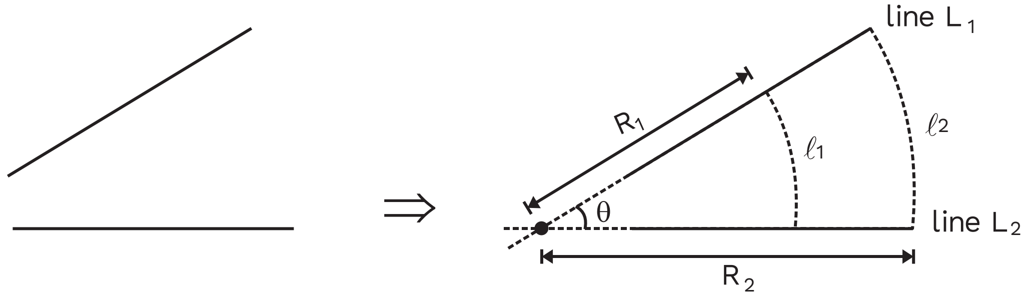
Mathematical Tools

DISCLAIMER

Trigonometry

Physics is basically a combination of theory and calcuations.
For "calculations", we need to learn some mathematical tools.

Angle :



θ = angular gap or angle between line L_1 and L_2.
By definition,

$$\boxed{\text{angle } (\theta) = \frac{\text{Arc}\,(\ell)}{\text{Radius (R)}}}$$

In above diagram

$$\theta = \frac{\ell_1}{R_1} = \frac{\ell_2}{R_2}$$

The unit of angle is 'radian' (rad.)

Angle has two directions → Anti-clockwise (positive)
→ Clockwise (Negative)

If we rotate line in anticlockwise direction

→ From OA to OC (semi-circle),

$$\theta_1 = \frac{\text{length ABC}}{\text{length OA}} = 3.141....\text{rad} = \pi\,\text{rad} \approx \frac{22}{7}\text{rad}.$$

→ From OA to OB (quarter-circle),

$$\theta_2 = \frac{\text{arc length AB}}{\text{radius length OA}} = 1.57....\text{rad} = \frac{\pi}{2}\text{rad}.$$

Similarly, for complete circle,

$$\text{angle} = \frac{\text{circumference of circle}}{\text{Radius}}$$

$$\theta_{\text{circle}} = 6.282....\text{rad} = 2\pi\ \text{rad}.$$

- Angle can also be measured in 'degree' where

$$\pi \text{ rad} = 180^\circ \quad \Rightarrow \quad 1 \text{ rad} = \frac{180^\circ}{\pi} \approx 57.3^\circ$$

$$\text{or} \quad 180^\circ = \pi \text{ rad} \quad \Rightarrow \quad 1^\circ = \left(\frac{\pi}{180}\right)\text{rad}$$

For example,

$$\frac{\pi}{6}\text{rad} = \frac{180^\circ}{6} = 30^\circ \quad \text{and} \quad \frac{\pi}{3}\text{rad} = \frac{180^\circ}{3} = 60^\circ$$

Trigonometric Ratio (T-Ratio) :

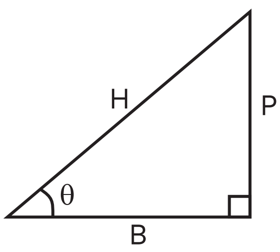
For a right-angled triangle,
Perpendicular (P) is that line which is just opposite to angle (θ), Hypotenuse (H) is that line which is just opposite to 90° angle while other line is base (B).

$$P^2 + B^2 = H^2$$

$$\sin\theta = \frac{P}{H}; \ \operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{H}{P}$$

$$\cos\theta = \frac{B}{H}; \quad \sec\theta = \frac{1}{\cos\theta} = \frac{H}{B}$$

$$\tan\theta = \frac{P}{B}; \quad \cot\theta = \frac{1}{\tan\theta} = \frac{B}{P}$$



B → Base
P → Perpendicular
H → Hypotenuse

Also,

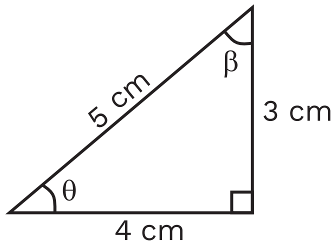
$$\tan\theta = \frac{P}{B} = \frac{P/H}{B/H} \quad \Rightarrow \quad \tan\theta = \frac{\sin\theta}{\cos\theta}$$

On measuring, the angle θ = 37° and β = 53°
For given triangle,

$$\tan\theta = \frac{3}{4} \ \text{while} \ \tan\beta = \frac{4}{3}$$

$$\sin\theta = \frac{3}{5} \ \text{while} \ \sin\beta = \frac{4}{5}$$

$$\cos\theta = \frac{4}{5} \ \text{while} \ \cos\beta = \frac{3}{5}$$



Trigonometric Identities :

$$\frac{P^2}{H^2}+\frac{B^2}{H^2}=\frac{H^2}{H^2} \Rightarrow \boxed{\sin^2\theta+\cos^2\theta=1}$$

$$\frac{P^2}{B^2}+\frac{B^2}{B^2}=\frac{H^2}{B^2} \Rightarrow \boxed{\tan^2\theta+1=\sec^2\theta}$$

$$\frac{P^2}{P^2}+\frac{B^2}{P^2}=\frac{H^2}{P^2} \Rightarrow \boxed{1+\cot^2\theta=\text{cosec}^2\theta}$$

Commonly used angles and their trigonometric ratios :

Angle (θ)	0°	30°	37°	45°	53°	60°	90°	180°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{1}{\sqrt{2}}$	$\frac{4}{5}$	$\frac{\sqrt{3}}{2}$	1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{4}{5}$	$\frac{1}{\sqrt{2}}$	$\frac{3}{5}$	$\frac{1}{2}$	0	–1
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	$\frac{3}{4}$	1	$\frac{4}{3}$	$\sqrt{3}$	∞	0

Trigonometrical formulae :

We remember values of $\sin\theta$, $\cos\theta$ and $\tan\theta$, for
θ = {0°, 30°, 37°, 45°, 53°, 60°, 90°, 180°}

For example : $\sin 30° = \frac{1}{2}$

But what if someone asks to calculate value of sin 120°.

→ Then we split the given angle in terms of
{0°, 30°, 37°, 45°, 53°, 60°, 90°, 180°}

For example : sin(120°) = sin(90° + 30°) = sin(180° – 60°)

- $\sin(2n\pi + \theta) = \sin\theta$, where n = 0, 1, 2, 3, ...

For example:

$\sin 390° = \sin(2\pi + 30°)$ [$\because$ 2π = 360°]

$= \sin(2n\pi + 30°)$ [Here, n = 1]

$= \sin 30°$

$\sin 390° = \frac{1}{2}$ $\left[\because \sin 30° = \frac{1}{2}\right]$

Similarly,

- $\cos(2n\pi + \theta) = \cos\theta$
- $\tan(2n\pi + \theta) = \tan\theta$
- $\sin(\pi - \theta) = +\sin\theta$
- $\cos(\pi - \theta) = -\cos\theta$
- $\tan(\pi - \theta) = -\tan\theta$

Q. **Find sin120°**

Sol. $\sin 120° = \sin(180° - 60°) = \sin(\pi - 60°) = +\sin 60° = \frac{\sqrt{3}}{2}$

Q. **Find tan 150°**

Sol. $\tan 150° = \tan(180° - 30°) = \tan(\pi - 30°) = -\tan 30° = -\frac{1}{\sqrt{3}}$

$\sin(\pi + \theta) = -\sin\theta$
$\cos(\pi + \theta) = -\cos\theta$
$\tan(\pi + \theta) = +\tan\theta$

Q. **Find sin210°**

Sol. $\sin 210° = \sin(180° + 30°) = \sin(\pi + 30°) = -\sin 30° = -\frac{1}{2}$

$\sin(2\pi - \theta) = -\sin\theta$
$\cos(2\pi - \theta) = +\cos\theta$
$\tan(2\pi - \theta) = -\tan\theta$

Q. **Find sin330°**

Sol. $\sin 330° = \sin(360° - 30°) = \sin(2\pi - 30°) = -\sin 30° = -\frac{1}{2}$

$$\sin\left(\frac{\pi}{2}+\theta\right) = +\cos\theta$$

$$\cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta$$

$$\tan\left(\frac{\pi}{2}+\theta\right) = -\cot\theta$$

Q. Find tan 120°

Sol. $\sin(120°) = \sin(90° + 30°) = \sin\left(\frac{\pi}{2}+30°\right) = +\cos 30° = \frac{\sqrt{3}}{2}$

$$\sin\left(\frac{\pi}{2}-\theta\right) = +\cos\theta$$

$$\cos\left(\frac{\pi}{2}-\theta\right) = +\sin\theta$$

$$\tan\left(\frac{\pi}{2}-\theta\right) = +\cot\theta$$

Q. Find cos30° using above rule

Sol. $\cos 30° = \cos(90° - 60°) = \cos\left(\frac{\pi}{2}-60°\right) = +\sin 60° = \frac{\sqrt{3}}{2}$

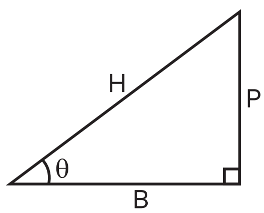
$\sin(-\theta) = -\sin\theta$
$\cos(-\theta) = +\cos\theta$
$\tan(-\theta) = -\tan\theta$

Q. Find sin (–60°)

Sol. $\sin(-60°) = -\sin 60° = -\frac{\sqrt{3}}{2}$

Maximum and minimum value

In this right-angle triangle $P \le H$ and $B \le H$



$\sin\theta = \dfrac{P}{H}$

$\because \quad P \le H \quad \Rightarrow \dfrac{P}{H} \le 1$

$\therefore \quad \sin\theta \le 1$

Similarly, for negative angle

$\sin\theta \ge -1$

Combining above two inequalities

$-1 \le \sin\theta \le 1$

Now, $\cos\theta = \dfrac{B}{H}$

$\Rightarrow \quad \cos\theta \le 1$

Considering both negative and positive angles

$-1 \le \cos\theta \le 1$

$\tan\theta = \dfrac{P}{B}$

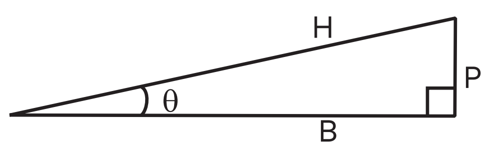
$-\infty < \tan\theta < \infty$

Small angle approximation :

In a right-angle triangle, if θ is very small

$P << H$ and $H \simeq B$ (see the diagram)

$\therefore$ We consider P as arc and H and B as radius



$\rightarrow \quad \sin\theta = \dfrac{P}{H}$

$\sin\theta = \dfrac{\text{Arc}}{\text{Radius}}$

But we know that θ(in radians) $= \dfrac{\text{Arc}}{\text{Radius}}$

$\therefore \quad \boxed{\sin\theta = \theta, \text{ if } \theta \to 0} \quad \Rightarrow$ 'θ' should be in radian

Trigonometry

Example :

sin 1° = 1 ✗ (incorrect) First convert 1° into radian $1^\circ = \frac{\pi}{180}$Rad $\sin 1^\circ = \sin\left(\frac{\pi}{180}\right) = \frac{\pi}{180}$	→ $\cos\theta = \frac{B}{H}$ If θ is very small then, H ≃ B $\Rightarrow \frac{B}{H} \simeq 1$ $\therefore$ $\boxed{\cos\theta = 1,\ \text{if}\ \theta \to 0}$	→ $\tan\theta = \frac{\sin\theta}{\cos\theta}$ For very small 'θ' sinθ = θ and cosθ = 1 ∴ For small 'θ' $\tan\theta = \frac{\theta}{1} = \theta$ $\boxed{\tan\theta = \theta,\ \text{if}\ \theta \to 0}$

Q. Find value of sin 5°

Sol. First convert 5° into radian.

$$1^\circ = \frac{\pi}{180}\ \text{Rad}$$

$$5^\circ = 5 \times \frac{\pi}{180}\ \text{Rad} = \frac{\pi}{36}\ \text{Rad}$$

$\frac{\pi}{36}$ Rad is a small angle

$$\therefore\ \sin 5^\circ = \sin\left(\frac{\pi}{36}\right)$$

$$\sin 5^\circ = \frac{\pi}{36}$$

- sin(A + B) = sinA cosB + cosA sinB
  sin(A – B) = sinA cosB – cosA sinB
- cos(A + B) = cosA cosB – sinA sinB
  cos(A – B) = cosA cosB + sinA sinB

Q. **Find sin15° using above rule**

Sol. sin15° = sin(45° – 30°) = sin45° cos30° – cos45° sin30°

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$$

$$\boxed{\sin 15^\circ = \left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)}$$

Q. **Find sin75° using above rule**

Sol. sin75° = sin(45° + 30°) = sin30° cos45° + cos30° sin45°

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$\boxed{\sin 75^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}}$$

Q. **Find sin120° using above rule**

Sol. sin120° = sin(60° + 60°)

= sin60° cos60° + cos60° sin60°

$$= \frac{\sqrt{3}}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

Q. **Find cos120° using above rule**

Sol. $\cos 120° = \cos(60° + 60°)$

$$= \cos 60° \cos 60° - \sin 60° \sin 60°$$

$$= \frac{1}{2}\cdot\frac{1}{2} - \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} = \frac{1}{4} - \frac{3}{4} = -\frac{1}{2}$$

$$\tan 120° = \frac{\sin 120°}{\cos 120°}$$

Q. **Find cos106° using above rule**

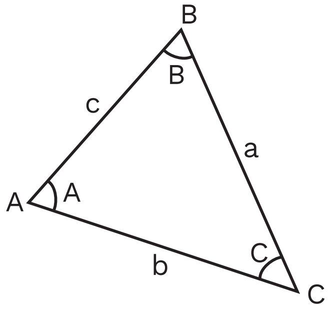
Sol. $\cos 106° = \cos(53° + 53°)$

$$= \cos 53° \cos 53° - \sin 53° \sin 53°$$

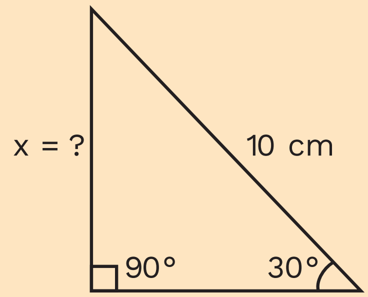
$$= \frac{3}{5}\cdot\frac{3}{5} - \frac{4}{5}\cdot\frac{4}{5} = \frac{-7}{25}$$

Sine formula :

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



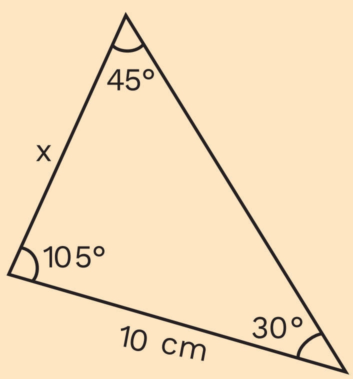
Q. **Find x in adjoining figure**



Sol. $\frac{\sin 90°}{10} = \frac{\sin 30°}{x}$

$\Rightarrow \quad x = 10 \times \frac{1}{2} = 5$

Q. **Find x in given figure**



Sol. $\frac{\sin 45°}{10} = \frac{\sin 30°}{x}$

$\Rightarrow \quad x = \frac{10 \times \sin 30°}{\sin 45°}$

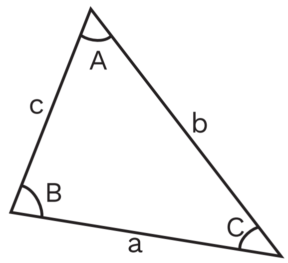
$= \frac{10 \times 1/2}{1/\sqrt{2}} = 5\sqrt{2}$

Cosine formula :

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$



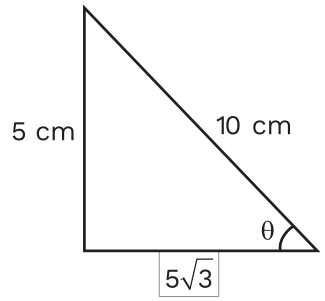
Q. **Find θ in given figure**

Sol. $$\cos\theta = \frac{100 + 75 - 25}{2 \times 10 \times 5\sqrt{3}}$$

$$= \frac{150}{20 \times 5\sqrt{3}} = \frac{3}{2\sqrt{3}}$$

$$\cos\theta = \frac{\sqrt{3}}{2}$$

$$\theta = 30°$$



$$-\sqrt{a^2 + b^2} \le a\sin\theta + b\cos\theta \le +\sqrt{a^2 + b^2}$$

For Example :

y = 2 sinθ + 3 cosθ

Here a = 2, b = 3

$$y_{max} = \sqrt{2^2 + 3^2} = 13$$

$$y_{min} = -\sqrt{13}$$

$$-\sqrt{13} \le 2\sin\theta + 3\cos\theta \le \sqrt{13}$$

Q. **y = 3 sinθ + 4 cosθ**
Find maximum value of y.

Sol. $$y_{max} = \sqrt{3^2 + 4^2}$$

$$= 5$$

Quadratic Equations

An equation of the form,

$ax^2 + bx + c = 0, \quad a \neq 0$ where

a, b and c are constants and x is variable is called a quadratic equation.
Number of Solutions = Maximum Power of 'x'
Solutions of equation are values of 'x' which when put in L.H.S, the L.H.S. will become zero.
Here, we have 2 solutions

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad x_1 + x_2 = -\frac{b}{a}$$

$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \qquad x_1 \cdot x_2 = +\frac{c}{a}$$

For example :

In equation,

$$2x^2 + 3x - 6 = 0$$

$$a = 2, b = 3, c = -6$$

$$x_1 = \frac{-3 + \sqrt{3^2 - 4 \times 2 \times (-6)}}{2 \times 2} = \frac{-3 + \sqrt{57}}{4}$$

$$x_2 = \frac{-3 - \sqrt{3^2 - 4 \times 2 \times (-6)}}{2 \times 2} = \frac{-3 - \sqrt{57}}{4}$$

$$x_1 + x_2 = \frac{-3}{2}$$

$$x_1 \cdot x_2 = \frac{-6}{2} = -3$$

1) If $b^2 - 4ac > 0$

   Two real and distinct solutions or roots

2) If $b^2 - 4ac < 0$.

