Function

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Function

Definition of Set:

A set is a collection of distinct objects, considered as an object in its own right.

Ex.

A = {1, 2, 3, 4}; B = {a, e, i, o, u}.

Note:

1. A set is generally represented by a capital letter.
2. The elements of set are written within the braces.
3. The numbers 2, 4 and 6 are distinct objects when considered separately, but when they are considered collectively, they form a single set of size three, written {2, 4, 6}.
4. No element in the set is repeated.
5. Set is a collection in which order of elements is not important.

Ex. $\{1, 2, 3\} \equiv \{3, 2, 1\}$.

Know the facts

There is no set like {1, 2, 1}.

Types of sets

Roster form / Tabular form

Keywords: List of elements
Ex.: {2,4,6}

Set Builder form

Keywords: Definition/property
Ex.: {x| x is one of the three smallest even numbers}

Roster form

Representation of a set that lists all the elements in the set, separated by commas, within braces.
Ex. {–3, –2, –1, 0, 1, 2}

Set Builder form

Mathematical notation for describing a set by enumerating its elements or stating the properties that its members must satisfy.
Ex. $\{x \mid -4 \le x < 3,\ x \in R\}$

Note:

1. Here '|' means "such that" and after "|", the property or definition is written.
2. We can also use ':' instead of "|".

Ex.

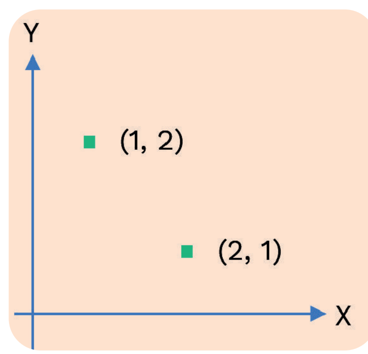
{x|x is a vowel} → Set Builder form
{a, e, i, o, u} → Roster form.
Both sets are same.

Ordered Pair

An ordered pair (a, b) is a pair of objects. The order in which the objects appear in the pair is significant.

Note:

1. The ordered pair (a, b) is different from the ordered pair (b, a) unless a = b.
2. It can be understood if the pair is taken as a point.
3. Here, (1, 2) and (2, 1) represent different points, so cannot be considered equal.



Know the facts

$(1, 2) \neq (2, 1)$

Cartesian product

It is product of 2 sets.

$$\underbrace{A \times B}_{\text{new set}} = \underbrace{\{(a,b) \mid a \in A \text{ and } b \in B\}}_{\text{element of new set}}$$

Ex. A = {a, b} and B = {1, 2, 3}
A × B = {(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)}
B × A = {(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)}

Point to Remember!!!

$$\underbrace{n(A \times B)}_{\text{no of elements in set } A\times B} = \underbrace{n(A)}_{\text{no of elements in set A}} \times \underbrace{n(B)}_{\text{no of elements in set B}}$$

Using this ${}^{2}C_1 \times {}^{3}C_1 = 2 \times 3 = 6$

Note:

It can be seen that all elements of $A \times B$ and $B \times A$ are not equal.

$\therefore$ $A \times B \neq B \times A$

A brief introduction to "Relation"

Any subset of $A \times B$ is a "Relation", from $A \to B$ (pronounced A to B).

Ex.

If $n(A \times B) = 45$, then number of possible relations in $A \to B$ is

$${}^{45}C_0 + {}^{45}C_1 + \ldots + {}^{45}C_{45} = 2^{45}$$

Point to Remember!!!

If possible, $n(A) = p$; $n(B) = q$, then number of possible relations in $A \to B$ is 2^{pq}.

Function

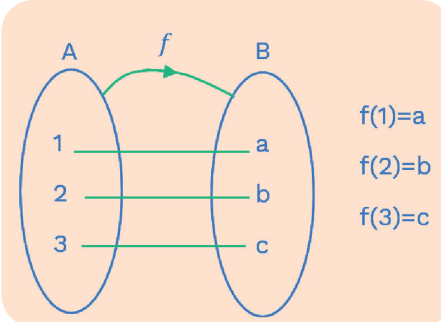
Definition:

A relation from a set A to a set B is called a function if

(i) Each element of set 'A' is associated with some element in set 'B'.

(ii) Each element of set 'A' has unique image in set 'B'.

Ex.



$\therefore f \equiv \{(1, a), (2, b), (3, c)\}$

So, it can be said that $f \subset A \times B$.

Image and Pre-image

If an element ($a \in A$) is associated with an element ($1 \in B$), then '1' is called, the

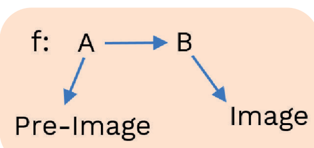
"f image of a" or "image of a under f"

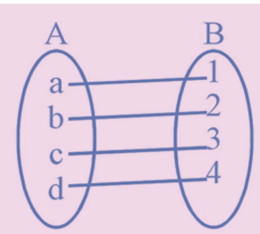
or

"the value of the function f at a"

or

"argument of a under the function f".

