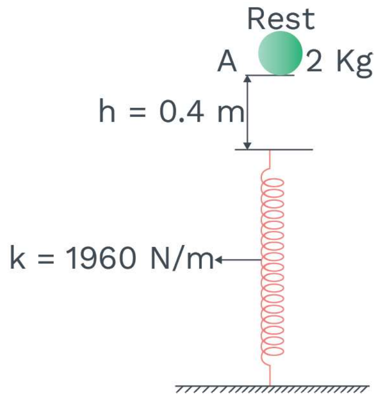
at $x = 0,\ v_i = 5(0)^{3/2} = 0$

at $x = 2$ m, $v_f = 5(2)^{3/2} = 10\sqrt{2}$ m/sec

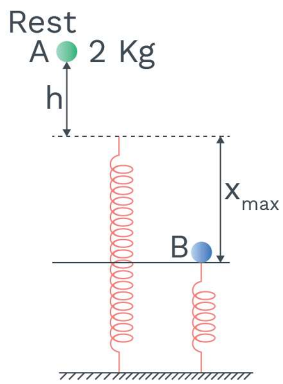
so, $W_{\text{all foces}} = \frac{1}{2}m[v_f^2 - v_i^2]$

$$= \frac{1}{2} \times 0.5[200 - 0] = 50 \text{ J}$$

Ex. Find maximum compression of spring during subsequent motion. If particle is released from rest from point A, which is at a height 0.4 m from free end of the spring. (Take g = 9.8 m/sec^2)



Sol. Apply W.E theorem from position A to B:



$$W_g + W_s = \frac{1}{2}m[v_f^2 - v_i^2]$$

$$mg(h + x_{max}) + \frac{1}{2}k[0^2 - x_{max}^2] = 0 - 0$$

$$2 \times 9.8(0.4 + x_{max}) + \frac{1}{2} \times 1960(0 - x_{max}^2) = 0$$

$$19.6(0.4 + x_{max}) = \frac{1}{2} \times 1960 \times x_{max}^2$$

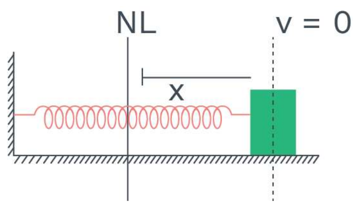
$$50\, x_{max}^2 - x_{max} - 0.4 = 0$$

So, $x_{max} = \frac{1}{10}$ m or 10 cm

Ex. A constant force is applied on the spring block system and displaces it from natural length. Find maximum elongation if spring constant is K.

Sol. WD = ΔK

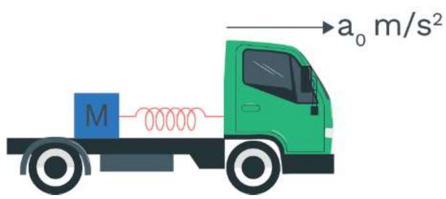
$(WD)_s + (WD)_f = K_f - K_i$



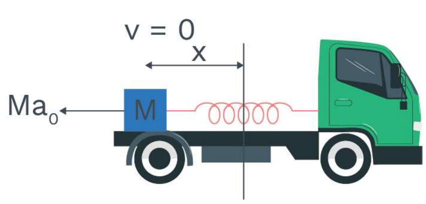
$$-\frac{1}{2}Kx^2 + F \cdot x = 0$$

$$x = \frac{2F}{K}$$

Ex. If truck start from rest with constant acceleration a_0. Find the maximum elongation in the spring of spring constant K.



Sol. Applying W.E. Theorem in Truck's reference frame



$$WD_{spring} + WD_{pseudo\ force} = \Delta K$$

$$-\frac{1}{2}Kx^2 + Ma_0 \cdot x = 0$$

$$x = \frac{2Ma_0}{K}$$

POTENTIAL ENERGY AND EQUILIBRIUM

A body is known as in translatory equilibrium, if net force acting on the body is zero i.e., $\vec{F}_{net} = 0$

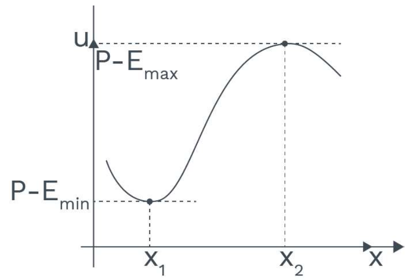
If the forces are conservative

$$F = -\frac{dU}{dr}$$

and for equilibrium F = 0,

so, $-\frac{dU}{dr} = 0 \Rightarrow \frac{dU}{dr} = 0$

At equilibrium position point slope of U-r graph is zero or the potential energy is optimum (minimum or maximum or constant)



TYPES OF EQUILIBRIUM

There are three types of equilibrium-

(i) Stable equilibrium

(ii) Unstable equilibrium

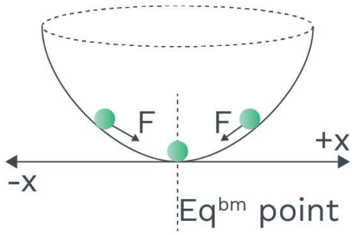
(iii) Neutral equilibrium

KEY POINTS

- Stable equilibrium
- Unstable equilibrium
- Neutral equilibrium

Stable equilibrium:

- Net force is zero.
- $\frac{dU}{dr} = 0$ or slope of U-r graph is zero.
- When displaced from its equilibrium position point , a net retarding forces starts acting on the particle, which has a tendency to bring the particle back to its equilibrium position point.
- PE in equilibrium position point is minimum as compared to its neighbouring points as $\frac{d^2U}{dr^2}$ is positive.
- When displaced from equilibrium position point the centre of gravity of the body comes down.



Concept Reminder

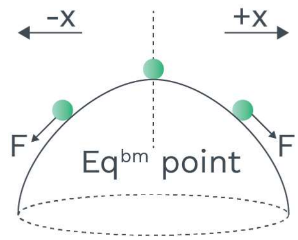
For stable equilibrium

$$\frac{dU}{dr} = 0 \text{ and } \frac{d^2U}{dr^2} > 0$$

Unstable equilibrium:

Net force is zero.

- $\frac{dU}{dr} = 0$ or slope of U-r graph is zero.
- When displaced from its equilibrium position point, a net force starts acting on the object which moves the object in the direction of displacement or away from the equilibrium position point.
- PE in equilibrium position point is maximum as compared to other positions as $\frac{d^2U}{dr^2}$ is negative.
- When displaced from equilibrium position point the centre of gravity of the body goes up.



Neutral equilibrium:

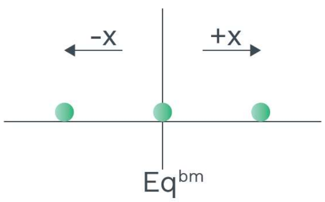
Net force is zero.