   Two imaginary and distinct solutions or roots

3) If $b^2 - 4ac = 0$

   Real and equal roots

- $\sqrt{-4} = \sqrt{4 \times (-1)} = 2 \times \sqrt{-1} = 2i$, Where $i^2 = -1$
- $\underbrace{x^2 + 2x - 8}_{\text{Expression}} = 0$

Same equation can be represent as $\underbrace{x^2 + 2x - 10}_{\text{Expression}} = -2$, $\underbrace{x^2 + 2x}_{\text{Expression}} = 8$

- When expression is equated with something it becomes equation.

  $x^2 + 2x - 8 = 0 \rightarrow$ Equation [Expression equated with 0]

  $x^2 + 2x = 8 \rightarrow$ Equation [Expression equated with 8]

- $y = x^2 \Rightarrow$ Equation of Parabola

When y = 4

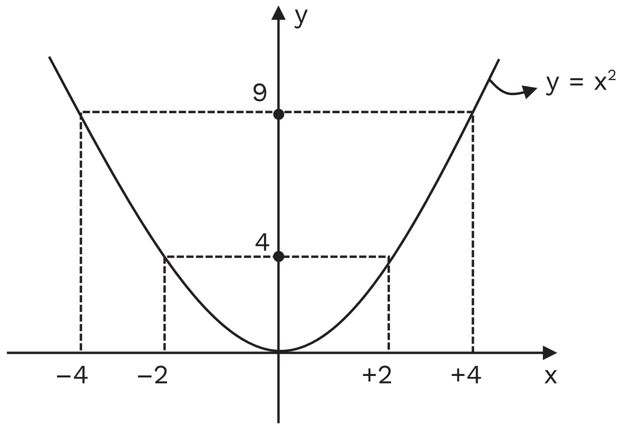
$\Rightarrow \; x^2 = 4$

$\Rightarrow \; x_1 = +2, x_2 = -2.$

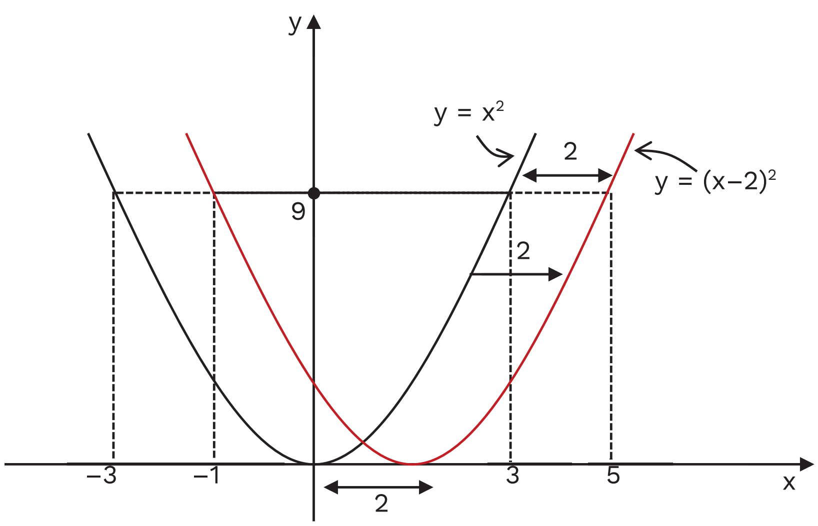
When y = 9

$\Rightarrow \; x^2 = 9$

$\Rightarrow \; x_1 = +3, x_2 = -3$



Origin shifting :



$y = x^2$ or $x^2 - y = 0$

Now the graph is pulled towards right by 2 units.

Initially y was zero at x = 0 But now y is zero at x = 2 Everywhere x will be replaced by (x – 2) New equation $\Rightarrow$ $y = (x-2)^2$	Initial equation: $y = x^2$ $y = 9 \rightarrow \begin{vmatrix} x_1 = +3 \\ x_2 = -3 \end{vmatrix}$ New equation: $y = (x-2)^2$ $y = 9 \rightarrow \begin{vmatrix} x_1 = +5 \\ x_2 = -1 \end{vmatrix}$

- Similarly, if curve is shifted by 'a' unit in left direction. Then new equation of parabola is $y = (x+a)^2$
- $y = x^2$ parabola is shifted downwards (towards negative y) by 2 units.

$y = x^2$,

replace y with y + 2

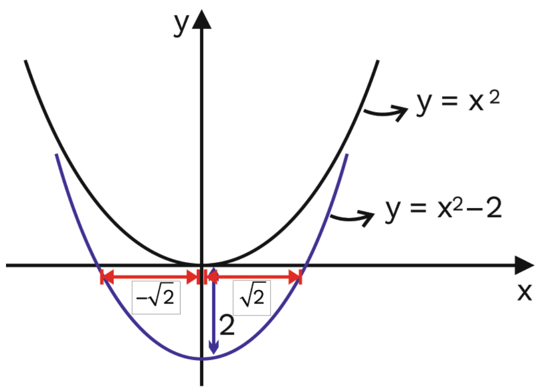
$\therefore \quad (y+2) = x^2$

$y = x^2 - 2$

$y = 0$

For $y = x^2$

$x^2 = 0 \quad \Rightarrow \quad x = 0$



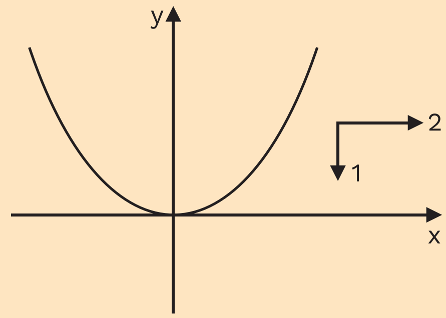
For $y = x^2 - 2$

$\Rightarrow \; x^2 - 2 = 0$

$x_1 = +\sqrt{2}, \; x_2 = -\sqrt{2}$

- Similarly, if parabola $y = x^2$ is pulled in upward direction (+y) by 'b' units, then new equation of parabola is $(y-b) = x^2 \quad \Rightarrow y = x^2 + b$

Q. **$y = x^2$ is pulled by 2 units towards +x direction and 1 unit toward –y direction. Find new equation of parabola.**



Sol. $y = x^2 \xrightarrow[\longrightarrow 2]{} y = (x-2)^2 \xrightarrow[\downarrow 1]{} y + 1 = (x-2)^2$

$y + 1 = (x-2)^2$

$\Rightarrow \quad y + 1 = x^2 + 4 - 4x$

$\Rightarrow \quad y = x^2 - 4x + 3$

$y = x^2 - 4x + 3 \rightarrow$ Parabola

Checking for $y = 0$

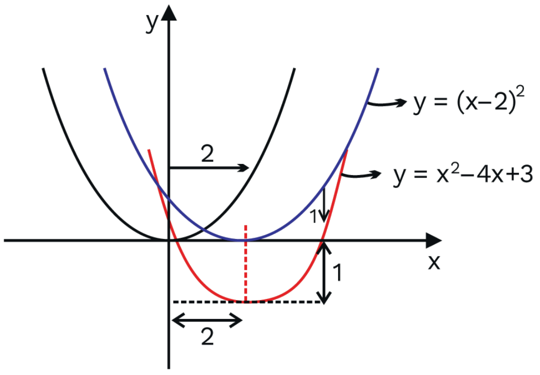
$\Rightarrow \quad x^2 - 4x + 3 = 0 \rightarrow$ Quadratic Equation

$\Rightarrow \quad x^2 - x - 3x + 3 = 0$

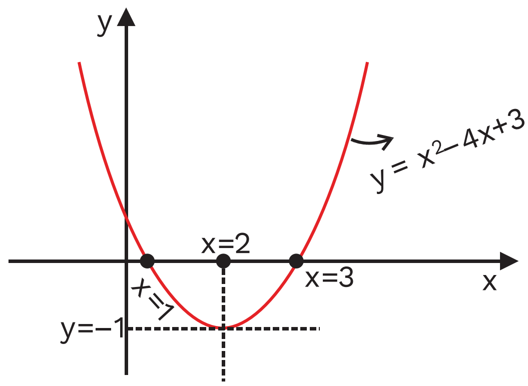
$\Rightarrow \quad x(x-1) - 3(x-1) = 0$

$\Rightarrow \quad (x-1)(x-3) = 0$

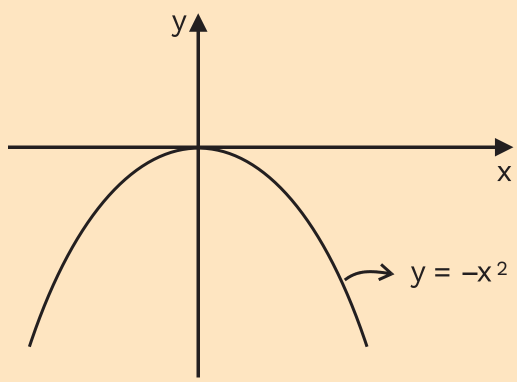
$\Rightarrow \quad x = 1$ and $x = 3$



Exact Graph ⇒

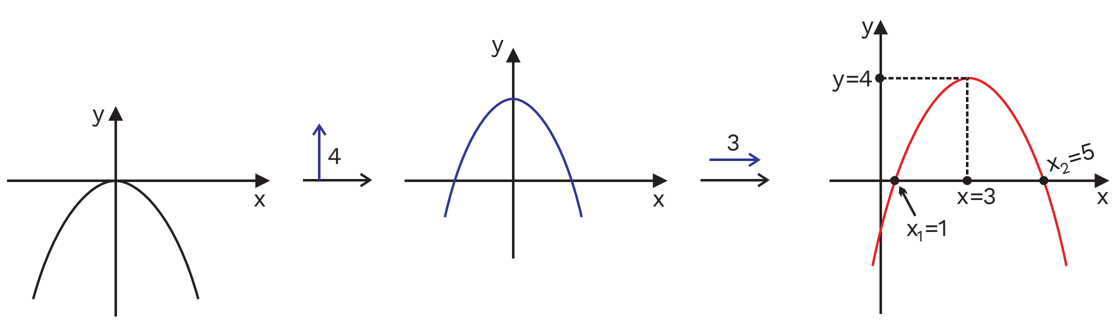


Q. $\mathbf{y = x^2}$



Now if above curve is pulled upwards by 4 units and towards right by 3 units, draw the new graph.

Sol. $y = -x^2 \xrightarrow{\uparrow 4} y - 4 = -x^2 \xrightarrow{\;3\;\rightarrow} y - 4 = -(x-3)^2$



$\Rightarrow\ y - 4 = -(x - 3)^2$ $\Rightarrow\ y = 4 - (x^2 + 9 - 6x)$ $\Rightarrow\ y = -x^2 + 6x - 5$

$y = 0 \Rightarrow -x^2 + 6x - 5 = 0$ $\Rightarrow\ x^2 - 6x + 5 = 0$ $\Rightarrow\ x^2 - 5x - x + 5 = 0$

$\Rightarrow\ x(x - 5) - 1(x - 5) = 0$ $\Rightarrow\ (x - 5)(x - 1) = 0$ $x_1 = 1,\ \ x_2 = 5$

Roots of quadratic equation (x_1, x_2) represent intersection of parabola with x-axis.

Quadratic Equations

Binomial Theorem

As per bionomial theorem, we can expand $(a + b)^n$ as,

$$(a+b)^n = (1)a^n b^0 + (n)a^{n-1} b^1 + \frac{n(n-1)}{1\times 2} a^{n-2} b^2 + \frac{n(n-1)(n-2)}{1\times 2\times 3} a^{n-3} b^3 + \frac{n(n-1)(n-2)(n-3)}{1\times 2\times 3\times 4} a^{n-4} b^4 + \ldots$$

After few terms coefficient of $a^{(\,)}\, b^{(\,)}$ will become zero, then stop writing next terms.

For example :

$$(a+b)^2 = 1a^2 b^0 + 2a^{2-1} b^1 + \frac{2(2-1)}{1\times 2} a^{2-2} b^2 + \frac{2(2-1)\cancelto{0}{(2-2)}}{1\times 2\times 3} a^{2-3} b^3$$

Here, n = 2

$$\boxed{(a+b)^2 = a^2 + 2ab + b^2}$$

$$(a+b)^3 = 1a^3 b^0 + 3a^{3-1} b^1 + \frac{3(3-1)}{2} a^{3-2} b^2 + \frac{3(3-1)(3-2)}{2\times 3} a^{3-3} b^3 + \frac{3(3-1)(3-2)(3-3)}{2\times 3\times 4} a^{3-4} b^4$$

Here, n = 3

$$\boxed{(a+b)^3 = a^3 + 3a^2 b + 3ab^2 + b^3}$$

Similarly, expansion of $(a+b)^4$, $(a+5)^5$, $(a+b)^6, \ldots$ can be written.

- $(99)^6 = (100-1)^6 = (a+b)^n = \ldots\ldots\ldots\ldots$

  a = 100

  b = – 1

  n = 6

- $$(1+x)^n = 1^n x^0 + n\,1^{(n-1)} x^1 + \frac{n(n-1)}{2} 1^{(n-2)} x^2 + \frac{n(n-1)(n-2)}{2\times 3} 1^{(n-3)} x^3 + \ldots$$

  $$(1+x)^n = 1 + nx + \frac{n(n-1)}{2} x^2 + \frac{n(n-1)(n-2)}{2\times 3} x^3 + \ldots$$

  Now, suppose 'x' is very small $(x \to 0)$

$\Rightarrow$ x^2 is very very small

$\Rightarrow$ x^3 is very very very small

Ignoring very small terms

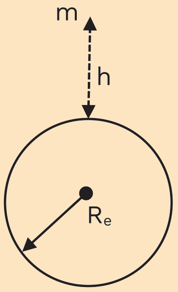
$\boxed{(1+x)^n = 1 + nx}$ $\Leftarrow$ Binomial approximation.