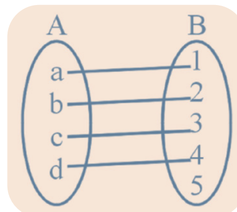




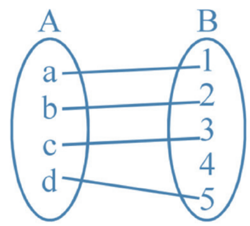
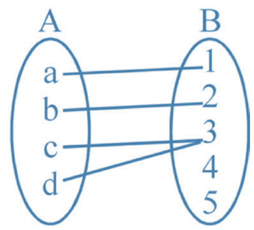
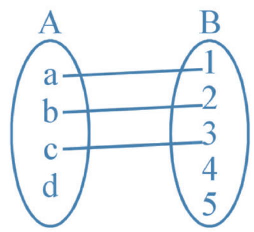
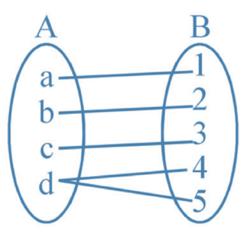
"a is called the pre-image of 1"

Ex.

f = {(a, 1), (b, 2), (c, 3), (d, 4)}
A = {a, b, c, d}, B = {1, 2, 3, 4, 5}
f: A $\rightarrow$ B



Ex.

A = {a, b, c, d}, B = {1, 2, 3, 4, 5} f: A $\rightarrow$ B		
(i)		is a function. Every element in A has a unique image in B.
(ii)		is a function. Every element in A has a unique image in B.
(iii)		is not a function. ('d' has no image in B)
(iv)		is not a function. ('d' does not have a unique image in B)

Domain, Codomain and Range

$A = \{a, b, c, d\}$, $B = \{1, 2, 3, 4, 5\}$

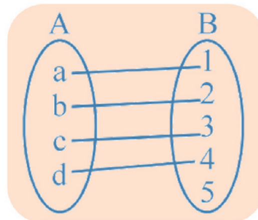
Domain $\rightarrow \{a, b, c, d\}$

Codomain $\rightarrow \{1, 2, 3, 4, 5\}$

Range $\rightarrow \{1, 2, 3, 4\}$

If f: A → B

Domain Codomain



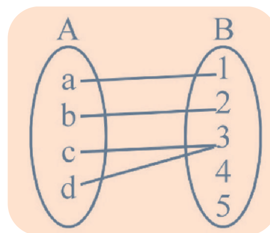
Range can be said to be the collection of functional outputs.

Ex.

Domain → {a, b, c, d}

Codomain → {1, 2, 3, 4}

Range → {1, 2, 3}



Note: Range $\subseteq$ Codomain

Note:
When only "rule of function" is given
(i) It is called "Real valued function".
(ii) Domain ≡ Set of real 'x' for which y is real (Input values).
(iii) Range ≡ Set of all real y values obtained after putting real x in domain (All output values).

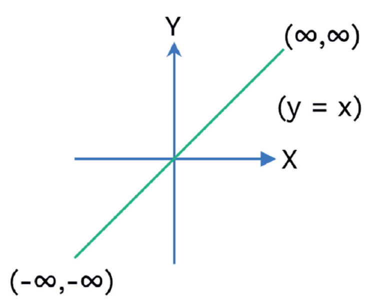
Q. **Find domain of**
(i) y = x

A. Here, it can be seen that all the values in $(-\infty, \infty)$ can be used as input as we move from left to right of graph.

At the same time, y achieves all values in $(-\infty, \infty)$ as we move from bottom to top of graph.

∴ Domain ≡ $x \in R$

Range ≡ $\mathbf{y} \in \mathbf{R}$



Q. **(ii) $y = 2x - 1$**

A. Domain $\equiv x \in R$

M-1: For range, it can be seen from graph

$\therefore$ Range $\equiv y \in R$

M-2: $y = 2x - 1$

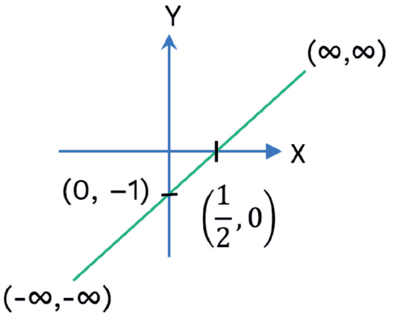
Since $x \in (-\infty, \infty)$

$y = 2(-\infty, \infty) - 1$

$= (-\infty, \infty) - 1$

$= (-\infty, \infty)$.