$\frac{dU}{dr} = 0$ or slope of U-r graph is zero.

When displaced from its equilibrium position point the body has neither the tendency to come back nor move away from the original position.

PE remains constant even if the body is moving to neighbouring points $\frac{d^2U}{dr^2} = 0$.

When displaced from equilibrium position point the centre of gravity of the body remains constant.



Concept Reminder

For unstable equilibrium

$\frac{dU}{dr} = 0$ and $\frac{d^2U}{dr^2} < 0$

Concept Reminder

For neutral equilibrium

$\frac{dU}{dr} = 0$ and $\frac{d^2U}{dr^2} = 0$

S.NO.	STABLE EQUILIBRIUM	UNSTABLE EQUILIBRIUM	NEUTRAL EQUILIBRIUM
1.	When displaced slightly, from its equilibrium position a net restoring force starts acting on the body which has a tendency to bring the body back to its equilibrium position.	When displaced slightly from its equilibrium position, a net force starts acting on the body which moves the body in the direction of displacement or away from the equilibrium position.	When displaced slightly from its equilibrium position the body has neither the tendency to come back to original position nor to move away from the original position.
2.	Potential energy in equilibrium position is minimum as compared to its neighboring points or $\frac{d^2U}{dr^2}$ = positive	Potential energy in equilibrium position is maximum as compared to its neighboring points or $\frac{d^2U}{dr^2}$ = negative	Potential energy remains constant even if the body is displaced from its equilibrium position or $\frac{d^2U}{dr^2} = 0$
3.	When displaced from equilibrium position the centre of gravity of the body goes up.	When displaced from equilibrium position the centre of gravity of the body comes down.	When displaced from equilibrium position the centre of gravity of the body remains at the same level.

Law of Conservation of Mechanical Energy:

- A system total mechanical energy is remain constant, if only conservative forces are acting on a system of particles and the work done by all other forces is zero.

  $\therefore \quad U_f - U_i = -W$

  From work energy theorem

  $W = k_f - k_i$

  $\therefore \quad U_f - U_i = -(k_f - k_i)$

  $\therefore \quad U_f + k_f = U_i + k_i$

  $\Rightarrow \quad U + K = \text{constant.}$

  The addition of kinetic energy and potential energy remains constant in any state.

- A object is projected vertically up from the ground. When it is at height 'h' point above the ground, its potential and kinetic energies are in the ratio x : y. If 'H' is the maximum height reached by the body, then

$$\frac{x}{y} = \frac{h}{H-h} \Rightarrow \frac{h}{H} = \frac{x}{x+y}$$

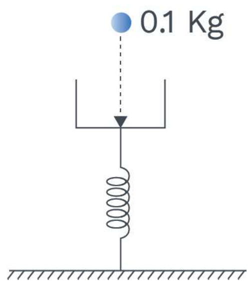
Definitions

- Mechanical energy E of a particle, object or system is defined as sum of kinetic energy K and potential energy U.
- E = K + U

Rack your Brain

The potential energy of particle in a force field is $U = \frac{A}{r^2} - \frac{B}{r}$ where A and B are positive constants and r is the distance of the particle from the center of field. Find the distance of particle for stable equilibrium.

Ex. A massless platform is kept on a light elastic spring as shown in figure. When a sand particle of 0.1kg mass is dropped on the pan from a height of 0.24m, the particle strikes the pan and the spring compresses by 0.01m. From what height should particle be dropped to cause a compression of 0.04 m.



Sol. By conservation of mechanical energy

$$mg(h+y) = \frac{1}{2}Ky^2$$

h = height of particle

y = compression of the spring

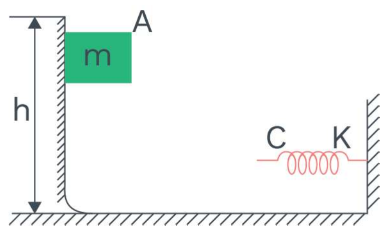
As here particle and spring remain same

$$\frac{h_1 + y_1}{h_2 + y_2} = \left(\frac{y_1}{y_2}\right)^2$$

$$\Rightarrow \quad \frac{0.24+0.01}{h_2+0.04} = \left(\frac{0.01}{0.04}\right)^2$$

$$\Rightarrow \quad h_2 = 3.96 \text{ m}$$

Ex. A small mass 'm' is sliding down on a smooth curved incline from a height 'h' and finally moves through a horizontal smooth surface. A light (massless) spring of force constant 'K' is fixed with a vertical rigid stand on the horizontal surface, as shown in the figure. Find the value for the maximum compression in the spring if mass 'm' is released from rest from height 'h' and hits the spring on the horizontal surface.



Sol. Conservation of energy b/w positions A and C

$$(PE_A)_{block} + KE_A = (PE_C)_{spring} + KE_C$$

$$mgh + 0 = \frac{1}{2}Kx^2 + 0$$

$$mgh = \frac{1}{2}Kx^2; \quad x = \sqrt{\frac{2mgh}{K}}$$

Ex. A vehicle of mass 1500 kg climbs up a hill 200 metre high. It then travelled on a level road with a speed of 30 ms^{-1}. Find out the potential energy gained by it and its total mechanical energy while running on the top of the hill.

Sol. m = 1500kg,

g= 9.8 ms^{-2}, h = 200m

Potential Energy gained,

U = 1500 × 9.8 × 200 = 2.94 × 10^6 J

$$KE = \frac{1}{2}mv^2 = \frac{1}{2} \times 1500 \times (30)^2$$

$$= 0.675 \times 10^6 \text{ J}$$

Total mechanical energy is

E = (0.675 + 2.94) × 10^6 = 3.615 × 10^6J

Ex. A particle is released from height H. At certain height from the ground its kinetic energy is twice its gravitational potential energy. Find the height and speed of particle at that height.

Sol. K.E = 2 PE

But KE = TE – PE

mg(H – h) = 2 mgh; mgH = 3 mgh

$$\Rightarrow \quad h = \frac{H}{3};$$

Also, KE = 2 PE,

$$\frac{1}{2}mv^2 = 2mg\left(\frac{H}{3}\right)$$

$$\Rightarrow \quad v = 2\sqrt{\frac{gH}{3}}$$

Ex. The potential energy of one kilogram particle free to move along X - axis is given by $U(x)=\left(\frac{x^4}{4}-\frac{x^2}{2}\right)$ J. The total mechanical energy of the particle is 2 J. Find the maximum speed of the particle.