Q. **In gravitation, approximae the force on m.**

Mass of earth = m_e

Radius of earth R_e = 6400 km

h = 10m



Sol. $\left(\frac{h}{R_e}\right)$ is very small

Force acting on particle of mass m, $F = \frac{Gm_em}{(R_e+h)^2} = \frac{Gm_em}{R_e^2\left(1+\frac{h}{R_e}\right)^2}$

$$F = \frac{Gm_em}{R_e^2}\left(1+\frac{h}{R_e}\right)^{-2} \qquad = \frac{Gm_em}{R_e^2}\left[1+(-2)\frac{h}{R_e}\right], \text{ (Here, x = –2)}$$

$$F = \frac{Gm_em}{R_e^2}\left[1-\frac{2h}{R_e}\right]$$

Q. **Find the value of $(0.99)^2$**

Sol. $(0.99)^2 = (1-0.01)^2 = \left(1-\frac{1}{100}\right)^2 = (1-x)^n$

$[1+(-x)]^n = 1+n(-x) = 1-nx$

$(0.99)^2 = 1-2\times\frac{1}{100} = 1-\frac{2}{100} = \frac{98}{100}$ = 0.98

Q. **Find the value of $(0.99)^7$**

Sol. $(0.99)^7 = \left(1-\frac{1}{100}\right)^7 = 1-7\cdot\left(\frac{1}{100}\right) = 1-\frac{7}{100} = \frac{93}{100} \simeq 0.93$

Logarithm

A logarithm is the power to which a number must be raised in order to get some other number e.g. The base ten logarithm of 100 is 2 because ten raised to the power of two is 100.

$$2^4 = 16 \Rightarrow \log_2 16 = 4$$

$$5^2 = 25 \Rightarrow \log_5 25 = 2$$

$$\boxed{a^b = c \Rightarrow \log_a c = b}$$

$$10^1 = 10 \Rightarrow \log_{10} 10 = 1$$

$$100,000 = 10^5 \Rightarrow \log_{10} 100,000 = 5$$

$$\log_{10} 1000 = 3$$

$$\log_4 64 = 3$$

$$\log_2 64 = 6$$

Rules :

$$\log(m \times n) = \log m + \log n$$

$$\log\left(\frac{m}{n}\right) = \log m - \log n$$

$$\log m^n = n \log m$$

$$\log_a b = \frac{\log_c b}{\log_c a}$$

Examples :

a. $\log_4 64 = 3$

b. $\log_4 64 = \dfrac{\log_2 64}{\log_2 4} = \dfrac{6}{2} = 3$

c. $\log_4 64 = \dfrac{\log_8 64}{\log_8 4} = \dfrac{2}{(2/3)} = 3$

d. $\log_8 4 = x \Rightarrow (8)^x = 4 \Rightarrow (8)^{2/3} = 4$

$\therefore \quad x = \dfrac{2}{3} = \log_8 4$

Q. **Find the value of $\log_{10}(5000)$**

Sol. $\log_{10}(5000) = \log_{10}(5 \times 1000)$

$= \log_{10} 5 + \log_{10} 1000$

$\log_{10}(5000) = 0.6989 + 3 = 3.6989$

$\Rightarrow\ 10^{3.6989} = 5000$

Values to remember :

$\log_{10} 2 = 0.3010,\ \log_{10} 3 = 0.4771,\ \log_{10} 5 = 0.6989,\ \log_{10} 7 = 0.8450$

Q. **Find the value of $\log_{10}(2500)$**

Sol. $\log_{10} 2500 = \log_{10}(5 \times 5 \times 100)$

$= \log_{10} 5 + \log_{10} 5 + \log_{10} 100$

$= 0.6989 + 0.6989 + 2$

$= 3.3978$

Q. **$\log_e 2500$ = ??**

Sol. $e = 2.7\ (2.7)^x = 2500\quad \Rightarrow x = ?$

$\log_{2.7}(2500) = ??$

$\boxed{\log_e M = 2.303 \log_{10} M}$

$\log_e 2500 = 2.303 \times \log_{10} 2500$

$\log_e 2500 = 2.303 \times 3.3978$

Examples :

a. $\log_{10} 25 = \log_{10} 5^2$

$= 2\log_{10} 5 \qquad [\log m^n = n\log m]$

$= 2 \times 0.6989$

b. $\log_{10} 42 = \log_{10}(2 \times 3 \times 7)$

$= \log_{10} 2 + \log_{10} 3 + \log_{10} 7$

$= 0.3010 + 0.4771 + 0.8450 = 1.6321$

c. $\log_e 42 = 2.303 \log_{10} 42 = 2.303 \times 1.6321$

d. $\log_5 42 = \dfrac{\log_{10} 42}{\log_5 42} = \dfrac{1.6321}{0.6989}$

e. $\log_{10} 1/8 = \log_{10}\left(\dfrac{1}{2^3}\right) = \log_{10} 2^{-3} = -3 \log_{10} 2$

$= -3 \times 0.3010 = -0.9030$

f. $\log_{10} \sqrt{24} = \dfrac{1}{2}\log_{10} 24 = \dfrac{1}{2}\log_{10}(2 \times 3 \times 2 \times 2)$

$= \dfrac{1}{2}\log_{10}(2^3 \times 3) = \dfrac{1}{2}[3\log_{10} 2 + \log_{10} 3] = \dfrac{1}{2}[3 \times 0.3010 + 0.4771] = 0.6901$

g. $\log_{10}\left(\dfrac{1}{2}\right) = -\log_{10} 2 = -0.3010$

h. $\log_e\left(\dfrac{1}{2}\right) = -2.303 \log_{10} 2 = -0.693$

Series

Arithmetic Progression (A.P.)

A series of the form of,

$$\underset{\substack{1^{st}\\ \text{term}}}{a}, \underset{\substack{2^{nd}\\ \text{term}}}{a+d}, \underset{\substack{3^{rd}\\ \text{term}}}{a+2d}, \underset{\substack{4^{th}\\ \text{term}}}{a+3d}, \underset{\substack{5^{th}\\ \text{term}}}{a+4d} \ldots\ldots\ldots\ldots, \underset{\substack{n^{th}\\ \text{term}}}{a+(n-1)d}, \ldots\ldots\ldots\ldots$$ is called an arithmetic progression.

where, $d = (2^{nd}\text{term}) - (1^{st}\text{term}) = (3^{rd}\text{term} - 2^{nd}\text{term}) = \ldots\ldots,$ is known as common difference.

For example :

3, 5, 7, 9, 11,......

1^{st} term, a = 3

2^{nd} term, a + d = 5

Common difference = 5 – 3 = 2

We can find n^{th} term as,

n^{th} term = a + (n – 1)d = 3 + (n – 1)2

4^{th} term = 3 + (4 – 1)2 = 3 + 6 = 9

5^{th} term = 3 + (5 – 1)2 = 3 + 8 = 11

1, 2, 3, 4, 5,......, n,......

Sum of n terms, $S = \frac{n}{2}[2a + (n-1)d]$

Q. For the given A.P,
3, 5, 7, 9, 11 , ...
Calculate sum upto 4 terms

Sol. $S = \frac{4}{2}[2 \times 3 + (4-1)2] = 2[6 + 6] = 24$

Check : 3 + 5 + 7 + 9 = 24

Geometric progression (G.P) :

A series of the form,

$$\underset{\substack{1^{st}\\ \text{term}}}{a}, \underset{\substack{2^{nd}\\ \text{term}}}{ar}, \underset{\substack{3^{rd}\\ \text{term}}}{ar^2}, \underset{\substack{4^{th}\\ \text{term}}}{ar^3} \ldots\ldots\ldots\ldots, \underset{\substack{n^{th}\\ \text{term}}}{ar^{n-1}}, \ldots\ldots\ldots\ldots, \text{ is called a geometirc progression (G.P.)}$$

Where, common ratio $= \dfrac{2^{nd}\text{term}}{1^{st}\text{term}} = \dfrac{3^{rd}\text{term}}{2^{nd}\text{term}} = \ldots\ldots = \dfrac{n^{th}\text{term}}{(n-1)^{th}\text{term}} = r$

For example :

If 1^{st} term, a = 4 & Common ratio, $r = \dfrac{1}{2}$

then G.P. is $4, 2, 1, \dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{5}, \ldots\ldots\ldots$

We can find n^{th} term of an G.P. as,

n^{th} term $= a \cdot r^{n-1} = 4\left(\dfrac{1}{2}\right)^{n-1}$

e.g., 5^{th} term $= 4\left(\dfrac{1}{2}\right)^{5-1} = 4\left(\dfrac{1}{2}\right)^{4} = \dfrac{4}{16} = \dfrac{1}{4}$

Sum of n terms of G.P :

$$\boxed{S = \frac{a(1-r^n)}{1-r}}$$

- For G.P. series having infinite number of terms

If $r < 1$, $S_\infty = \dfrac{a}{1-r}$

If $r > 1$, $S_\infty = \infty$

Q. **$4, 2, 1, \dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{8}, \ldots\ldots\ldots\infty$. Find sum.**

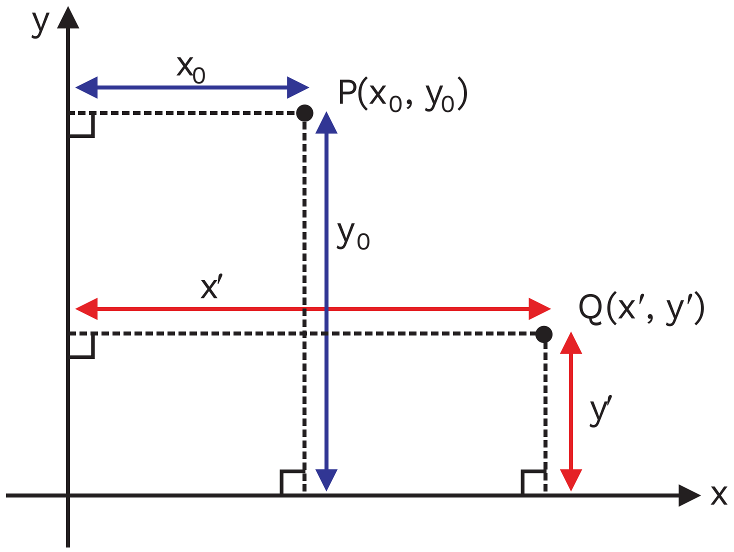
Sol. Here r < 1 for this ∞ series $\left(r = \dfrac{1}{2}\right)$

So, $S = \dfrac{a}{1-r} = \dfrac{4}{1-\dfrac{1}{2}} = 8$

S = 8

Co-Ordinate Geometry

Let us consider two points P and Q in XY plane as shown.



x-coordinate of point P = x_0 = Perpendicular distance of ‘P’ from y-axis

y-coordinate of point P = y_0 = Perpendicular distance of ‘P’ from x-axis

x-coordinate of point Q = x' = Perpendicular distance of ‘Q’ from y-axis

y-coordinate of point Q = y' = Perpendicular distance of ‘Q’ from x-axis

Distance Formula :

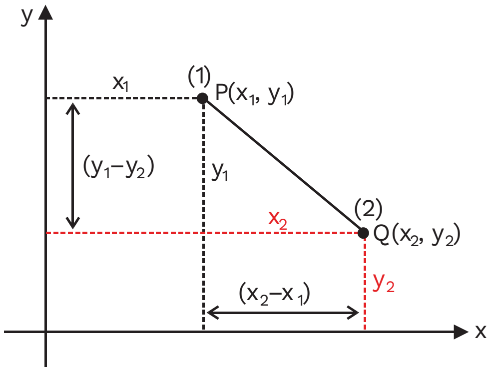
Distance between points P and Q

$PQ = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$

$\therefore$ Distance between two points having

co-ordinates (x_1, y_1) and (x_2, y_2) is given by

$\text{Distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$



Straight Line :

Equation of straight line $\Rightarrow \boxed{y = mx + c}$

c = Intercept on y-axis

m = slope = tan θ

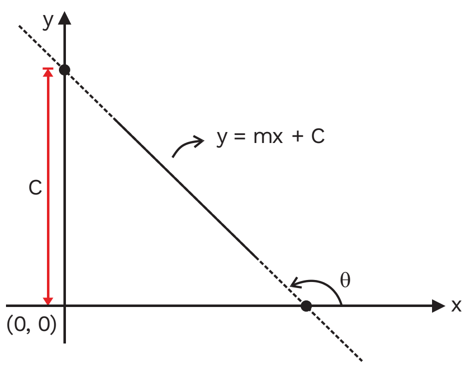
where 'θ' is angle made by line with +ve x-axis.

If m is positive

$\Rightarrow \begin{cases} \text{x increases then y also increases} \\ \text{x decreases then y also decreases} \end{cases}$

If m is negative

$\Rightarrow \begin{cases} \text{x increases then y decreases} \\ \text{x decreases then y increases} \end{cases}$



For example :

Case a.

Here C is positive when moving from $A \to B$,

x increases but y decreases $\Rightarrow$ m < 0

When moving from $B \to A$,

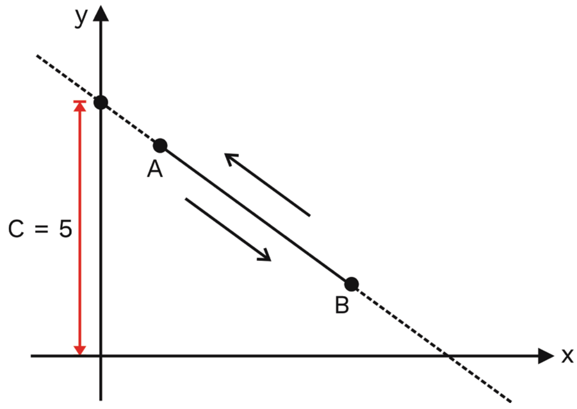
x decreases but y increases $\Rightarrow$ m < 0

$\therefore$ Slope of given line is negative.

$\therefore$ Equation of given line could be

$\boxed{y = -2x + 5}$

m = –2 (–ve) ; C = +5 (+ve)



Case b.

Here C is negative

When moving from $A \to B$

$x \to$ increases and $y \to$ increases

$\Rightarrow$ m > 0

When moving from $B \to A$

$x \to$ decreases and $y \to$ decreases

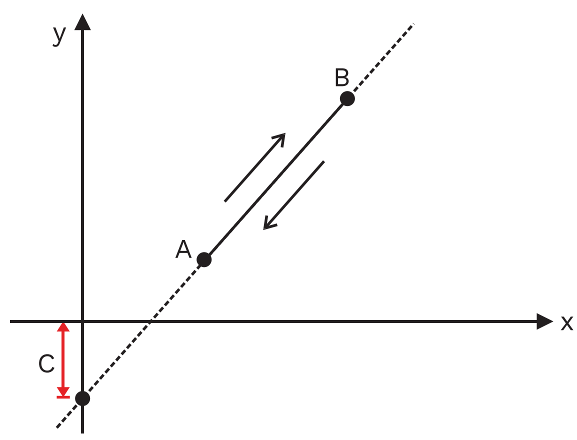
$\Rightarrow$ m > 0

$\therefore$ Equation of given line could be

$\boxed{y = +3x - 2}$

m = +3 (+ve)

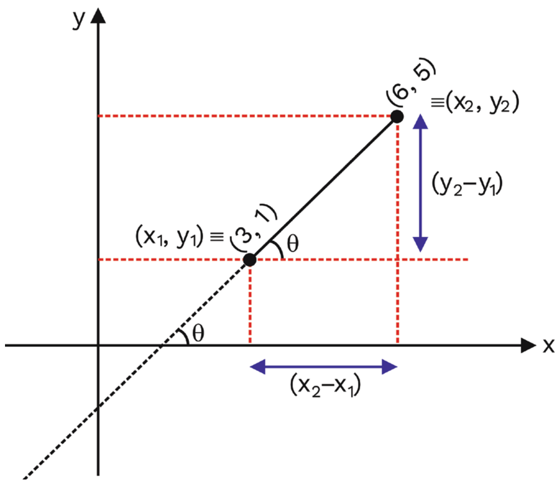
c = –2 (–ve)



- **How to write the equation of given line ?**

Let, the equation of the given line is, y = mx + c

$$\boxed{\text{Slope} = m = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1}}$$



See the triangle

$$\therefore \quad m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-1}{6-3} = \frac{4}{3}$$

$$m = \frac{4}{3}$$

$$\therefore \quad y = \frac{4}{3}x + C$$

As point (3, 1) lies on straight line, it must satisfy the equation of straight line i.e.

When x = 3, we get y = 1

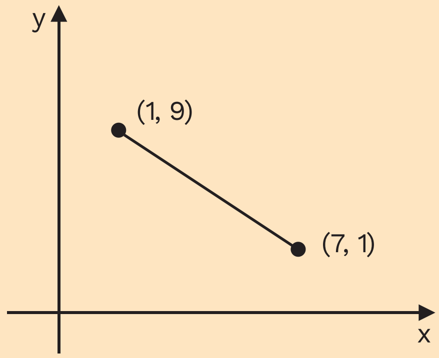
$$1 = \frac{4}{3}\cdot 3 + C$$

$$\Rightarrow \quad C = 1 - 4 = -3$$

$$\Rightarrow \quad y = \frac{4}{3}x - 3$$

[Point (6, 5) will also satisfy the equation of straight line]

Q. **Find equation of straight line**



Sol. Let, $(x_2, y_2) = (1, 9)$ and $(x_1, y_1) = (7, 1)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9-1}{1-7} = \frac{8}{-6} = \frac{-4}{3}$$

$$y = mx + c$$

$$y = \frac{-4}{3}x + C$$

Now, Point (1, 9) will satisfy this equation.

$$\Rightarrow \quad 9 = \frac{-4}{3} \cdot 1 + C \quad \Rightarrow \quad C = 9 + \frac{4}{3} = \frac{31}{3}$$

$$\Rightarrow \quad \boxed{y = \frac{-4}{3}x + \frac{31}{3}} \quad \Rightarrow \quad 3y = -4x + 31$$

- Another way to write equation of straight line is

$$\boxed{y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)}$$

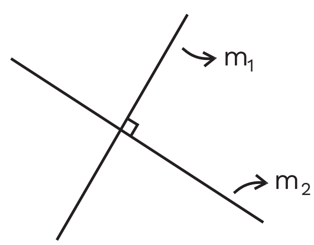
Here, $x_1 = 7, y_1 = 1, x_2 = 1, y_2 = 9$

$$\Rightarrow \quad y - 1 = \frac{9-1}{1-7}(x-7) \quad \Rightarrow \quad y - 1 = \frac{-4}{3}(x-7)$$

$$\Rightarrow \quad 3y - 3 = -4x + 28 \Rightarrow \boxed{3y + 4x = 31}$$

- If two lines having slopes m_1 and m_2 are perpendicular to each other, then

$$\boxed{m_1 \cdot m_2 = -1}$$



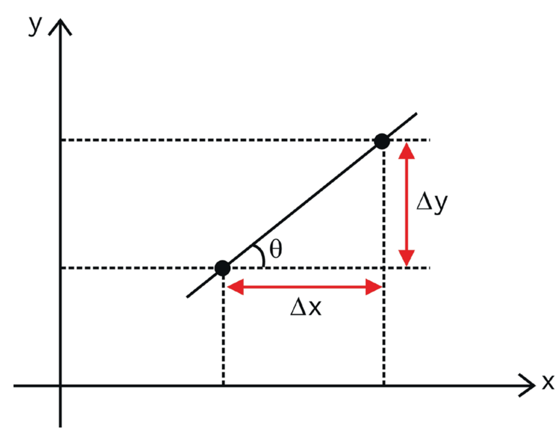
$$\boxed{m_1 \cdot m_2 = -1}$$

- Slope = m = $\tan\theta = \frac{\Delta y}{\Delta x}$

  Δ = Change

  = final value – Initial value

  Slope $= \frac{\Delta y}{\Delta x}$ = Rate of change of y w.r.t x

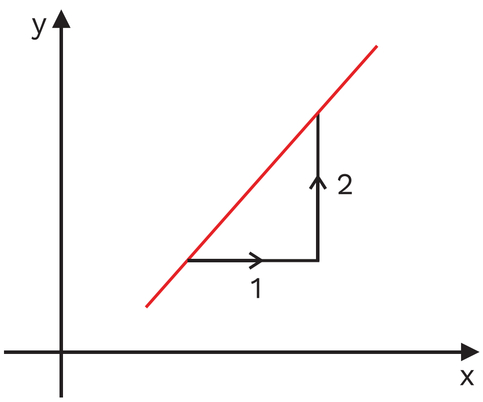


For example :

If Slope = 2

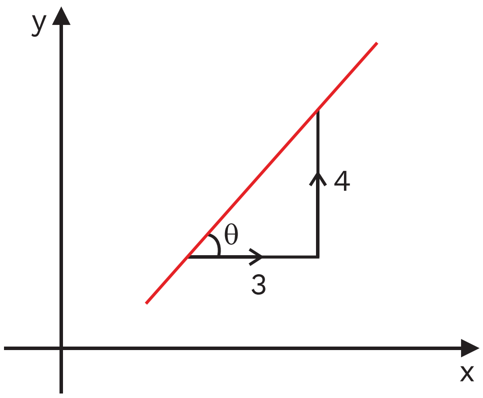
then, $\frac{\Delta y}{\Delta x} = 2 \Rightarrow \Delta y = 2\Delta x$

If x increases by 1 then y increases by 2



For slop, $m = \frac{4}{3} = \tan\theta = \frac{\Delta y}{\Delta x}$

$\Delta y = \frac{4}{3}\Delta x$



- If Δx is very small then

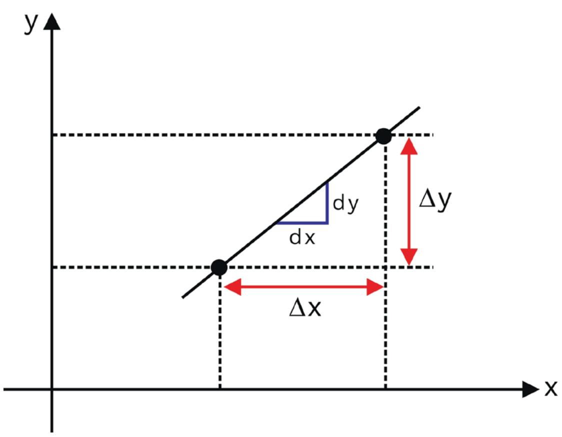
  Slope $= m = \tan\theta = \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$

  $\boxed{\frac{dy}{dx} = \text{differentiation}}$

- $\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = m = \text{Slope} = \tan\theta = \frac{dy}{dx}$

  = Rate of change of y w.r.t x

  = Differentiation of y w.r.t x



Co-Ordinate Geometry

Differentiation

FUNCTION

A function is a relation between a set of inputs and a set of permissible outputs with the property that each input is related to exactly one output.

e.g., $y = t^2 + t$

On changing 't', y changes

$\therefore$ y is a function of 't' $\Rightarrow$ $y = f(t)$

Conversely, on changing y, t changes

$\therefore$ t is a function of y $\Rightarrow$ $t = g(y)$

- $y = x^2$

  $\Rightarrow$ $y = f(x)$ and $x = g(y)$
- $y = \sin x$

  $y = f(x)$
- $y = x^2 + x$ and $x = 2t$

  $y = x^2 + x = (2t)^2 + (2t) = 4t^2 + 2t$

  Here, $y = f(x)$ and $x = g(t)$

  Combining the two relations,

  $y = f(x) = f[g(t)]$ $[\because x = g(t)]$

  $y = f[g(t)] = fog(t) \leftarrow$ Composite function

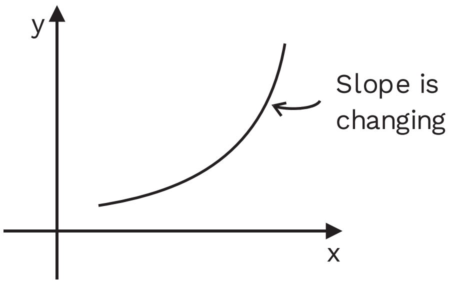
DIFFERENTIAL CALCULUS

We know that, $\text{slope} = \tan\theta = \dfrac{\Delta y}{\Delta x}$ [when slope is constant]

and, $\text{slope} = \tan\theta = \dfrac{dy}{dx}$ [when slope changes]

Now, $\dfrac{dy}{dx}$ = Rate of change of 'y' w.r.t 'x'

= Differentiation of 'y' w.r.t 'x'

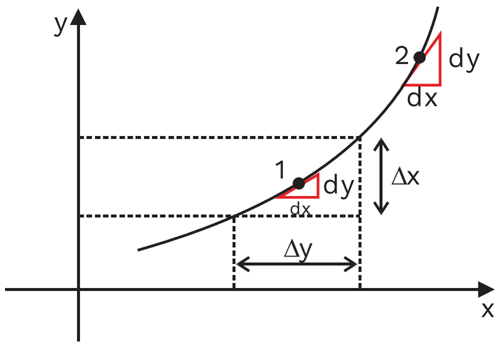


- Slope at point (1)

We can't get correct value of slope by taking big interval for a curved graph.

$\therefore \quad \frac{\Delta y}{\Delta x} \neq$ correct slope

$\frac{dy}{dx}$ = more correct slope (for very small interval)



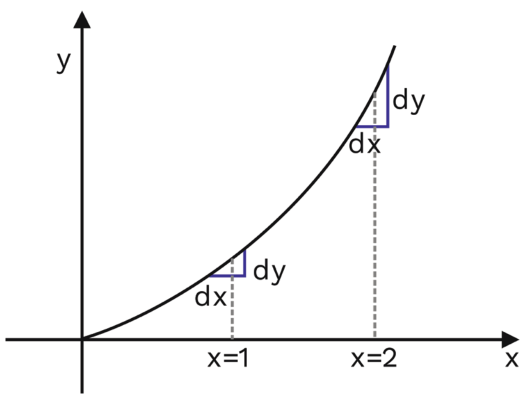
Similarly, by calculating $\frac{dy}{dx}$ at point (2), we can get slope at point 2

- Changing slope = curve

Constant slope = straight line

- $y = x^2$, calculate slope at x = 1 and x = 2

$$\left(\frac{dy}{dx}\right)_{x=1} < \left(\frac{dy}{dx}\right)_{x=2}$$



It is difficult to find slope $\left(\frac{dy}{dx}\right)$ by drawing graph for every function. To ease it, Newton developed formulae in calculus book. We will use those formulae to calculate $\frac{'dy'}{dx}$ for functions whose graphs cannot be drawn easily.

Functions	Differentiation	Examples
$y = x^n$	$\frac{dy}{dx} = nx^{n-1}$	$y = x^7 \to \frac{dy}{dx} = 7x^{7-1} = 7x^6$
$y = x$	$\frac{dy}{dx} = 1$	$y = x^1 \to \frac{dy}{dx} = 1 \cdot x^{1-1} = 1 \cdot x^0 = 1$
$y = ax$	$\frac{dy}{dx} = a$	$y = 7x \to \frac{dy}{dx} = 7 \cdot \frac{dx}{dx} = 7 \cdot 1 = 7$
$y = \sin x$	$\frac{dy}{dx} = \cos x$	
$y = \cos x$	$\frac{dy}{dx} = -\sin x$	
$y = \tan x$	$\frac{dy}{dx} = \sec^2 x$	
$y = \operatorname{cosec} x$	$\frac{dy}{dx} = -\operatorname{cosec} x \cdot \cot x$	
$y = \sec x$	$\frac{dy}{dx} = \sec x \tan x$	
$y = \cot x$	$\frac{dy}{dx} = -\operatorname{cosec}^2 x$	
$y = a^x$ a=constant	$\frac{dy}{dx} = a^x \cdot \log_e a$	$y = 7^x \to \frac{dy}{dx} = 7^x \cdot \log_e 7 = 7^x \ell n 7$, $e \simeq 2.7$
$y = e^x$	$\frac{dy}{dx} = e^x$	$y = e^x \to \frac{dy}{dx} = e^x \log_e e = e^x \cdot 1 = e^x$
$y = \log_e x = \ell nx$	$\frac{dy}{dx} = \frac{1}{x}$	
$y = \sin^{-1} x$	$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$	
$y = \sec^{-1} x$	$\frac{dy}{dx} = \frac{1}{\lvert x \rvert \sqrt{x^2 - 1}}$	
$y = \tan^{-1} x$	$\frac{dy}{dx} = \frac{1}{1+x^2}$	

Rules :

- $y = \text{const.} \Rightarrow \frac{dy}{dx} = 0$
- $y = x \Rightarrow \frac{dy}{dx} = 1$
- $y = u + v$, where $u = f(x)$ and $v = g(x)$
- $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

For example,

a. $y = x^2 + \sin x$

Here $u = x^2$, $v = \sin x$

then, $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} = 2x + \cos x$

b. $y = c \times u$, where c = constant and $u = f(x)$

then, $\frac{dy}{dx} = c \cdot \frac{du}{dx}$

c. $y = 3 \sin x$

then, $\frac{dy}{dx} = 3 \cdot \frac{d}{dx}(\sin x) = 3\cos x$

Product Rule :

For $y = u \times v$, where $u = f(x)$ and, $v = g(x)$

$$\frac{dy}{dx} = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$$

For example,

If $y = x^2 \sin x$

then, $\frac{dy}{dx} = x^2 \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(x^2)$

$\frac{dy}{dx} = x^2 \cdot \cos x + \sin x \cdot 2x = x[x\cos x + 2\sin x]$

Quotient Rule :

- $y = \frac{u}{v}$, where $u = f(x)$ and, $v = g(x)$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$$

For example,

If $y = \dfrac{\sin x}{x^2}$,

then, $\dfrac{dy}{dx} = \dfrac{x^2 \cdot \cos x - \sin x \cdot 2x}{x^4}$

Q. **If $y = \dfrac{e^x}{x^3}$, then find $\dfrac{dy}{dx}$**

Sol. $\dfrac{dy}{dx} = \dfrac{x^3 \dfrac{d}{dx}(e^x) - e^x \dfrac{d}{dx}(x^3)}{(x^3)^2} = \dfrac{x^3 \cdot e^x - e^x \cdot 3x^2}{x^6} = \dfrac{x^2(x-3)e^x}{x^6}$

$\dfrac{dy}{dx} = \dfrac{(x-3)e^x}{x^4}$

Q. **If $y = x^3 \cdot \tan x$, then find $\dfrac{dy}{dx}$**

Sol. $\dfrac{dy}{dx} = x^3 \cdot \dfrac{d}{dx}(\tan x) + \tan x \cdot \dfrac{d}{dx}(x^3)$

$= x^3 \cdot \sec^2 x + \tan x \cdot (3x^2)$

$= x^2(x \sec^2 x + 3 \tan x)$

Q. **If $y = \sqrt{2x}$, $\dfrac{dy}{dx} = ?$**

Sol. $y = \sqrt{2x} = \sqrt{2} \cdot x^{1/2}$ (Here, n = 1/2)

$\dfrac{dy}{dx} = \sqrt{2}\dfrac{d}{dx}(x^{1/2}) = \sqrt{2} \cdot \dfrac{1}{2} \cdot x^{\frac{1}{2}-1} = \dfrac{\sqrt{2}}{2} \cdot x^{\frac{-1}{2}}$

$\dfrac{dy}{dx} = \dfrac{1}{\sqrt{2x}}$

Chain Rule :

In calculus, the chain rule is a formula to compute the derivative of a composite function.

$\rightarrow$ $y = 4\sin(3x)$

$$\frac{dy}{dx} = ?$$

Let, $t = 3x$ ...(1) $\Rightarrow \frac{dt}{dx} = 3$

$y = 4\sin t$ ...(2) $\Rightarrow \frac{dy}{dt} = 4\frac{d}{dt}(\sin t) = 4\cos t$

$$\frac{dy}{dt} * \frac{dt}{dx} = \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = 4\cos t \cdot 3 = 12\cos t \Rightarrow \frac{dy}{dx} = 12\cos(3x)$$

Q. $y = 4e^{x^2-2x}$ $\left[y = e^x, \frac{dy}{dx} = e^x\right]$

Find $\frac{dy}{dx} = ?$

Sol. $t = x^2 - 2x \Rightarrow \frac{dt}{dx} = 2x - 2$

$y = 4e^t \Rightarrow \frac{dy}{dt} = 4 \cdot e^t$

$$\frac{dy}{dx} = \frac{dy}{dt} * \frac{dt}{dx}$$

$$= 4e^t \cdot (2x - 2)$$

$$= 4e^{x^2-2x} \cdot (2x - 2)$$

Q. $y = \ell n(\cos 3x)$, find $\frac{dy}{dx}$.

Sol. $P = 3x$ ...(i) $\Rightarrow \frac{dP}{dx} = 3$

$t = \cos 3x = \cos P$ ...(ii) $\Rightarrow \frac{dt}{dP} = -\sin P$

$y = \ell nt$ ...(iii) $\Rightarrow \frac{dy}{dt} = \frac{1}{t}$

$$\underbrace{\frac{dy}{dt} \times \frac{dt}{dP} \times \frac{dP}{dx}}_{\text{Chain}} = \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{t} \times 3 \times (-\sin P)$$

$$= \frac{1}{\cos 3x} \cdot 3 \cdot (-\sin 3x)$$

$$\frac{dy}{dx} = -3\tan 3x$$

Q. $y = \sqrt{\log_e x} = (\log_e x)^{1/2}$. Find $\frac{dy}{dx}$

Sol. $t = \log_e x \Rightarrow \frac{dt}{dx} = \frac{1}{x}$

$y = t^{1/2} \Rightarrow \frac{dy}{dt} = \frac{1}{2} \cdot t^{-1/2} = \frac{1}{2\sqrt{t}}$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{1}{2\sqrt{t}} \cdot \frac{1}{x}$$

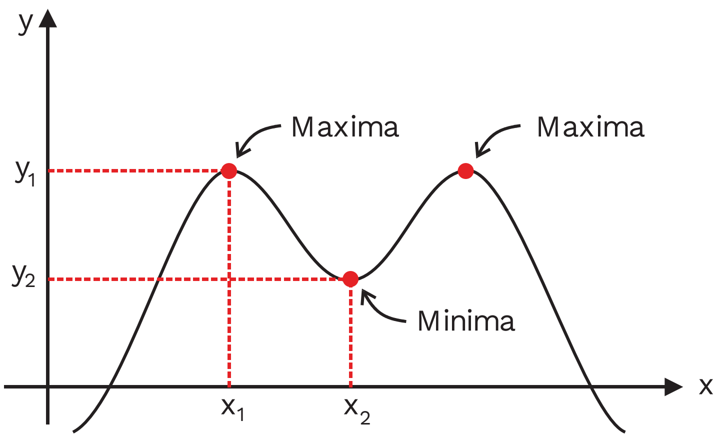
$$\frac{dy}{dx} = \frac{1}{2\sqrt{\log_e x}} \cdot \frac{1}{x}$$

Maxima-Minima :

$y = f(x)$

Maxima : Value of y at that point is greater than values of y at immediate points in the left and right.

Minima : Value of y at minima is less than values of y at immediate points in the left and right.



There can be many number of maxima and minima in a single curve. That is why it is also called as local maxima and local minima.

If on increasing x, y also increases.

$\Rightarrow$ dx > 0 then dy > 0

$$\Rightarrow \frac{dy}{dx} > 0$$

or If on decreasing x, y also decreases.