Sol. For maximum value of U, $\frac{dU}{dx}=0$

$$\therefore \quad \frac{4x^3}{4}-\frac{2x}{2}=0 \text{ or } x=0,\ x=\pm 1$$

At x = 0,

$$\frac{d^2U}{dx^2}=-1$$

At x = ± 1,

$$\frac{d^2U}{dx^2}=2$$

Hence U is minimum at x = ± 1 with value

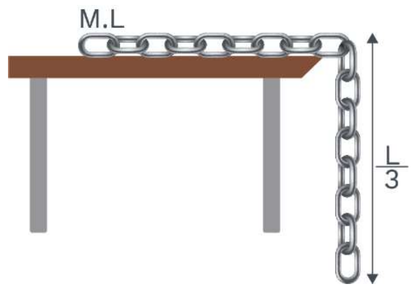
$$U_{min}=\frac{1}{4}-\frac{1}{2}=-\frac{1}{4}J$$

$$K_{max}+U_{min}=E$$

$$\text{or} \quad K_{max}=\frac{1}{4}+2 \Rightarrow K_{max}=\frac{9}{4}$$

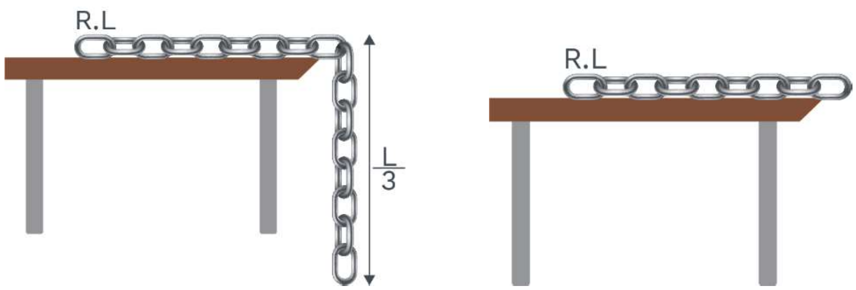
$$\Rightarrow \quad \frac{1}{2}mv^2=\frac{9}{4} \Rightarrow v_{max}=\frac{3}{\sqrt{2}}ms^{-1}$$

Ex. Find W.D in lifting full length of chain on table ?



Sol. $U_i=0-\left(\frac{M}{3}\right)g\times\left(\frac{L}{6}\right);\quad U_f=0$

$$WD = U_f - U_i = 0 - \left(-\frac{MgL}{18}\right) = \frac{MgL}{18}$$



Ex. Find W.D in stretching a storing of spring constant K from extension x_0 to extension 3 x_0 ?

Sol. WD by us against spring = $\Delta U = U_f - U_i$

$$= \frac{1}{2}k(3x_0)^2 - \frac{1}{2}k(x_0)^2$$

$$= \frac{1}{2}k[9x_0^2 - x_0^2]$$

$$= \frac{1}{2}k(8x_0^2) = 4kx_0^2$$

Ex. Two springs have spring constants k_1 and k_2. Then find ratio of W.D in stretching these springs:

(i) If they are stretched to same extension.

(ii) If they are stretched by same stretching force.

Sol. (i) $WD = \frac{1}{2}kx^2$

If x = same, then $WD \propto k$

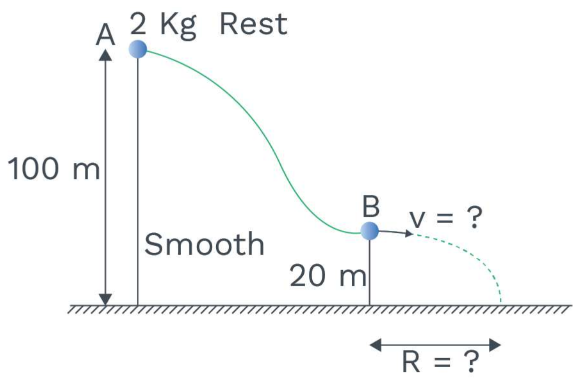
$$\Rightarrow \frac{W_1}{W_2} = \frac{k_1}{k_2}$$

(ii) $WD = \frac{1}{2}kx^2 = \frac{1}{2}\frac{k^2x^2}{k} = \frac{F^2}{2k}$

If stretching force, F = same then $WD \propto \frac{1}{k}$

$$\Rightarrow \frac{W_1}{W_2} = \frac{k_2}{k_1}$$

Ex. Particle is released from rest from top of inclined plane then final value of R?



Sol. Apply come from A to B-

$$k_1 + U_1 = k_2 + U_2$$

$$0 + 2 \times 10 \times 100 = \frac{1}{2} \times 2 \times v^2 + 2 \times 10 \times 20$$

$\Rightarrow$ $2000 - 400 = v^2$

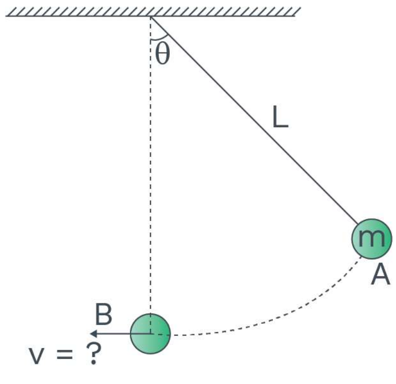
$1600 = v^2$

$v = 40$ m/sec

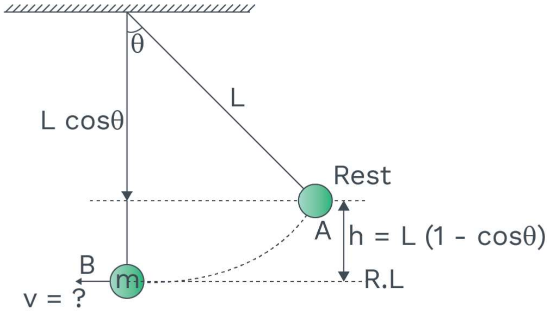
and $R = v\sqrt{\frac{2h}{g}} = 40\sqrt{\frac{2 \times 20}{10}} = 40 \times 2$

$\Rightarrow$ $R = 80$ m

Ex. Particle is released from rest from position A then find velocity of particle when string becomes vertical ?



Sol. Apply come from A to B-



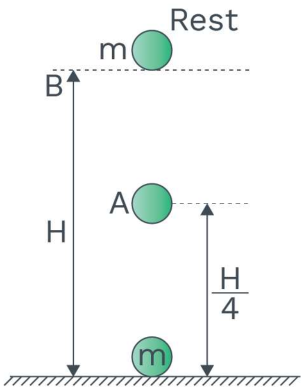
$$K_1 + U_1 = K_2 + U_2$$

$$0 + mg \times L(1 - \cos\theta) = \frac{1}{2}mv^2 + 0$$

$$\sqrt{2gL(1 - \cos\theta)} = v$$

Ex. A particle of mass 'm' is projected vertically upwards so that it reach to a maximum height 'H', then find ratio of its K.E to P.E when it was at a height $\frac{H}{4}$ from ground ?