$\Rightarrow$ dx < 0 then dy < 0

$$\Rightarrow \frac{dy}{dx} > 0$$

Similarly, If on increasing x, y decreases or on decreasing x, y increases

$\Rightarrow$ dx > 0 then dy < 0 or dx < 0 then dy > 0

$$\Rightarrow \frac{dy}{dx} < 0$$

Maxima :

At maxima, slope, m = 0

$$\boxed{\frac{dy}{dx} = 0} \qquad ...(1)$$

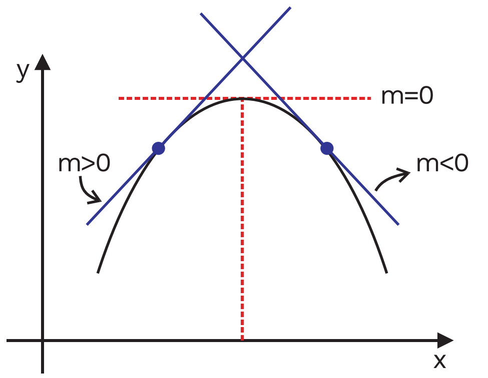
$\frac{d^2y}{dx^2}$ = Double differentiation

Also as x increases, m decreases

(+ve → 0 → −ve)

$$\therefore \frac{dm}{dx} < 0,\ m = \frac{dy}{dx} \Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) < 0$$

$$\Rightarrow \boxed{\frac{d^2y}{dx^2} < 0} \qquad ...(2)$$



Minima :

At minima, slope m = 0

$$\boxed{\frac{dy}{dx} = 0} \qquad ...(1)$$

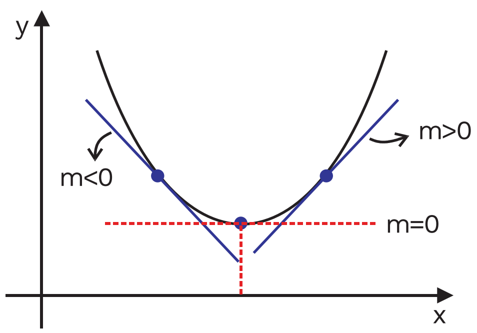
Also, as x increases, m increases

($-v_e \to 0 \to +v_e$)

$$\therefore \frac{dm}{dx} > 0,\ m = \frac{dy}{dx}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) > 0$$

$$\Rightarrow \boxed{\frac{d^2y}{dx^2} > 0} \qquad ...(2)$$

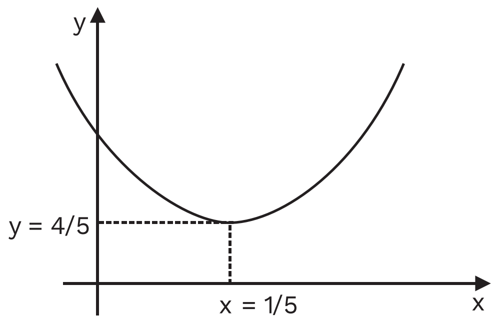


Q. **y = 5x² – 2x + 1, find maxima or minima.**

Sol. $m = \frac{dy}{dx} = 10x - 2$

At maxima/minima, $\frac{dy}{dx} = 0$

$\Rightarrow \; 10x - 2 = 0 \;\Rightarrow\; x = \frac{2}{10} = \frac{1}{5}$



At $x = \frac{1}{5}$ there can be maxima or minima.

$\frac{dm}{dx} = \frac{d^2y}{dx^2} = +10$

As $\frac{d^2y}{dx^2} > 0$,

So, at $x = \frac{1}{5}$, there is a minima.

$y_{min} = 5\left(\frac{1}{5}\right)^2 - 2\left(\frac{1}{5}\right) + 1 \Rightarrow y = \frac{4}{5}$

Q. **For y = x³ – 3x², find max/min.**

Sol. $\frac{dy}{dx} = 3x^2 - 6x$

At maxima or minima, $\frac{dy}{dx} = 0$

$\Rightarrow \; 3x^2 - 6x = 0$

$\Rightarrow \; 3x(x-2) = 0 \quad \left| \begin{array}{l} x = 0 \to \text{max/ min} \\ x = 2 \to \text{max/ min} \end{array} \right.$

Let's check for maxima or minima.

$$\frac{dm}{dx} = \frac{d^2y}{dx^2} = 6x - 6$$

At $x = 0 \Rightarrow \frac{d^2y}{dx^2} = -6$

As $\frac{d^2y}{dx^2} < 0$ at $x = 0$

So, at x = 0, there is a maxima.

$y_{max} = 0^3 - 3(0)^2 = 0$

At $x = 2$, $\frac{d^2y}{dx^2} = +6$

So, at $x = 2$, $\frac{d^2y}{dx^2} > 0$

∴ At x = 2, there is a minima

$ymin = 2^3 - 3(2)^2 = -4$

Q. **$y = \sin x + \cos(2x)$, find $\frac{d^2y}{dx^2}$.**

Sol. For $y = \sin x + \cos(2x)$

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x) + \frac{d}{dx}(\cos 2x) = \cos x + \underbrace{\frac{d}{dx}(\cos 2x)}_{\text{Chain Rule}}$$

Let $z = \cos 2x$

and, $t = 2x \rightarrow \frac{dt}{dx} = 2$

$$z = \cos t \rightarrow \frac{dz}{dt} = -\sin t$$

$$\frac{d}{dx}(\cos 2x) = \frac{dz}{dx} = \frac{dz}{dt} \cdot \frac{dt}{dx} = -\sin t \cdot 2$$

$$\frac{d}{dx}(\cos 2x) = -2\sin 2x \Rightarrow \frac{dy}{dx} = \cos x - 2\sin 2x$$

Now,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\cos x) - 2\frac{d}{dx}(\sin 2x)$$

$$\frac{d^2y}{dx^2} = -\sin x - 2\frac{d}{dx}(\sin 2x)$$

Let m = sin2x

and, $t = 2x \rightarrow \frac{dt}{dx} = 2$

$\Rightarrow \quad m = \sin t \rightarrow \frac{dm}{dt} = \cos t$

$\Rightarrow \quad \frac{d}{dx}(\sin 2x) = \frac{dm}{dx} = \frac{dm}{dt} \times \frac{dt}{dx} = \cos t \times 2 = 2\cos 2x$

$\Rightarrow \quad \frac{d^2y}{dx^2} = -\sin x - 4\cos 2x$

Integration

INTEGRAL CALCULUS

Integration is reverse process of differentiation.

$$\begin{tikzcd}[column sep=huge] y = x^3 \arrow[r, bend left, "\text{Differentiation w.r.t } x \;\; \frac{dy}{dx}"] & 3x^2 \arrow[l, bend left, "\text{Integration w.r.t } x \;\; \int 3x^2dx"] \end{tikzcd} \qquad x^3 = \int 3x^2\,dx$$

For example :

$$\int x^7dx = ?$$

Let, $\int x^7dx = y$

$$\frac{dy}{dx} = x^7 \qquad \text{...(1)}$$

Let, $y = ax^8$

$$\frac{dy}{dx} = 8ax^7 \qquad \text{...(2)}$$

Comparing (1) and (2)

$$1 = 8a \quad \Rightarrow \quad a = \frac{1}{8}$$

$$\therefore \quad y = \frac{x^8}{8}$$

$$\int x^7dx = \frac{x^8}{8}$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad \Leftarrow \text{'C' is constant}$$

But why this constant

$$y = x^3 + C$$

$$\frac{dy}{dx} = 3x^2 + 0 = 3x^2$$

$$\therefore \quad \int 3x^2dx = x^3 + C$$

Similarly, $\int x^7dx = \dfrac{x^8}{8} + C$

Think that integration is reverse process of differentiation and you can easily predict the results given below.

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$\int 1 dx = x + C$$

$$\int a\,dx = ax + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = +\sin x + C$$

$$\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \operatorname{cosec}^2 x\,dx = -\cot x + C$$

$$\int \frac{1}{x} dx = \ell nx + C$$

$$\int e^x dx = e^x + C$$

$$\int a^x dx = \frac{a^x}{\log_e a} + C$$

$$\int \frac{1}{1+x^2} dx = \tan^{-1}(x) + C$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}(x) + C$$

$$\int \frac{1}{|x|\sqrt{x^2-1}} dx = \sec^{-1}(x) + C$$

Rules :

- $\int a\,dx = ax + C$
- $\int (u+v)\,dx = \int u\,dx + \int v\,dx$

For example :

$$\int (x^2 + \sin x)\,dx = \int x^2 dx + \int \sin x\,dx = \frac{x^3}{3} - \cos x + C$$

- $\int a \cdot f(x) dx = a \int f(x)\,dx$

For example :

$$\int 3x^3dx = 3\int x^3dx \quad = 3\frac{x^4}{4} + C \quad = \frac{3}{4}x^4 + C$$

Substitution Method

Q. **Find the integration,**

$$\mathbf{I = \int \sin^3\theta \cdot \cos\theta \, d\theta}$$

Sol. Put $u = \sin\theta \quad \Rightarrow \quad \frac{du}{d\theta} = \cos\theta \quad \Rightarrow \quad du = \cos\theta \, d\theta$

$$I = \int u^3 \, du \quad = \frac{u^4}{4} + C$$

$$I = \frac{\sin^4\theta}{4} + C$$

Q. **Find the integration**

$$\mathbf{I = \int \frac{\tan^5\theta}{\cos^2\theta} d\theta}$$

Sol. $u = \tan\theta$

$$\frac{du}{d\theta} = \sec^2\theta \Rightarrow \; du = \sec^2\theta \, d\theta$$

$$I = \int \tan^5\theta \cdot \sec^2\theta \, d\theta$$

$$= \int u^5 \cdot du \quad = \frac{u^6}{6} + C$$

$$I = \frac{\tan^6\theta}{6} + C$$

Rule :

- $\int \frac{du}{u} = \ell nu + C$

Q. $\mathbf{I = \int \frac{1}{\sin\theta} \times \cos\theta\, d\theta}$ **, find I.**

Sol. $u = \sin\theta \Rightarrow du = \cos\theta\, d\theta$

$$I = \int \frac{1}{u} \cdot du = \ell nu + C$$

$$I = \ell n \sin\theta + C$$

$$\therefore \int \cot\theta\, d\theta = \ell n \sin\theta + C$$

Q. **Find the integral**

$$\mathbf{I = \int \frac{dx}{5 - 2x}}$$

Sol. Put, $t = 5 - 2x$, so that,

$$\Rightarrow \frac{dt}{dx} = -2 \quad \Rightarrow \frac{dt}{-2} = dx$$

$$I = \frac{-1}{2} \int \frac{dt}{t} = -\frac{1}{2} \ell nt + C$$

$$I = -\frac{1}{2} \ell n(5 - 2x)$$

Q. **Evaluate the integral**

$$\mathbf{I = \int \frac{\sqrt{x}}{1 + x\sqrt{x}} dx}$$

Sol. Put, $t = 1 + x\sqrt{x} = 1 + x^{3/2}$

So that, $\dfrac{dt}{dx} = 0 + \dfrac{3}{2}x^{\frac{3}{2}-1} = \dfrac{3}{2}\sqrt{x} \Rightarrow \dfrac{2}{3}dt = \sqrt{x}\,dx$

$$I = \frac{2}{3}\int \frac{dt}{t} = \frac{2}{3}\ell nt + C$$

$$I = \frac{2}{3}\ell n(1 + x\sqrt{x}) + C$$

Q. **Evaluate the integral**

$$\mathbf{I = \int \cos^3\theta \sin\theta \, d\theta}$$

Sol. $t = \cos\theta \Rightarrow \dfrac{dt}{d\theta} = -\sin\theta \Rightarrow dt = -\sin\theta\, d\theta$

$$I = \int t^3(-dt) = -\frac{t^4}{4} + C$$

$$I = \frac{\cos^4\theta}{4} + C$$

Q. **Find integration**

$$\mathbf{I = \int \frac{e^t dt}{\sqrt{1 - e^t}}}$$

Sol. Put $x = 1 - e^t$, so that

$$\frac{dx}{dt} = 0 - e^t \Rightarrow dx = -e^t dt$$

$$I = -\int \frac{dx}{\sqrt{x}} = -\int x^{-1/2}dx$$

$$I = -\left(\frac{x^{\frac{-1}{2}+1}}{\frac{-1}{2}+1}\right) + C$$

$$I = -2x^{1/2} + C = -2\sqrt{x} + C = -2\sqrt{1 - e^t} + C$$

Definite Integration :

In definite integration, we use limits of integration and get a constant value for the constant of integration (c). For the function, y = x^3,

$$\int_{x=1}^{x=2} y\,dx = \int_1^2 x^3\,dx = \left[\frac{x^4}{4}\right]_{\underbrace{1}_{\text{lower limit}}}^{\overbrace{2}^{\text{upper limit}}} = \left[\left(\frac{2^4}{4}\right) - \left(\frac{1^4}{4}\right)\right] = 4 - \frac{1}{4} = \frac{15}{4}$$

Q. **Evaluate the integral,**

$$I = \int_1^2 (2y+1)^7\,dy$$

Sol. Put t = 2y +1

So that, $\frac{dt}{dy} = 2 + 0 = 2 \Rightarrow \;\; dy = \frac{dt}{2}$

So, $I = \int t^7 \frac{dt}{2} = \frac{1}{2}\int t^7 dt = \frac{1}{2}\left[\frac{t^8}{8}\right] = \frac{1}{2}\cdot\frac{1}{8}[t^8]$

$$I = \frac{1}{16}\left[(2y+1)^8\right]_1^2$$

$$I = \frac{1}{16}\left[(5^8) - (3^8)\right] = \frac{5^8 - 3^8}{16} = 24004$$

Area Under the Curve : (Geometrical Meaning of Integration)

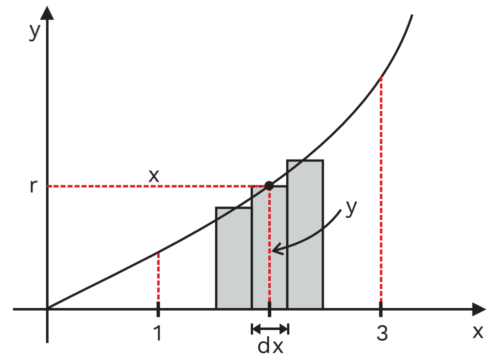
$y = x^2$

$$I = \int_1^3 y\,dx = \int_1^3 x^2\,dx = \left[\frac{x^3}{3}\right]_1^3 = \frac{1}{3}\left[(3^3) - (1^3)\right]$$

$$I = \frac{1}{3}(26) = \frac{26}{3}$$

$$I = \int_1^3 y\,dx$$

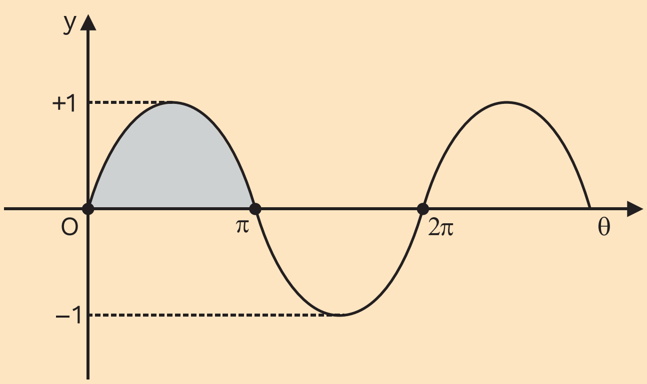
= Area under y vs x curve from x = 1 to x = 3



Q. **Find area under $y = x^3 + 3$, with x-axis from x = 1 to x = 3.**

Sol. $\text{Area} = \int_1^3 y\,dx = \int_1^3 (x^3 + 3)dx = \left[\frac{x^4}{4}\right]_1^3 + 3[x]_1^3 = \left(\frac{3^4 - 1^4}{4}\right) + 3(3 - 1) = 26$ units

Q. **For $y = \sin\theta$**
Find area under y vs from θ = 0 to $\theta = \pi$



Sol. $\text{Area} = \int_0^\pi y\,d\theta = \int_0^\pi \sin\theta\,d\theta$

$= -[\cos\theta]_0^\pi = -[\cos\pi - \cos 0] = -[(-1) - (1)]$

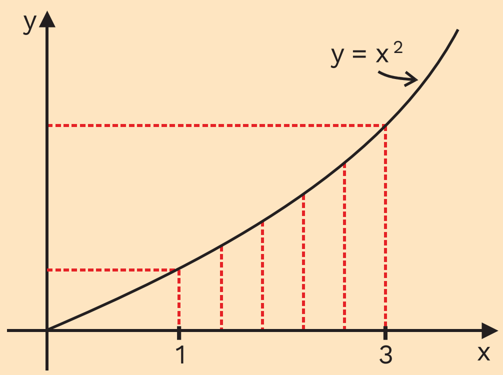
Area = 2 unit

Average Value :

Average value of a function can be found using the relation,

$$\boxed{y_{avg} = \langle y \rangle = \frac{\int_{x_1}^{x_2} y\,dx}{\int_{x_1}^{x_2} dx}}$$

Q. **For y = f(x) = x²**
Find average value of y from x = 1 to x = 3



Sol.

$$\langle y \rangle = \frac{\int_1^3 y\,dx}{\int_1^3 dx} = \frac{\int_1^3 x^2\,dx}{\int_1^3 dx} = \frac{\left[\frac{x^3}{3}\right]_1^3}{[x]_1^3} = \frac{\frac{1}{3}[27-1]}{[3-1]} = \frac{26}{6} \Rightarrow \langle y \rangle = \frac{13}{2}$$

Q. **For y = 2 sinθ**
Find average of y from θ = 0 to q = π.

Sol.

$$\langle y \rangle = \frac{\int_0^\pi y\,d\theta}{\int_0^\pi d\theta} = \frac{\int_0^\pi 2\sin\theta\,d\theta}{\int_0^\pi d\theta} = \frac{2[-\cos\theta]_0^\pi}{[\theta]_0^\pi} = \frac{2(2)}{[\pi-0]} = \frac{4}{\pi}$$

$$\langle y \rangle = \frac{4}{\pi}$$

Vectors

Physical Quantities :

Definition : The quantities by means of which, we described the laws of physics are called physical quantities.