Sol. Apply come from A to B-



$$K_1 + U_1 = K_2 + U_2$$

$$K_1 + \frac{mgH}{4} = 0 + mgH$$

$$\Rightarrow \quad K_1 = mgH - \frac{mgH}{4}$$

$$\text{So,} \quad \frac{K_1}{U_1} = \frac{\frac{3mgH}{4}}{\frac{mgH}{4}} = \frac{3}{1}$$

Ex. A chain of mass 'm' and length L is held on a frictionless table in such a way that $1/n^{th}$ part is hanging below the edge of table. Calculate the work done to pull the hanging part of the chain.

Sol. Required work done = change in potential energy of chain

Concept Reminder

$$P_{inst} = \frac{dW}{dt} = \vec{F}\cdot\vec{v}$$

Now, let Potential energy (U) = 0 at table level

So potential energies of chain initially and finally are respectively

$$U_i = -mg\left(\frac{L}{2n}\right) = -\left(\frac{M}{L}\right)\frac{L}{n}g\left(\frac{L}{2n}\right) = -\frac{MgL}{2n^2},\quad U_f = 0$$

$$\therefore \text{ required work done } = U_f - U_i = \frac{MgL}{2n^2}$$

POWER

- The rate of doing work is called power. Power or average power is given by

$$P_{avg} = \frac{\text{work done}}{\text{time}}$$

Power is a scalar quantity.

SI Unit: watt (W) or J/s.

CGS Unit: erg/sec.

Other units: mega watt, kilo watt and horse power.

One horse power (hp) = 746 watt

- **Instantaneous Power:**

$$P = \underset{\Delta t \to 0}{Lt}\left(\frac{\Delta W}{\Delta t}\right)$$

It is also calculated by

$$P = Fv\cos\theta = \vec{F}\cdot\vec{v}$$

- **Relation between P_{avg} and P_{inst} :**

$$P_{avg} = \frac{W}{t} = \frac{mv^2}{2t} = \frac{1}{2}mv\left(\frac{v}{t}\right) = \frac{1}{2}mav = \frac{1}{2}\vec{F}\cdot\vec{v}$$

$$P_{avg} = \frac{1}{2}P_{inst}$$

Rack your Brain

If $\vec{F} = (60\hat{i} + 15\hat{j} - 3\hat{k})N$ and $\vec{v} = (2\hat{i} - 4\hat{j} + 5\hat{k})$ m/s. Then find out instantaneous power.

KEY POINTS

- Average power
- Instantaneous power

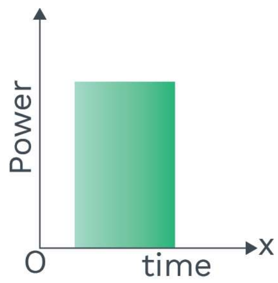
- The area under P-t graph gives work done

$$P = \frac{dW}{dt}$$

$$\therefore \quad W = \int P\, dt$$

The slope of W-t curve represent instantaneous power

$$P = \frac{dW}{dt} = \tan\theta$$



Concept Reminder

Practical uniform power is horse-power (hp)

1 hp = 746 watt

Applications on Power:

- The power of a machine gun firing n bullets each of mass m with a velocity 'v' in a time interval 't' is given by

$$P = \frac{n\left(\frac{1}{2}mv^2\right)}{t} = \frac{nmv^2}{2t}$$

- A crane lifts a body of mass 'm' with a constant velocity v from the ground, its power is $P = Fv = mgv$
- Power of lungs of a boy blowing a whistle is $P = \frac{1}{2}$ (mass of air blown per sec) $(\text{velocity})^2$.
- Power of a heart pumping blood = (pressure) (volume of blood pumped/sec).
- A conveyor belt is running with a constant speed v horizontally and gravel is falling on it at a rate

Rack your Brain

The heat of a man pumps 5 litres of blood through arteries per minute of a pressure of 150 mm of mercury. If the density of mercury be 13.6×10^3 kg/m^3 and g = 10 m/s^2 then find power (in watt) of $\frac{dm}{dt}$. Then additional force required to maintain speed v is $F = v\frac{dm}{dt}$ and additional power required to drive the belt is,

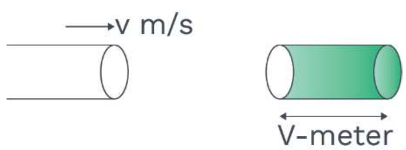
$$P = Fv = v^2\frac{dm}{dt}$$

- **Power of pump:**
  As variable mass system with constant velocity

$$\vec{F} = \vec{v}\cdot\frac{dm}{dt}$$

$$Power_{pump} = \vec{F}\cdot\vec{v}$$

$$\vec{v}\cdot\frac{dm}{dt}\cdot\vec{v} = v^2\frac{dm}{dt}$$



Area of cross section = a m^2
Density of fluid = ρ kg/m^3

mass is 1 second = $\frac{dm}{dt} = \rho av$

Force on fluid by pump = $v\frac{dm}{dt} = \rho av^2$

Power of pump = $v^2\frac{dm}{dt} = \rho av^3$

- A vehicle of mass m is driven with constant acceleration along a straight level road against a constant external resistance R when the velocity is v.
  Power of engine is

$$P = Fv = (R + ma)v$$

- If rated power P of a device and if its efficiency is x%, useful power is (output power)

Concept Reminder

If a pump motor is used to deliver water at a certain rate from a given pipe then to obtain 'n' times water from same pipe in same time power of motor should be increased by n^3 times.

$$P' = \frac{x}{100}P$$

- If a motor lifts water (liquid) from a well of depth 'h' and delivers with a velocity 'v' in a time t then power of the motor

$$P = \frac{mgh + \frac{1}{2}mv^2}{t}$$

- If a particle of mass 'm' starts from rest and accelerated uniformly to a velocity v_0 in a time t_0, then the work done on the body in a time 't' is given by

$$W = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{v_0 t}{t_0}\right)^2$$

$$a = \frac{v_0}{t_0}; \quad v = at = \left(\frac{v_0}{t_0}\right)t$$

Instantaneous power,

$$P = Fv = mav$$

$$\therefore \quad P = m\frac{v_0}{t_0}\left(\frac{v_0}{t_0}\right)t = m\frac{v_0^2}{t_0^2}t$$