Example : Length, Mass, Time, Force

→ Physical quantities can be measured.

→ Emotions, Feelings, Thinking, Pain are not physical quantities, because they cannot be measured.

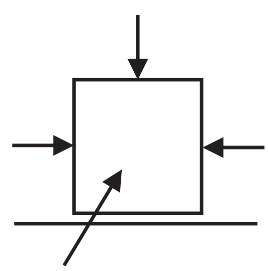
Physical Quantities

- Magnitude only, then it is known as ratio. Ex: Refractive index
- Magnitude and unit, then it is known as scalar. Ex: Mass, charge, current
- Magnitude, unit and direction, then it is known as vector. Ex: velocity, Force, Torque

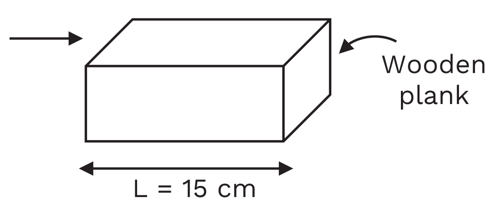
→ If someone asks about your mass and your answer is 60 kg, then person is satisfied because he got the complete answer.

→ If someone asks about location (position) of your school and you only tell the distance of school from your current location, then person will not be satisfied and he will ask about the direction also. Hence position is a vector quantity.

→ A block is kept on a platform and someone asks you to apply a force of 10N on the block. You will ask the person that in which direction you have to apply the force.



∴ Force is a vector quantity because to completely define a force, information of magnitude and direction, both are required.



Length of wooden plank is 15 cm and direction is not required to specify the length. Hence length is a scalar quantity.

- A Physical quantity is vector, if

  → It has magnitude

  → It has direction

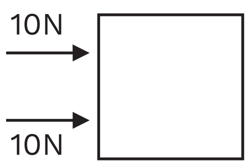
  → It follows rules of vector addition.

- Current is a scalar quantity.

  → Current has magnitude as well as direction but it does not follow laws of vector addition.

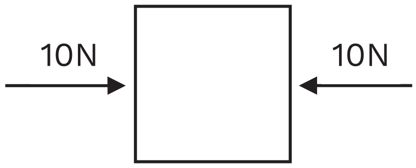
- Vector changes on changing its direction. Value of addition of vectors also changes on changing direction.

  For example : We apply two forces, each of magnitude 10 N in same direction as shown in figure.



Result : Block will move

Now, we apply two forces, each of magnitude 10N but in opposite directions as shown in figure.

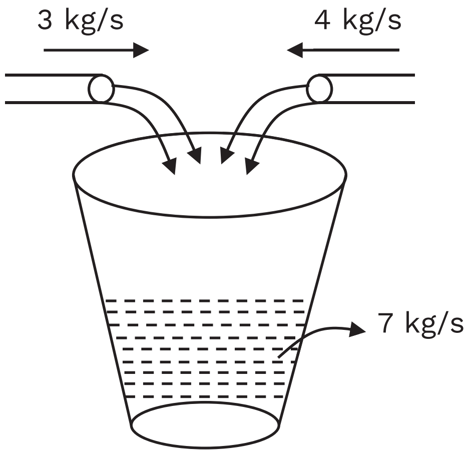


Result : Block will not move

→ Result changes on changing the direction of vector.

∴ Force is a vector quantity.

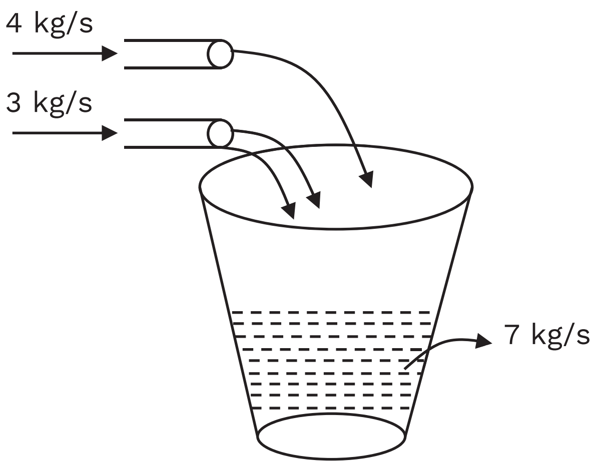
- Water is being filled in a bucket with the help of two pipes.
  3 kg/s & 4 kg/s are mass flow rates of water in pipes. Bucket is filled at a rate 7 kg/s.
  [If it was like 3 lit/s and 4 lit/s then it is volume flow rate of water in pipe. Bucket will be filled at 7 lit/s.]



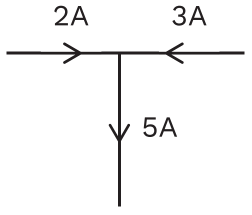
Here also, bucket is filled at 7 kg/s rate.

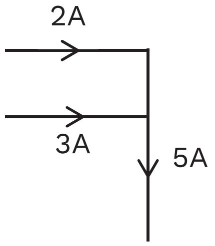
→ No change in result in above cases on changing the direction of flow.

∴ Mass flow rate and volume flow rate are scalar quantities.

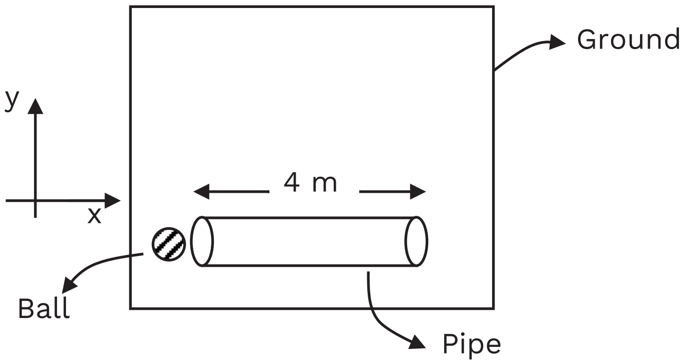


Current :

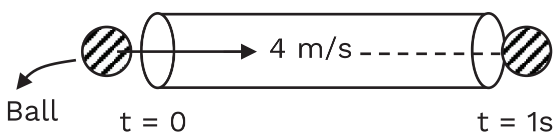




As there is no change in result, hence current is a scalar quantity.
Now, let's take a ball and pipe setup as shown below.



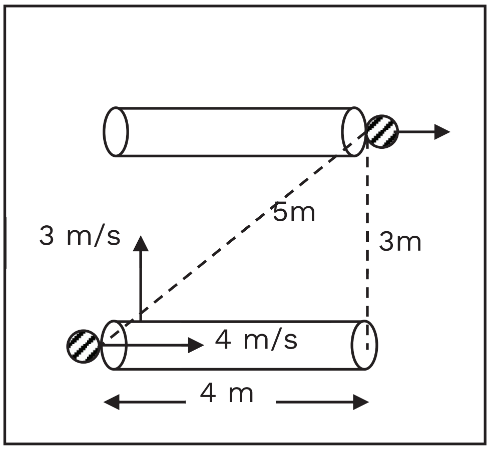
Case I : Ball is given a speed of 4 m/s in x-direction.



Ball will come out of pipe after 1s.

$$t = \frac{4}{4} = 1 \text{ sec}$$

Case II : Ball is given a speed of 4 m/s in x-direction and simultaneously pipe is given a speed of 3 m/s in y-direction.



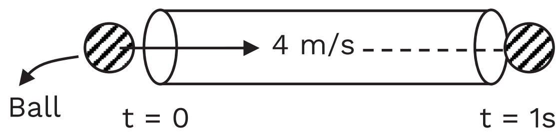
Ball will again come out of pipe after 1s, but this time at a different location.

Ball travels 5m in 1s

∴ Velocity of ball is 5m/s

→ Here ball has been given two velocities, velocity of 4 m/s in x-direction and velocity of 3 m/s in y-direction.

→ Vector addition of two velocities 3 m/s and 4 m/s gives a velocity of 5 m/s in this case.



Case III: Ball is given a velocity of 4 m/s in x-direction and simultaneously pipe is also given a velocity of 3m/s in x-direction.

Ball again comes out of pipe after 1s. This time ball travels 7 m and velocity of ball is 7 m/s.

→ Here resultant of two velocities 3 m/s and 4m/s is 7 m/s.

→ Result changes on changing the direction of velocity.

∴ Velocity is a vector quantity.

VECTOR:

Definition: A physical quantity is said to be a vector is in addition of magnitude and unit, it

→ has specified direction.

→ Obeys the parallelogram law of vector addition.

$R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$

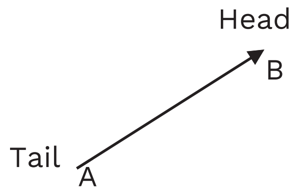
→ And its addition is commutative i.e.

$\vec{A} + \vec{B} = \vec{B} + \vec{A}$

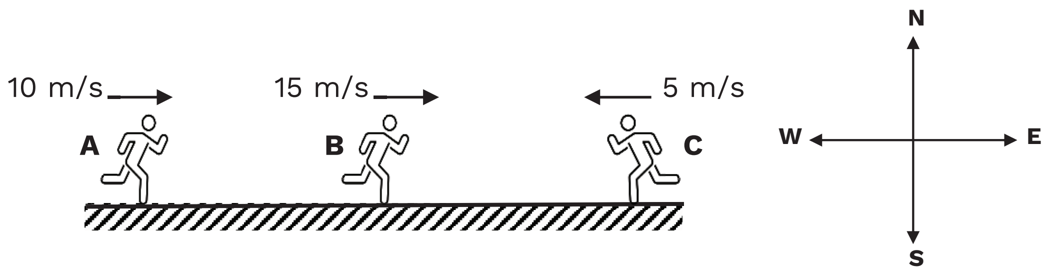
Geometrical representation of vector

Length of arrow : Magnitude of vector quantity.

Direction of arrow head gives its direction.



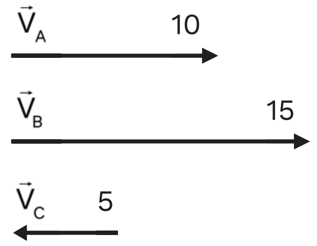
For example, 3 Persons A, B and C are running on road as shown.



Velocity of A, $(\vec{V}_A) = 10\,m/s$, towards East

Velocity of B, $(\vec{V}_B) = 15\,m/s$, towards East

Velocity of C, $(\vec{V}_C) = 5\,m/s$, towards West



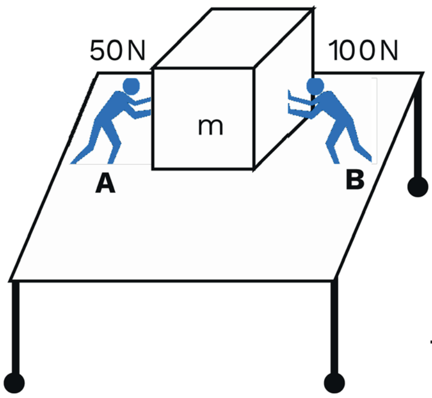
Length of arrow represents the magnitude of vector. Direction of arrow represents the direction of vector.

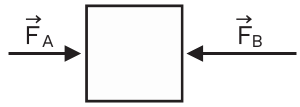
For example :

Two persons A and B are applying forces of 50 N and 100 N respectively on the block as shown.

$\vec{F}_A$ = Force applied by A on block.

= Force applied by B on block.





Types of vector :

→ Zero vector

→ Parallel vector

→ Antiparallel vector

→ Equal vector

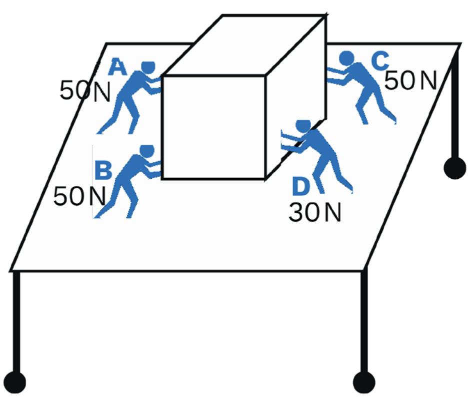
→ Opposite vector

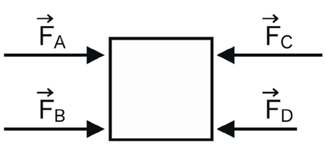
→ Unit vector

→ Co-linear vector

→ Co-planar vector

Let's see these various types in the example below:
Four persons A, B, C and D are applying forces on a block as shown in diagram.





Equal vectors: Vectors having same magnitude as well as same direction.

$\rightarrow \vec{F}_A$ & $\vec{F}_B$ are equal vectors.

Opposite vectors : Vectors having same magnitude but opposite direction.

$\rightarrow \vec{F}_A$ & $\vec{F}_C$ are opposite vectors.

$\rightarrow \vec{F}_B$ & $\vec{F}_C$ are opposite vectors.

Parallel vectors : Vector having same direction but different magnitudes.

$\rightarrow \vec{F}_C$ & $\vec{F}_D$ are parallel.

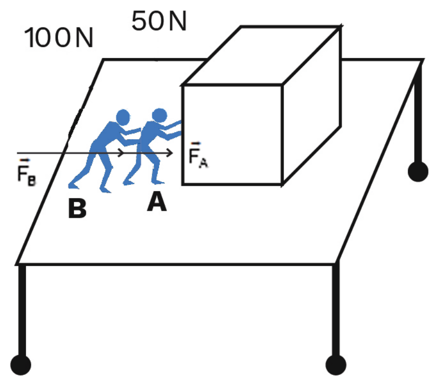
Antiparallel vectors : Vectors having different magnitude but they are in opposite directions.

$\rightarrow \vec{F}_A$ & $\vec{F}_D$ are antiparallel.

$\rightarrow \vec{F}_B$ & $\vec{F}_D$ are antiparallel.

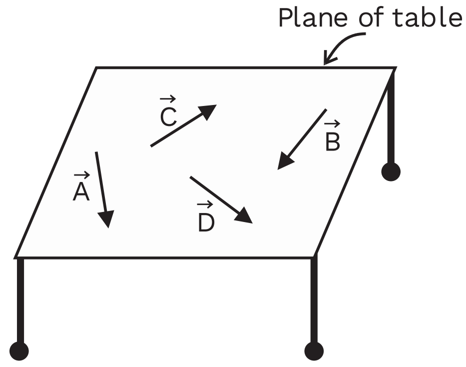
Collinear vectors : Vectors along same line.

$\rightarrow \vec{F}_A$ & $\vec{F}_B$ are collinear vectors.



Co-planar vectors: Vectors on same plane.

$\rightarrow \vec{A}, \vec{B}, \vec{C}, \& \vec{D}$ are co-planar.



Unit Vector :

$\rightarrow$ A vector with magnitude of unity is called unit vector.

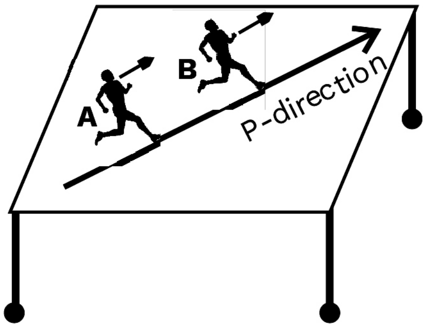
$\rightarrow$ Unit vector in direction of $\vec{a}$ is, $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$

Let's have an example to undersatand unit vector.

Two persons A & B are running along direction 'P' with velocities 10 m/s and 6 m/s respectively.

$\rightarrow$ $V_A = 10\,m/s$ along P-direction

$\rightarrow$ $V_B = 6\,m/s$ along p-direction.

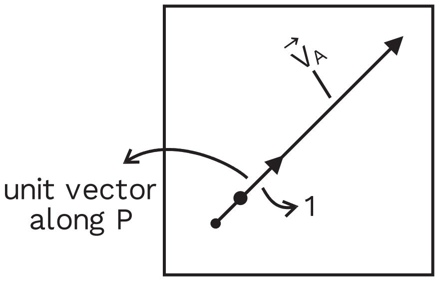


Viewing from the Top

$$\hat{V}_P = \frac{\vec{V}_A}{|\vec{V}_A|} = \text{unit vector in p-direction}$$

$$\vec{V}_B = 6.\hat{V}_P = 6.\frac{\vec{V}_A}{|\vec{V}_A|} = \frac{6}{10}\vec{V}_A$$

$$\vec{V}_B = 0.6\vec{V}_A$$



$\rightarrow$ If you want to find a unit vector in any direction then

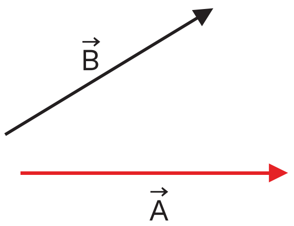
(a) take a vector in that direction.

(b) Divide the vector by its magnitude.