- A motor pump is useful to deliver water (liquid) at a certain rate from a given pipe. To obtain n times water from the same pipe at the same time by what amount of (a) force and (b) power of the motor should be increased. If a liquid of density 'ρ' is flowing through a pipe of cross section 'A' at speed 'v' the mass coming out per second will be

$$\frac{dm}{dt} = Av\rho$$

To get 'n' times water at the same time

$$\left(\frac{dm}{dt}\right)' = n\left(\frac{dm}{dt}\right)$$

$$\Rightarrow \quad A'v'\rho' = n(Av\rho)$$

As the pipe and liquid are not changed,

$$\rho = \rho; \; A' = A \text{ and } v' = nv$$

Rack your Brain

How much water a pump of 2 kW can raise in one minute to a height of 10 m? (take $g = 10\ ms^{-2}$)

as $F = v\frac{dm}{dt}$

$$\Rightarrow \quad \frac{F'}{F} = \frac{v\left(\frac{dm}{dt}\right)'}{v\left(\frac{dm}{dt}\right)} = \frac{(nv)\left(n\frac{dm}{dt}\right)}{v\left(\frac{dm}{dt}\right)} = n^2$$

as P = Fv

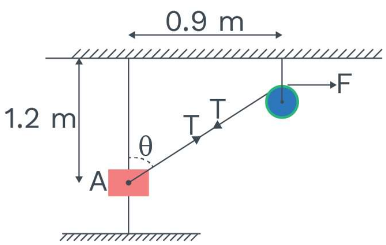
$$\Rightarrow \quad \frac{P'}{P} = \frac{F'v'}{Fv} = \frac{(n^2F)(nv)}{Fv} = n^3$$

$\therefore \quad F' = n^2F$

$\therefore \quad P' = n^3P$

To get n times of water force must be increased n^2 times while power n^3 times.

Ex. The 50 N collar starts from rest at A and is lifted with a constant speed of 0.6 m/s along the smooth rod. Determine the power developed by the force F at the instant shown.



Sol. Since the collar is lifted with a constant speed

$T\cos\theta - mg = 0$

$\Rightarrow \quad T\cos\theta = mg = 5 \times 10$

Now, $P = \vec{F}\cdot\vec{v} = T\cos\theta \times v$

Here T = F

$P = 50 \times v = 50 \times 0.6 = 30\,W$

Ex. A machine delivers power to an object which is directly proportional to velocity of the object. If the object starts with a velocity which is almost negligible, find the distance covered by the object in attaining a velocity v.

Sol. Power

$$P = Fv\cos\theta = Fv = m\left(\frac{dv}{dt}\right)v \Rightarrow P \propto v$$

$$mv\frac{dv}{dt} = K_0 v$$

Where K_0 = constant

$$m\frac{dv}{dt} = K_0 \Rightarrow m\left(\frac{dv}{dx}\right)\frac{dx}{dt} = K_0$$

$$mv\frac{dv}{dx} = K_0 \quad \Rightarrow vdv = \left(\frac{K_0}{m}\right)dx$$

Integrating

$$\frac{v^2}{2} = \left(\frac{K_0}{2}\right)x \quad \Rightarrow x = \frac{1}{2}\frac{mv^2}{K_0}$$

Ex. Find out the power of an engine which can draw a train of '400' metric ton up the inclined plane of 1 in 98 at the rate 10 ms^{-1}. The resistance due to friction acting on the train is '10' N per ton.

Sol. Given,

$$\sin\theta = \frac{1}{98}; \quad m = 400 \times 10^3 kg$$

Friction force,

$$F = 10 \times 400 = 4000 \text{ N}$$

Velocity, $v = 10\ ms^{-1}$

$\therefore$ Power, $P = (mg\sin\theta + f)v$

$$\therefore \quad P = \left[\left(400 \times 10^3 \times 9.8 \times \frac{1}{98}\right) + 4000\right] \times 10$$

$$= 440000 \text{ W} = 440 \text{ KW}$$

Ex. A pipe (hose) has a diameter of 2.5 cm and is required to direct a jet of water to a height of at least 40 m. Find out the minimum power of the pump needed for this hose.

Sol. Volume of water ejected per sec

$$Av = \pi\left(\frac{d}{2}\right)^2 \times \sqrt{2gh} \ \ m^3/s$$

$\therefore \quad v = \sqrt{2gh}$

Mass ejected per sec is

$$M = \frac{1}{4}\pi d^2 \times \sqrt{2gh}\,\rho \text{ Kg / s}$$

Kinetic energy of water leaving horse / sec

$$KE = \frac{1}{2}mv^2 = \frac{1}{8}\pi d^2 \times (2gh)^{3/2} \times \rho$$

$$= \frac{1}{8} \times 3.14 \times (2.5 \times 10^{-2})^2 \times (2 \times 9.8 \times 40)^{3/2} \times 1000$$

$= 21.5$ KJ

Ex. A particle of mass m accelerates uniformly from rest to velocity v_0 in time t_0, find the instantaneous power delivered to particle when velocity is $\frac{v_0}{2}$.

Sol. Acceleration $a = \frac{v_0}{t_0}$

Force $F = \frac{mv_0}{t_0}$

Instantaneous power

$$P = F \cdot \frac{v_0}{2} = \left(\frac{mv_0}{t_0}\right)\frac{v_0}{2} = \frac{mv_0^2}{2t_0}$$

Ex. 54000 litre water falls per hour from height 100 m and one-third of gravitational potential energy is converted into electrical energy. How much power is generated?

Sol. $\frac{\Delta U_w}{\Delta t} = \frac{mgh}{t} = 54 \times 10^3 \times 10 \times 100$

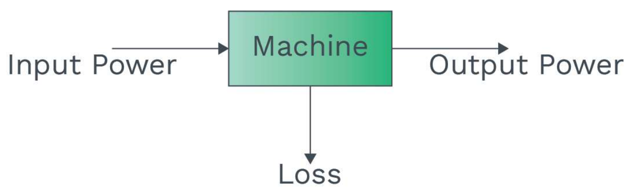
$\frac{\Delta U_w}{\Delta t} = 54 \times 10^6$ J/hr.

Electrical energy per hour $= \frac{1}{3}\left(\frac{\Delta U}{\Delta t}\right)$

$$P_{electrical} = \frac{E_{electrical}}{t} = \frac{18 \times 10^6}{60 \times 60} = 5\,Kw$$

EFFICIENCY

Ratio of output power to Input power.



$$\text{Efficiency} = \frac{\text{Output power}}{\text{Input power}}$$

Ex. A ball throwing cannon, lifts ball and throws with 72 km/hr Speed. If balls are raised upto 10 m. then find Input power of cannon if it lifts 8 balls per minute and its efficiency is 80%. Mass of each ball is 1 kg.

Sol. $P = \frac{W}{t} = \frac{\Delta k + \Delta u_g}{t}$

$$= 8\left[\frac{\left[\frac{1}{2} \times 1 \times (20)^2 - 0\right] + 1 \times 10 \times 10}{60}\right]$$

$$P_{output} = \frac{8 \times 300}{60} = 40\ \text{watt}$$

$$\frac{P_{output}}{P_{input}} = 0.8$$

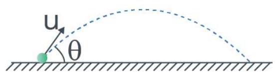
$$P_{input} = \frac{40}{0.8} = 50\ \text{watt}$$

Ex. (1) Find average power generated by gravitational force till particle reach maximum height.

(2) Instantaneous power of gravitational force.

(a) At striking point

(b) At maximum height.



Sol. (1) $P_{avg} = \frac{WD}{\Delta t} = \frac{-Mgh_{max}}{T/2}$ [T is time of flight]

$$= \frac{-Mg \times u^2 \sin^2\theta}{2g \times u \sin\frac{\theta}{g}} = \frac{-Mgu\sin\theta}{2}$$

(2) $P = \vec{F}\cdot\vec{v}$

(a) $P = -mg\,\hat{j}\cdot(u\cos\theta\,\hat{i} - u\sin\theta\,\hat{j})$

$= mg \times [u\sin\theta]$

(b) $P = -mg\,\hat{j}\cdot u\cos\theta\,\hat{i} = 0$

Note: If power is constant then average power is same as instantaneous power

$$P_{inst} = \overline{P} = \vec{F}\cdot\vec{v}$$

Ex. A Car accelerates under constant power P, find the velocity of automobile as a function of time t and as a function of position, given that mass of car is M.

$$P = \vec{F}\,.\,\vec{v}$$

For time

$$P = M\frac{d\vec{v}}{dt}\,.\,\vec{v}$$

For position

$$P = M\frac{\vec{v}\,d\vec{v}}{dx}\,\vec{v}$$

Sol. (1) $\int_0^v v\cdot dv = \frac{P}{M}\int_0^t dt$

$$v^2 = \frac{2P}{M}t$$

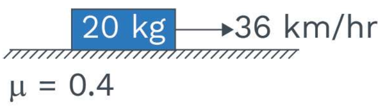
$$v = \sqrt{\frac{2P}{M}}\,t^{1/2}$$

(2) $\int_0^v v^2\,dv = \int_0^s \frac{P}{M}dx$

$$v^3 = \frac{3Px}{M}$$

$$v = \left(\frac{3P_0}{M}x\right)^{1/3}$$

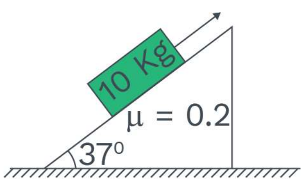
Ex. A block is dragged by a machine with constant speed of 36 km/hr as shown find power output of machine.



Sol. $P = \vec{F} \cdot \vec{v}$

$P = \mu mg \times 10 = 80 \times 10 = 800 \text{ watt}$

Ex. A cart is pulled by a machine along the inclined plane. with constant speed of 72 km/hr. Find out power of machine.



Sol. $P = \vec{F} \cdot \vec{v}$

$= [mg\sin\theta + \mu mg\cos\theta] \cdot \vec{v}$

$= [60 + 0.2 \times 80] \times 20 = 1520$ watt

Ex. A pump of power 1 H.P. is ejecting water with constant velocity of 10m/s find mass of water coming out of pipe in one seconds.

Sol. $P = v^2 \dfrac{dm}{dt}$

$$\Rightarrow \quad \frac{dm}{dt} = \frac{P}{v^2} = \frac{1 \times 746}{100} = 7.46 \text{ kg / s}$$

Ex. An engine pumps water of density r, through a hose pipe. Water leaves the hose pipe with a velocity v. Find:

(a) Rate at which kinetic energy is imparted to water

(b) Power of the engine.

Sol. (a) Rate of change of kinetic energy

$$\frac{dE_k}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = \frac{1}{2}v^2\frac{dm}{dt}$$

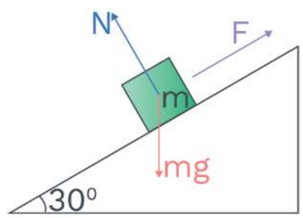
$$= \frac{1}{2}v^2 \frac{d}{dt}(\rho Ax) = \frac{1}{2}\rho Av^2 \frac{dx}{dt} = \frac{1}{2}\rho Av^3$$

(b) Power

$$Fv = \left(v\frac{dm}{dt}\right)v = v^2\frac{dm}{dt} = v^2(\rho Av) = \rho Av^3$$

Ex. A body of mass 2 kg is pulled up on a smooth incline plane of angle 30° with horizontal. If the body moves with an acceleration of 1 m/s^2, find the power delivered by the pulling force at a time 4 seconds after motion starts. Find out the average power delivered during these 4 seconds after the motion starts?

Sol. The forces acting on the body are shown in figure.



Resolving forces parallel to incline,

$$F - mg\sin\theta = ma$$

$\Rightarrow$ $$F = mg\sin\theta + ma$$

$$= 2\times 9.8\times \sin 30° + 2\times 1 = 11.8\ N$$

The velocity after 4 seconds

= u + at = 0 + 1 × 4 = 4 m/s

Power delivered by force at t = 4 seconds

= Force × Velocity = 11.8 × 4 = 47.2 W

The displacement during 4 seconds is given by the formula

$$v^2 = u^2 + 2as$$

$$v^2 = 0 + 2\times 1\times s$$

$\therefore$ s = 8 m

Work done in four seconds

= Force × distance = 11.8 × 8 = 94.4 J

$\therefore$ Average power delivered

$$= \frac{\text{work done}}{\text{time}} = \frac{94.4}{4} = 23.6\ W$$

Q1 **EXAMPLES**

A body is displaced from $\vec{r}_A = (2\hat{i} + 4\hat{j} - 6\hat{k})$ to $\vec{r}_B = (5\hat{i} - 4\hat{j} + 2\hat{k})$ under a

Sol: **constant force $\vec{F} = (4\hat{i} + 3\hat{j} - \hat{k})$. Find the work done.**

Work done $W = \vec{F}.\vec{s}$; $W = \vec{F}.(\vec{r}_B - \vec{r}_A)$

$$W = (4\hat{i} + 3\hat{j} - \hat{k}).\left[(5\hat{i} - 4\hat{j} + 2\hat{k}) - (2\hat{i} + 4\hat{j} - 6\hat{k})\right]$$

$$W = (4\hat{i} + 3\hat{j} - \hat{k}).(3\hat{i} - 8\hat{j} + 8\hat{k})$$

Q2 $W = 12 - 24 - 8 = -20$ units

A force $\vec{F} = 2x\hat{i} + \hat{j} + 2z\hat{k}$ N is acting on a particle. Find the work-done by the

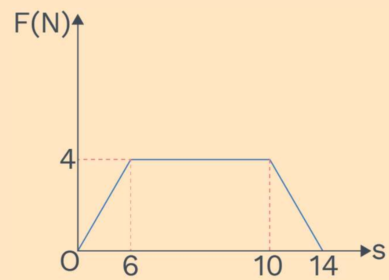
Sol: **force in displacing the body from (1, 2, 3) m to (3, 6, 1) m.**