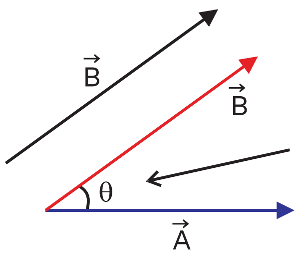
(c) We are left with only direction.

Addition of Vectors :

- Angle between two vectors



$\rightarrow$ We can shift a vector parallel to itself.
$\rightarrow$ To find the angle between two vectors we have to place them tail to tail.

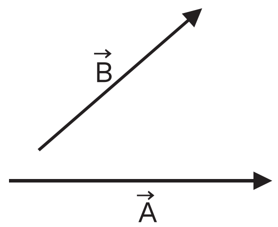


Laws of vector addition :

1) Triangle law
2) Parallelogram law

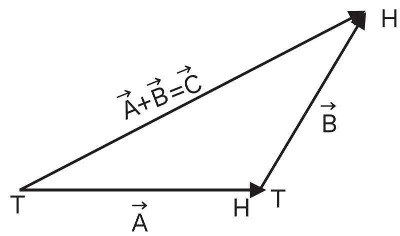
Triangle law :

Given that angle between $\vec{A}$ and $\vec{B}$ is 'θ'. Find $\vec{A} + \vec{B} = \vec{C}$?

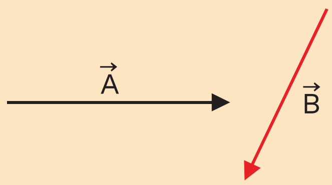


$\underset{1^{st}}{\vec{A}} + \underset{2^{nd}}{\vec{B}} = \vec{C}$

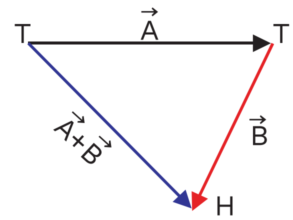
$\rightarrow$ Draw the 1st vector as it is.
$\rightarrow$ Shift the 2nd vector as it is and keep the tail of 2nd vector on the head of 1st vector.
$\rightarrow$ Join the tail of 1st vector with head of 2nd vector and it will give '$\vec{A} + \vec{B}$'



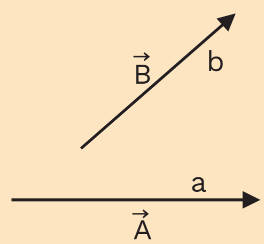
Q. **Find $\vec{A} + \vec{B}$**



Sol.



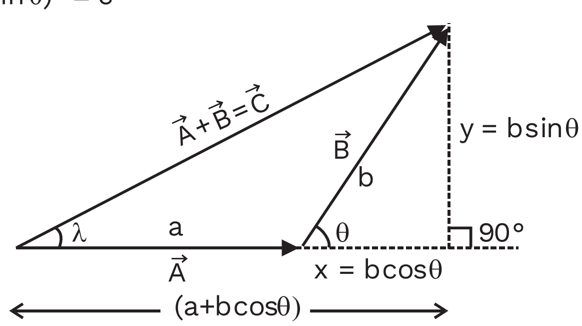
Q. **Angle between $\vec{A}$ & $\vec{B}$ is 'θ' and $|\vec{A}| = a$, $|\vec{B}| = b$. Find $\vec{A} + \vec{B}$?**



Sol. $\frac{x}{b} = \cos\theta \Rightarrow x = b\cos\theta$

$\frac{y}{b} = \sin\theta \Rightarrow y = b\sin\theta$

$(a + b\cos\theta)^2 + (b\sin\theta)^2 = c^2$

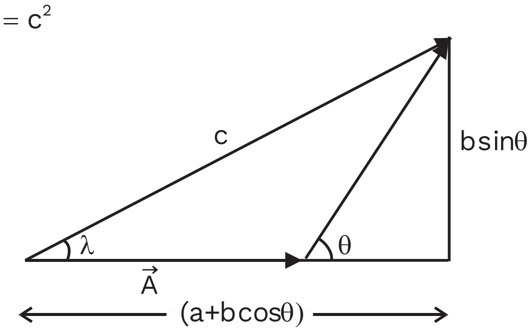


$$\Rightarrow a^2 + b^2\cos^2\theta + 2ab\cos\theta + b^2\sin^2\theta = c^2$$

$$\Rightarrow a^2 + b^2(\sin^2\theta + \cos^2\theta) + 2ab\cos\theta = c^2$$

$$\Rightarrow c^2 = a^2 + b^2 + 2ab\cos\theta$$

$$\boxed{c = \sqrt{a^2 + b^2 + 2ab\cos\theta}}$$



Angle of $\vec{B}$ with $\vec{A}$ is 'θ'

Angle of $\vec{C}$ with $\vec{A}$ is 'λ'

$$\boxed{\tan\lambda = \frac{b\sin\theta}{a + b\cos\theta}}$$

Q. **Angle between two vectors $\vec{A}$ and $\vec{B}$ is 60°. $|\vec{A}| = 5$, $|\vec{B}| = 10$. Then find**

Sol. $|\vec{A} + \vec{B}|$

$|\vec{A}| = a = 5$, $|\vec{B}| = b = 10$, $\theta = 60°$

$$c = \sqrt{a^2 + b^2 + 2ab\cos\theta} = \sqrt{5^2 + 10^2 + 2\cdot 5\cdot 10\cdot\cos 60^\circ}$$

$$c = \sqrt{25 + 100 + 100\times\frac{1}{2}} = \sqrt{175}$$

$$|\vec{A} + \vec{B}| = c = \sqrt{175}$$

Angle between $\vec{A}$ & $\vec{C}$

$$\tan\lambda = \frac{b\sin\theta}{a + b\cos\theta} = \frac{10\times\frac{\sqrt{3}}{2}}{5 + 10\times\frac{1}{2}} = \frac{10\frac{\sqrt{3}}{2}}{5+5} = \frac{\sqrt{3}}{2}$$

$$\tan\lambda = \frac{\sqrt{3}}{2}$$

$$\lambda = \tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

Q. $|\vec{A}| = 4$, $|\vec{B}| = 3$ and $|\vec{A}+\vec{B}| = 5$. Find angle between $\vec{A}$ and $\vec{B}$.

Sol. $|\vec{A}+\vec{B}| = c = \sqrt{a^2+b^2+2ab\cos\theta}$

$c^2 = 5^2 = 4^2 + 3^2 + 2\times4\times3\times\cos\theta$

$\Rightarrow \quad 25 + 24\cos\theta = 25 \Rightarrow \quad 24\cos\theta = 25-25 = 0$

$\Rightarrow \quad \cos\theta = 0 \Rightarrow \boxed{\theta = 90^\circ}$

Q. Angle between $\vec{A}$ and $\vec{B}$ is 120°, $|\vec{A}| = a$ and $|\vec{B}| = a$. Find $|\vec{A}+\vec{B}|$ and angle between $\vec{C}$ & $\vec{A}$.

Sol.

$c = \sqrt{a^2+b^2+2ab\cos\theta}$

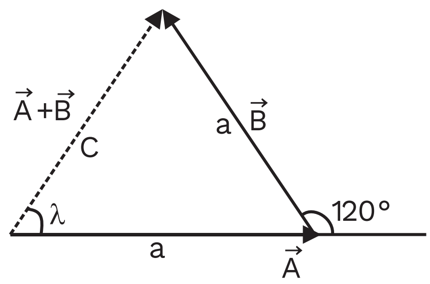
$c = \sqrt{a^2+a^2+2a\cdot a\cos120^\circ}$

$\cos120^\circ = -\frac{1}{2}$

$c = \sqrt{a^2+a^2+2\cdot a\cdot a\times\left(-\frac{1}{2}\right)}$

$c = \sqrt{a^2+a^2-a^2} \quad \Rightarrow c = a$

$\tan\lambda = \frac{a\sin120^\circ}{a+a\cos120^\circ} \Rightarrow \tan\lambda = \sqrt{3} \Rightarrow \quad \lambda = 60^\circ$

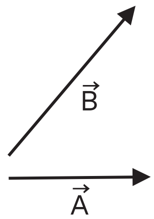


Parallelogram law :

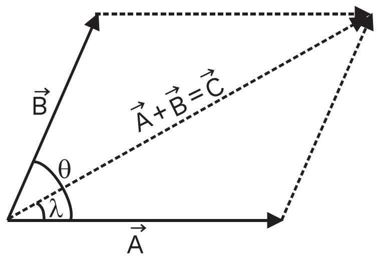
Angle between $\vec{A}$ and $\vec{B}$ is ‘θ’.

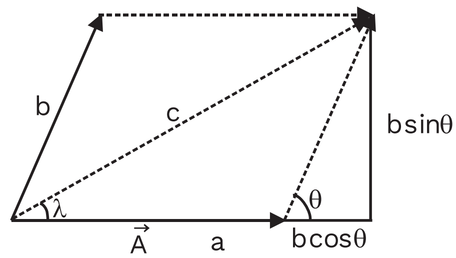
Find, $\vec{A}+\vec{B} = \vec{C}$

As per parallelogram law,



Draw the two vectors such that their tails coincide. Draw a line parallel to $\vec{A}$ passing through head of $\vec{B}$. Draw a line parallel to $\vec{B}$ passing through head of $\vec{A}$. Point of intersection of these lines when joined from the common tail point of $\vec{A}$ and $\vec{B}$, it gives $\vec{A}+\vec{B}$.



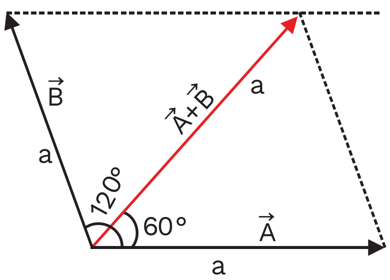


$$c = \sqrt{a^2 + b^2 + 2ab\cos\theta} \;\;\&\;\; \tan\lambda = \frac{b\sin\theta}{a + b\cos\theta}$$

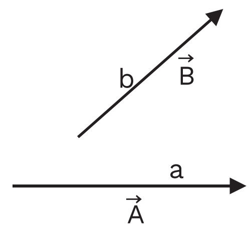
Q. **$|\vec{A}| = a$, $|\vec{B}| = a$, Angle between $\vec{A}$ & $\vec{B}$, $\theta = 120°$. Find $\vec{A} + \vec{B} = \vec{C}$**

Sol. $|\vec{C}| = a$ ...... (diagonal of the parallelogram)

Angle of $\vec{C}$ with $\vec{A}$ is 60°.



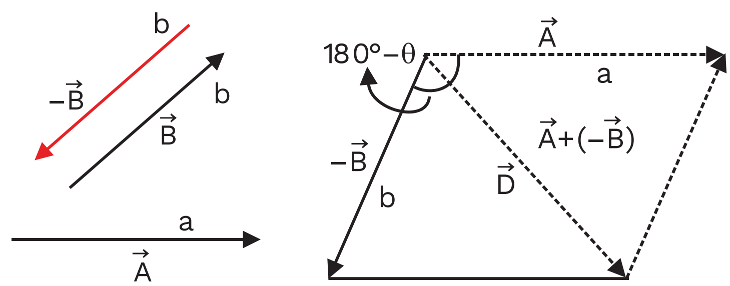
Subtraction of vectors: Subtraction is also a type of addition.



$\vec{A}$ **,** $\vec{B}$ are given. Angle between $\vec{A}$ & $\vec{B}$ is ‘θ’ .

Find $\vec{A}-\vec{B}=\vec{D}$

$\vec{A}-\vec{B}=\vec{A}+(-\vec{B})$

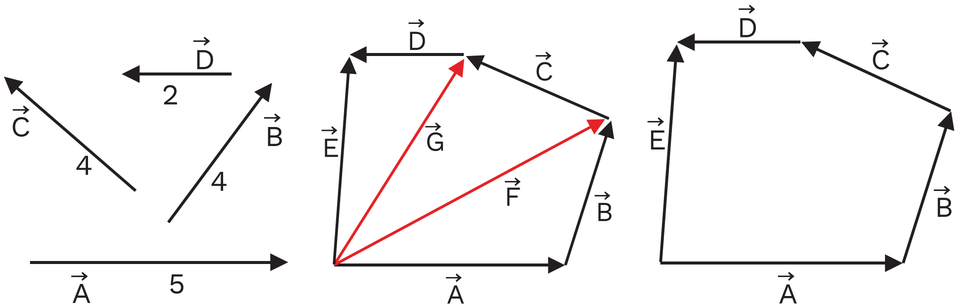


$$|\vec{D}|=\sqrt{a^2+b^2+2ab\cos(180^\circ-\theta)}$$

$$\boxed{d=\sqrt{a^2+b^2-2ab\cos\theta}}$$

Polygon law :

$\vec{A}+\vec{B}+\vec{C}+\vec{D}=\vec{E}$



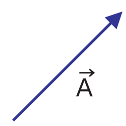
$\vec{A}+\vec{B}=\vec{F}$

$\Rightarrow\ \vec{F}+\vec{C}+\vec{D}=\vec{E}$

$\vec{F}+\vec{C}+\vec{G}$

$\Rightarrow\ \vec{G}+\vec{D}=\vec{E}$

Multiplication of a vector by a scalar (number) :



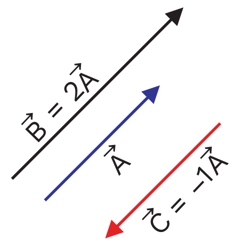
$\vec{B} = 2\vec{A}$
In general
$\vec{B} = n\vec{A}$

If ‘n’ is positive

- magnitude becomes ‘n’ times
- direction remains same

If ‘n’ is negative

- magnitude becomes |n| times
- direction becomes opposite



$\vec{C} = -1\vec{A}$

Components : Resolution of a Vector :

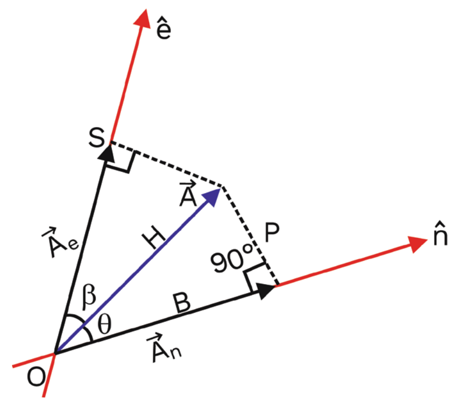
$\frac{B}{H} = \frac{A_n}{A} \Rightarrow A_n = A \cdot \left(\frac{B}{H}\right)$

$\Rightarrow \boxed{A_n = A\cos\theta}$

'$\vec{A}_n$' is component of '$\vec{A}$' in '$\hat{n}$' direction.

$OS = A_e = A\cos\beta$

'$\vec{A}_e$' is component of '$\vec{A}$' in $\hat{e}$ direction



- Meaning of component is effect.

For example,

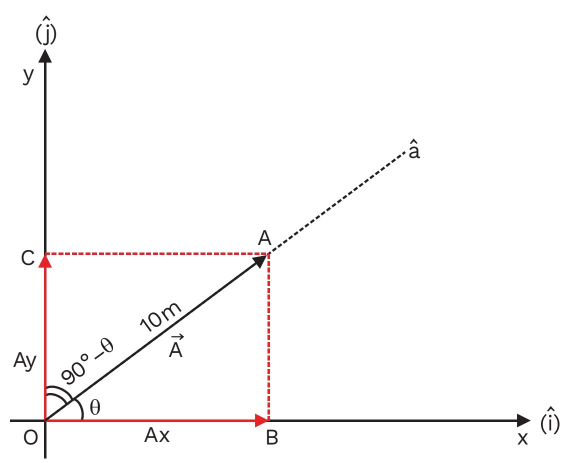
A person moves by 10 m in '$\hat{a}$' direction as shown.

$A_x = A\cos\theta$

$\vec{A}_x = A\cos\theta\hat{i}$

$OC = A_y = A\cos(90° - \theta) = A\sin\theta$

$\vec{A}_y = A\sin\theta\hat{j}$

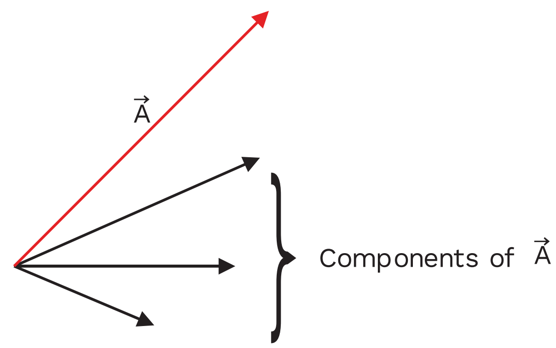


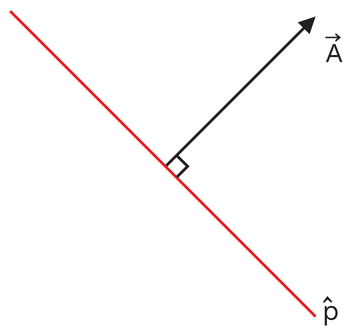
•A vector can have many components.

Component of a vector at 90° is “zero”.

$$A_{90^\circ} = A\cos 90^\circ = 0$$

If a person moves in y-direction then its x co-ordinate will not change.



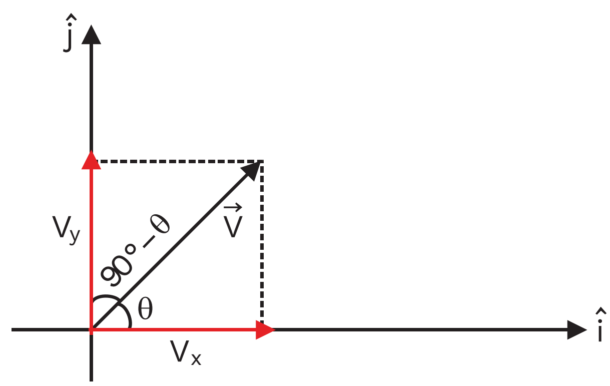


$\theta = 90^\circ$

$\vec{A}_p = 0$

Rectangular Components :

Any set of two components at 90° with each other are called rectangular components.



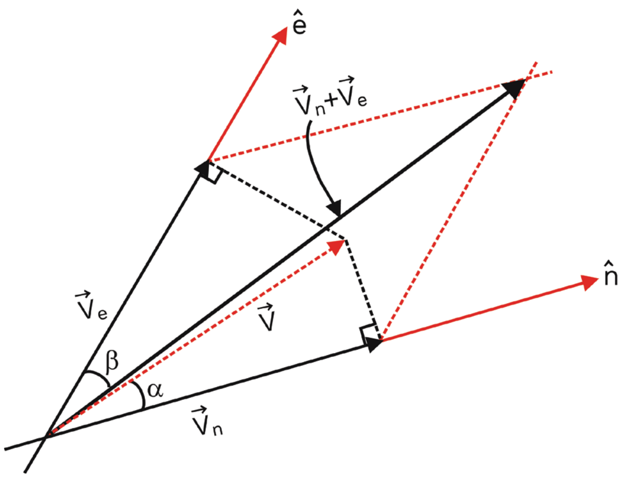
$$V_x = V\cos\theta$$
$$V_y = V\cos(90^\circ - \theta) = V\sin\theta$$
$$\Rightarrow \vec{V}_x = V\cos\theta\,\hat{i}$$
$$\vec{V}_y = V\sin\theta\,\hat{j}$$

- When rectangular components are added, we get original vector.

$$\vec{V} = \vec{V}_x + \vec{V}_y$$

$$\vec{V} = V\cos\theta\,\hat{i} + V\sin\theta\,\hat{j}$$

- For $\alpha + \beta \neq 90°$ $\Rightarrow$ $\vec{V}_e + \vec{V}_n \neq \vec{V}$



→ "A vector can have many sets of rectangular components."

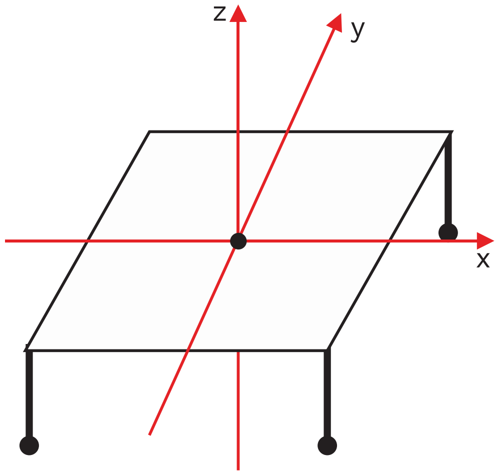
Coordinate System :

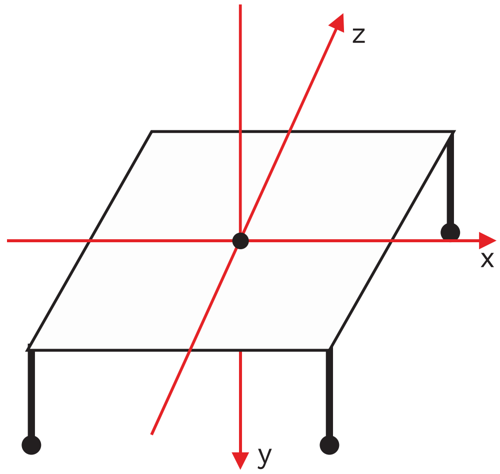
Right handed coordinate system : x, y, z axes are selected by "Right Hand Rule".

— Put your hand along x-axis

— Curl your four fingers towards y-axis

— Thumb gives direction of z-axis





3-Rectangular Components :

$$V_x = V\cos\alpha$$

$$V_y = V\cos\beta$$

$$V_z = V\cos\gamma$$

$$OS = \sqrt{V_x^2 + V_z^2}$$

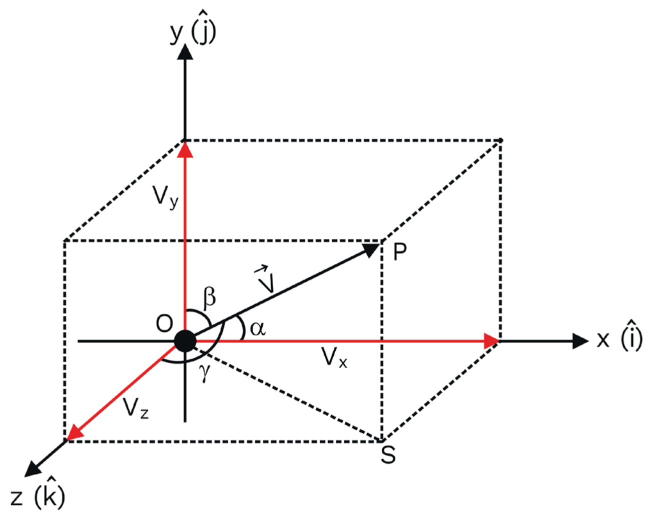
$$OP = \sqrt{OS^2 + SP^2}$$

$$SP = V_y$$

$$OP = \sqrt{V_x^2 + V_y^2 + V_z^2}$$

$$V = \sqrt{V_x^2 + V_y^2 + V_z^2}$$

$$\vec{V} = V\cos\alpha\,\hat{i} + V\cos\beta\,\hat{j} + V\cos\gamma\,\hat{k}$$



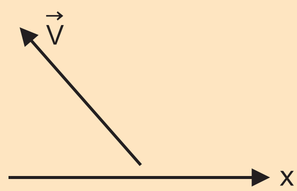
Now,

$$V = \sqrt{(V\cos\alpha)^2 + (V\cos\beta)^2 + (V\cos\gamma)^2}$$

$$\Rightarrow V^2 = V^2[\cos^2\alpha + \cos^2\beta + \cos^2\gamma]$$

$$\boxed{\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1}$$

Q. **$\vec{V}$ makes an angle 135° with x-axis write $\vec{V}$ in terms of $\hat{i}$ and $\hat{j}$**

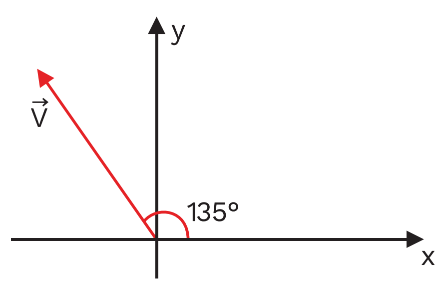


Sol. $\vec{V} = V\cos\theta\,\hat{i} + V\sin\theta\,\hat{j}$, Here, $\theta = 135°$

$$\vec{V} = V\cos(135°) + V\sin(135°)$$

$$= V\left(-\frac{1}{\sqrt{2}}\right)\hat{i} + V\left(\frac{1}{\sqrt{2}}\right)\hat{j}$$

$$\vec{V} = -\frac{V}{\sqrt{2}}\hat{i} + \frac{V}{\sqrt{2}}\hat{j}$$



Q. $\vec{A} = 2\hat{i} + 3\hat{j} + \hat{k}$, draw $\vec{A}$. Also find magnitude of $\vec{A}$ i.e. $|\vec{A}|$

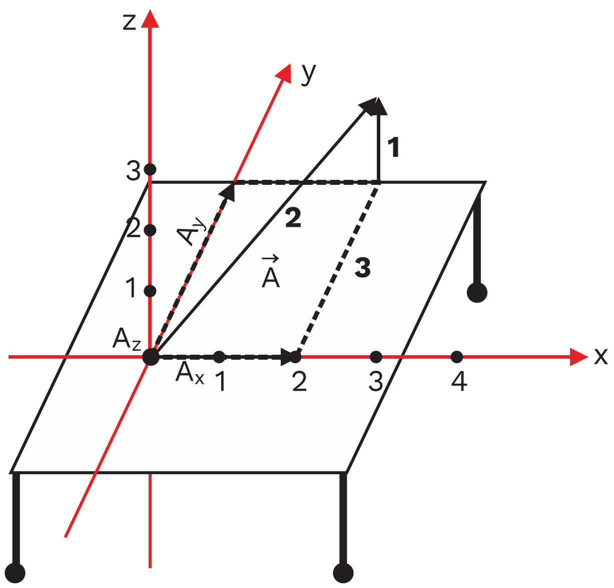
Sol. $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

$A_x = 2,\ A_y = 3,\ A_z = 1$

$|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$

$= \sqrt{2^2 + 3^2 + 1^2}$

$|\vec{A}| = \sqrt{14}$



Product of Vectors :

1) Vector Product (Cross Product) $\Rightarrow$ Result is a vector.

2) Scalar Product (Dot Product) $\Rightarrow$ Result is a scalar.

- Vector Product of $\vec{A}$ and $\vec{B} \Rightarrow \quad \vec{A} \times \vec{B} = \vec{C}$ (vector)
- Scalar Product of $\vec{A}$ and $\vec{B} \Rightarrow \quad \vec{A} \cdot \vec{B} = D$ (scalar)

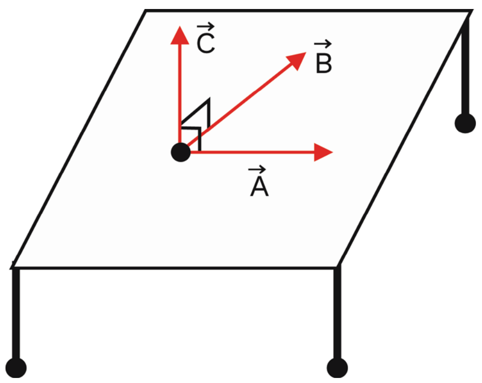
$\vec{A} \times \vec{B} = (ab\sin\theta)\,\hat{e}$

$\vec{A} \cdot \vec{B} = ab\cos\theta$

Where $|\vec{A}| = a$ and $|\vec{B}| = b$

and 'θ' is angle between $\vec{A}$ and $\vec{B}$

Cross Product :



$\vec{A} \times \vec{B} = \vec{C}$

- $\vec{C}$ is perpendicular to $\vec{A}$ and $\vec{B}$. both
- $\vec{C}$ is perpendicular to plane formed by $\vec{A}$ and $\vec{B}$.
- Direction of $\vec{C}$ is given by right hand thumb rule.

$\vec{A} \times \vec{B} = \vec{C}$

To find direction of $\vec{C}$

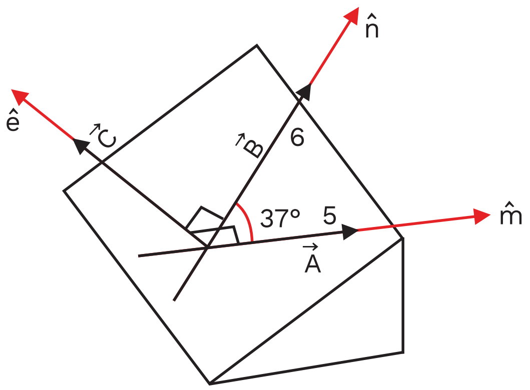
Place your stretched right palm perpendicular to the plane of $\vec{A}$ and $\vec{B}$ in such a way that the fingers are along $\vec{A}$ and when the fingers are closed, they go towards $\vec{B}$. Then, direction of thumb gives the direction of $\vec{C}$.

OR $\vec{A} \times \vec{B} = \vec{C} = AB\sin\theta\hat{n}$

- Direction of $\hat{n}$ is found out by same method.

Q. **If $\vec{A} = 5\hat{m}$ and $\vec{B} = 6\hat{n}$. Angle between $\hat{m}$ and $\hat{n}$ is 37°. Find $\vec{A}\times\vec{B} = \vec{C}$.**

Sol.



$$\vec{A}\times\vec{B} = \vec{C} = AB\sin\theta\,\hat{e}$$

$$= 5\times 6\times \sin 37°\hat{e}$$

$$= 5\times 6\times \frac{3}{5}\hat{e}$$

$$\vec{C} = 18\,\hat{e}$$

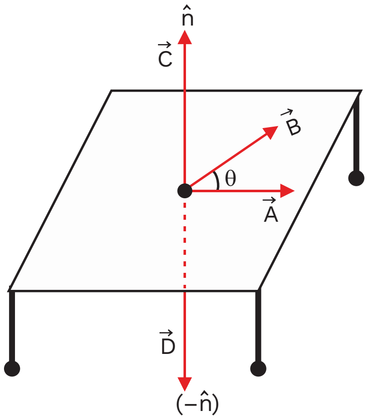
$\vec{A}\times\vec{B} = \vec{C} = ab\sin\theta\,\hat{n}$

$\vec{B}\times\vec{A} = \vec{D} = ba\sin\theta\,(-\hat{n})$

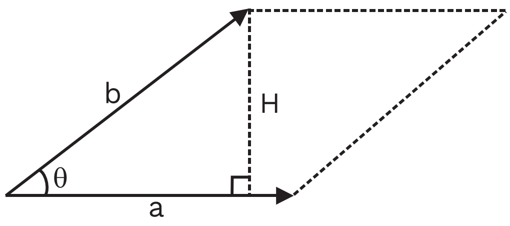
$\vec{A}\times\vec{B}$ and $\vec{B}\times\vec{A}$ have same magnitude, but opposite direction.

$|\vec{A}\times\vec{B}| = |\vec{B}\times\vec{A}| = ab\sin\theta$, but

$$\boxed{(\vec{A}\times\vec{B}) = -(\vec{B}\times\vec{A})}$$



$|\vec{A}\times\vec{B}| = ab\sin\theta$ = Base × Height = Area of Parallelogram



Dot Product :

$\vec{A}\cdot\vec{B} = ab\cos\theta$

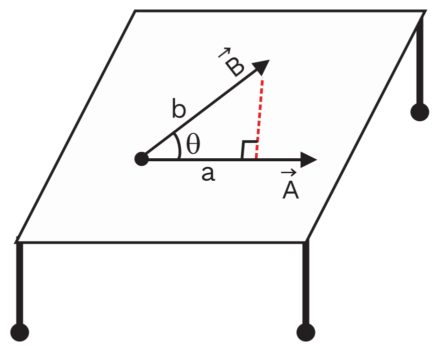
$\vec{A}\cdot\vec{B} = a(b\cos\theta)$

= a × length of projection of $\vec{B}$ on $\vec{A}$

= a × component of $\vec{B}$ along $\vec{A}$

$\vec{A}\cdot\vec{B} = (a\cos\theta)b$

= length of projection of $\vec{A}$ on $\vec{B}\times b$



Q. $\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{B} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

Find : 1) $\vec{A}\times\vec{B} = \vec{C}$ 2) $\vec{A}\times\vec{B} = d$

Sol.

1) $$|\vec{A}\times\vec{B}| = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= (a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

If $\vec{A} = 2\hat{i} + 3\hat{j} - 2\hat{k}$

$\vec{B} = 2\hat{i} + 4\hat{j} + 1\hat{k}$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -2 \\ 2 & 4 & 1 \end{vmatrix}$$

$$= [3\times1 - 4\times(-2)]\hat{i} - [2\times1 - 2\times(-2)]\hat{j} + [2\times4 - 2\times3]\hat{k}$$
$$= 11\hat{i} - 6\hat{j} + 2\hat{k}$$

2) $\vec{A} \cdot \vec{B} = d = a_1b_1 + a_2b_2 + a_3b_3$

If $\vec{A} = 2\hat{i} + 3\hat{j} - 2\hat{k}$

$\vec{B} = 2\hat{i} + 4\hat{j} + \hat{k}$

$\vec{A} \cdot \vec{B} = 2 \times 2 + 3 \times 4 + (-2) \times 1 = 4 + 12 - 2 = 14$

- Angle between $\vec{A}$ and $\vec{B}$

$$\vec{A} \cdot \vec{B} = ab\cos\theta$$

$$\cos\theta = \frac{\vec{A} \cdot \vec{B}}{ab}$$

$a = |\vec{A}|$ and $b = |\vec{B}|$

$\cos\theta = \dfrac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|}$, where '$\theta$' is angle between $\vec{A}$ and $\vec{B}$

$$|\vec{A}| = \sqrt{2^2 + 3^2 + (-2)^2} = \sqrt{17}$$

$$|\vec{B}| = \sqrt{2^2 + 4^2 + 1^2} = \sqrt{21}$$

$$\cos\theta = \frac{14}{\sqrt{17}\sqrt{21}}.$$

- When two vectors are perpendicular (θ = 90°), their dot product is zero.

$$\vec{A} \cdot \vec{B} = AB\cos 90° = 0$$