Work done $$W = \int_{x_1}^{x_2} F_x dx + \int_{y_1}^{y_2} F_y dy + \int_{z_1}^{z_2} F_z dz$$

$$W = \int_1^3 2x dx + \int_2^6 dy + \int_3^1 2z dz$$

$$W = 2\left[\frac{x^2}{2}\right]_1^3 + [y]_2^6 + 2\left[\frac{z^2}{2}\right]_3^1 = 4\text{ J}$$

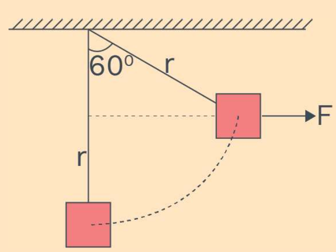
Q3 **The force acting on an object varies with the distance travelled by the object as shown in the figure. Find the work-done by the force in moving the object from x = 0 m to x = 10 m.**



Sol: Work done = Area under F-s curve.

$$W = \left(\frac{1}{2} \times 6 \times 4\right) + (4 \times 4) = 28J.$$

Q4 **A 20 kg block is pulled along a frictionless surface in the form of an arc of a circle of radius 5 m. The applied force is 100 N. Find the work done by**
(a) applied force and
(b) gravitational force in displacing through an angle 60°



Sol: Work done by applied force W = Fr sinθ

$$W = 100 \times 5 \times \sin 60° = 100 \times 5 \times \frac{\sqrt{3}}{2} = 433\,J$$

Work done by gravitational force

$$W = -mgr(1 - \cos\theta)$$

$$W = -20 \times 9.8 \times 5(1 - \cos 60°)$$

$$W = -98 \times 10\left(1 - \frac{1}{2}\right) = -490\,J$$

Q5 **A uniform chain of length 4 m is kept on a table such that a length of 60 cm hangs freely from the edge of the table. The total mass of the chain is 8 kg. What is the work done in pulling the entire chain back onto the table ?**

Sol: $M = 8\,kg, L = 4\,m, l = 0.6\,m, g = 10\,m/s^2$

Work done $W = mg\frac{l}{2} = \left(\frac{M}{L}\right)l\,g\frac{l}{2}$

$W = \left(\frac{8}{4}\right) \times 0.6 \times 10 \times \frac{0.6}{2} = 3.6\,J$

Q6 **Find the work done in lifting a body of mass 10 kg and specific gravity 2.8 to a height of 10 m in water ? (g = 10 m/s²)**

Sol: Given specific gravity $\frac{\rho_b}{\rho_w} = 2.8$

$\rho_b = 2.8 \times \rho_w = 2.8 \times 1000 = 2800$

Work done $W = mgh\left(1 - \frac{\rho_w}{\rho_b}\right) = 10 \times 10 \times 10\left(1 - \frac{1000}{2800}\right)$

$W = 642.8\ J$

Q7 **A block of mass 'm' is lowered with the help of a rope of negligible mass through a distance 'd' with an acceleration of g/2. Find the work-done by the rope on the block?**

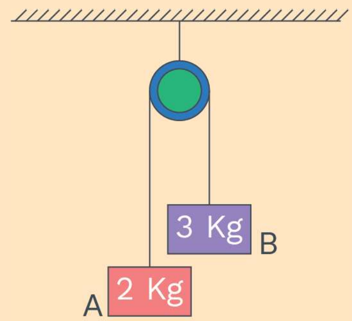
Sol: During lowering a block, the tension in rope is

$T = m(g - a)$ and $s = d$

Work done $W = -m(g - a)d$

$W = -m\left(g - \frac{g}{2}\right)d = -\frac{mgd}{2}$

Q8 **If the system shown is released from rest. Find the net work done by tension in first two second ($g = 10\ m/s^2$)**



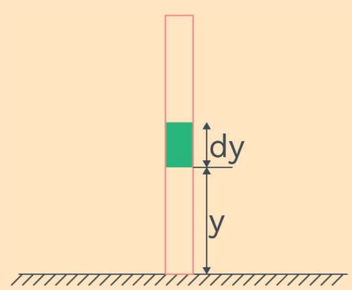
Sol: $a = \left(\frac{m_2 - m_1}{m_1 + m_2}\right)g = \left(\frac{3-2}{2+3}\right)10 = 2\,m/s^2$

$T = \frac{2m_1m_2g}{m_1 + m_2} = \frac{2 \times 2 \times 3 \times 10}{2+3} = 24\,N$

For each block $s = \frac{1}{2}at^2 = \frac{1}{2} \times 2 \times 4 = 4\,m$

$\therefore \quad W_{net} = W_1 + W_2 = Ts - Ts = 0$

Q9 **A rod of mass 'm' and length L is held vertical. Find out its gravitational potential energy with respect to zero potential energy at the lower end ?**



Sol: Choose a small element of length dy, then

Mass of the element $dm = \left(\frac{m}{L}\right)dy.$

The potential energy of the element dU = (dm)g (y)

Potential energy of the entire rod

$$U = \int_0^L \left(\frac{m}{L}\right)(dy).gy = \frac{m}{L} g \int_0^L y dy$$

$$U = \frac{m}{L} g \left(\frac{y^2}{2}\right)_0^L = \frac{mgL}{2}$$

Q10 **Under the action of force 1 kg body moves such that its position 'x' varies as a function of time t given by $x = \frac{t^3}{3}$, x is in metre and t in second. Calculate the work done by the force in first two seconds.**

Sol: From work-energy theorem W = ΔKE

$$x = \frac{2t^3}{3}, \text{ Velocity } v = \frac{dx}{dt} = 2t^2$$

At $t = 0,\ v_1 = 0,\ \text{At } t = 2s,\ v_2 = 8m/s$

$$W = \frac{1}{2}m\left(v_2^2 - v_1^2\right) = \frac{1}{2} \times 1\left(8^2 - 0\right) = 32J$$

Q11 **In a molecule, the potential energy between two atoms is given by $U(x) = \frac{a}{x^{12}} - \frac{b}{x^6}$. Where 'a' and 'b' are positive constants and 'x' is the distance between atoms. Find the value of 'x' at which force is zero and minimum P.E at that point.**

Sol: Force is zero $\Rightarrow \frac{dU}{dx} = 0$

i.e., $a(-12)x^{-13} - b(-6)x^{-7} = 0$

$$\frac{-12a}{x^{13}} + \frac{6b}{x^7} = 0 \Rightarrow \frac{12a}{x^{13}} = \frac{6b}{x^7}$$

$$\Rightarrow \quad x^6 = \frac{2a}{b} \quad \therefore \quad x = \left(\frac{2a}{b}\right)^{\frac{1}{6}}$$

Substituting the value of x

$$\Rightarrow \qquad U_{min} = a\left(\frac{b}{2a}\right)^{\frac{12}{6}} - b\left(\frac{b}{2a}\right)^{\frac{6}{6}}$$

$$U_{min} = a\left(\frac{b^2}{4a}\right) - \left(\frac{b^2}{2a}\right) \Rightarrow U_{min} = \frac{-b^2}{4a}$$

Q12 **A vehicle of mass 1500 kg climbs up a hill 200 metre hight. It then run on a level road with a speed of 30 ms⁻¹. Find out the potential energy gained by it and its total mechanical energy while running on the top of the hill.**

Sol: $m = 1500\,kg,\ g = 9.8\,ms^{-2}, h = 200\,m$

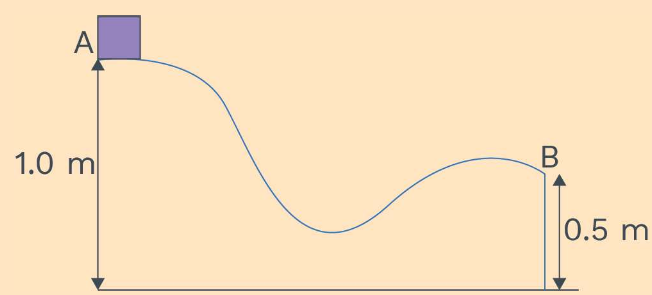
P.E. gained, $U = mgh = 1500 \times 9.8 \times 200 = 2.94 \times 10^6$ J

$$K.E. = \frac{1}{2}mv^2 = \frac{1}{2} \times 1500 \times (30)^2 = 0.675 \times 10^6 J$$

Total mechanical energy

$$E = K + U = (0.675 + 2.94) \times 10^6 = 3.615 \times 10^6 J$$

Q13 **Figure shows a particle sliding on a frictionless track which terminates in a straight horizontal section. If the particle starts slipping from the point A, how far away from the track will the particle hit the ground ?**



Sol: Applying the conservation law of mechanical energy for the points A and B,

$$mgH = \frac{1}{2}mv^2 + mgh$$

$$g - \frac{v^2}{2} = \frac{g}{2} \text{ or } v^2 = g \Rightarrow v = \sqrt{g} = 3.1 ms^{-1}$$

After point B the particle exhibits projectile motion with $\theta = 0°$ and $y = -0.5 m$

Horizontal distance travelled by the body

$$R = u\sqrt{\frac{2h}{g}} = 3.1 \times \sqrt{\frac{2 \times 0.5}{9.8}} = 1m$$

Q14 **A machine delivers power to a body which is directly proportional to velocity of the body. If the body starts with a velocity which is almost negligible, find the distance covered by the body in attaining a velocity v.**

Sol: Power $P = Fv\cos 0 = Fv = m\left(\frac{dv}{dt}\right)v,\ P \propto v$

$$mv\frac{dv}{dt} = K_0 v, \text{ Where } K_0 = \text{constant}$$

$$m\frac{dv}{dt} = K_0\ ;\ m\left(\frac{dv}{dx}\right)\frac{dx}{dt} = K_0$$

$$m\frac{dv}{dx} = K_0\ ;\ vdv = \left(\frac{K_0}{m}\right)dx$$

Integrating $\int_0^v vdv = \int_0^x \left(\frac{K_0}{m}\right)dx$

$$\frac{v^2}{2} = \left(\frac{K_0}{m}\right)x \Rightarrow x = \frac{1}{2}\frac{mv^2}{K_0}$$

Q15 **Find the power of an engine which can draw a train of 400 metric ton up the inclined plane of 1 in 98 at the rate 10 ms^{-1}. The resistance due to friction acting on the train is '10' N per ton.**

Sol: Given $\sin\theta = \frac{1}{98}$; $m = 400 \times 10^3$ kg

frictional force $f = 10 \times 400 = 4000 N$

velocity $v = 10\,\text{ms}^{-1}$

$\therefore$ Power $P = (mg\sin\theta + f)v$

$$\therefore \quad P = \left[\left(400 \times 10^3 \times 9.8 \times \frac{1}{98}\right) + 4000\right] \times 10$$

$= 440000$ W $= 440$ KW

Q16 **Two bodies of mass m_1 and m_2 are moving with velocities 1 ms^{-1} and 3 ms^{-1} respectively in opposite directions. If the bodies undergo one dimensional elastic collision, the body of mass m_1 comes to rest. Find the ratio of m_1 and m_2.**

Sol:

$u_1 = 1\,m/s, u_2 = -3\,m/s, v_1 = 0$

$$v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)u_1 + \left(\frac{2m_2}{m_1 + m_2}\right)u_2$$

$$0 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)1 + \left(\frac{2m_2}{m_1 + m_2}\right)(-3)$$

$$m_1 - m_2 = 6m_2 \,;\; m_1 = 7m_2 \,;\; \frac{m_1}{m_2} = \frac{7}{1}.$$

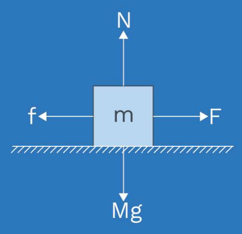
MIND MAP

WORK

$W = \int \vec{F}.d\vec{s}$

$W = \int F \cos\theta \, ds$
(θ is angle between force $\vec{F}$ and displacement $\vec{s}$)

NATURE OF WORK DONE



Applied force F > limiting friction

Positive Work ($0^\circ \leq \theta < 90^\circ$)

Work done by F is positive

Negative Work ($90^\circ < \theta \leq 180^\circ$)

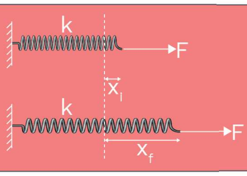
Work done by f is negative

Zero Work Done ($\theta = 90^\circ$)

Work done by Mg and N is zero

WORK DONE BY SPRING FORCE

$W_s = \frac{1}{2} k (x_i^2 - x_f^2)$

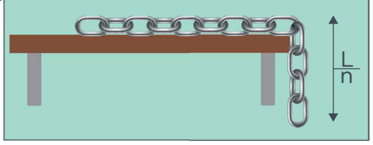


WORK-ENERGY THEOREM

$W_{total} = \Delta k$
$= k_f - k_i$

Work done in pulling the chain

$W = \frac{MgL}{2n^2}$



ENERGY

It is defined as capcacity of body for doing work

KINETIC ENERGY

Energy in a body due to its motion.

$k = \frac{1}{2} mv^2$

POTENTIAL ENERGY

Energy of a system due to configuration of the system or position of particles in system