$\therefore$ $\mathbf{y \in R}$.



Q. **(iii) $y = 3x + 4$**

A. Similarly,

Domain $\equiv x \in R$

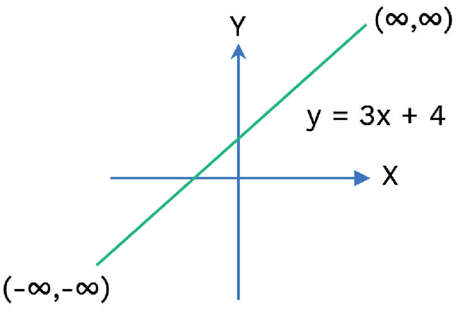
Range $\equiv y \in R$

So, In general, for $y = ax + b$

(linear, $a \neq 0$),

Domain $\equiv x \in R$

Range $\equiv \mathbf{y} \in \mathbf{R}$.



Solving Trick (for domain):

Try to think of values of x for which real value of y does not exist.

Q. **Find the domain of**

(i) $y = \frac{1}{x}$

A. Domain: $x \in (-\infty, 0) \cup (0, \infty)$

(at $x = 0$, y does not exit)

For range, draw the graph and check especially for endpoints of intervals.

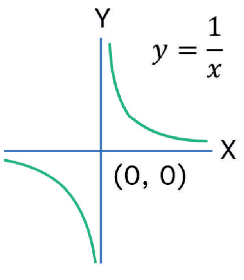
$f(-\infty) \approx 0^-$

$f(0^-) = -\infty$

$f(0^+) = +\infty$

$f(\infty) \approx 0^+$

$y \neq 0$



Function

$\therefore$ Range: $y \in R - \{0\}$

Alternate Method: $x = \frac{1}{y} \Rightarrow y \neq 0$

So, graph of $xy = 1$ or $xy = c^2$ is given:

Q. **(ii)** $y = \frac{1}{2x-1}$

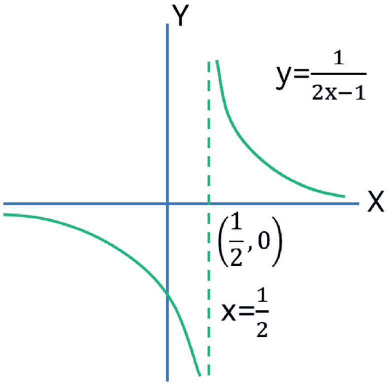
A. Domain: $2x - 1 \neq 0$ or $x \in R - \left\{\frac{1}{2}\right\}$

For range,

$y\left(x - \frac{1}{2}\right) = \frac{1}{2} \Rightarrow YX = \frac{1}{2}$ $\left(Y = y; X = x - \frac{1}{2}\right)$

So, using shifting of origin, it can be seen that function is similar to $xy = c^2$

$\therefore$ Range: $y \in R - \{0\}$



Q. **(iii)** $y = \frac{1}{3x+4}$

A. $3x + 4 \neq 0 \Rightarrow x \neq \frac{-4}{3}$

$\therefore$ Domain: $x \in R - \left\{\frac{-4}{3}\right\}$

Similarly, range: $y \in R - \{0\}$

Point to Remember!!!

Range of $f(x) = \frac{1}{ax+b}, (a \neq 0)$ will always be $R - \{0\}$.

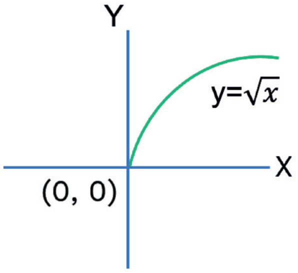
Q. **(iv)** $y = \sqrt{x}$

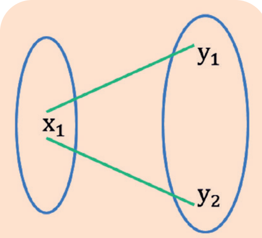
A. Since square root of a negative value is not real, in case of $y = \sqrt{f(x)}$, to find domain, make $f(x) \geq 0$.

Using concept, Domain of $y = \sqrt{x}$ is $x \in [0, \infty)$.

From graph

Range: $y \in [0, \infty)$





Not a function

Point to Remember!!!

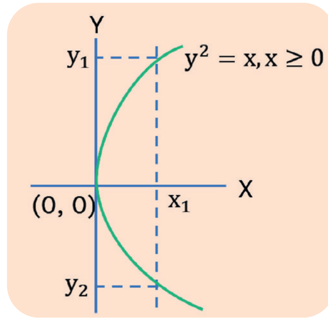
$y^2 = x$ is curve not function.
For $x \geq 0$

$\Rightarrow y = \pm\sqrt{x}$

So, for same value of x, there is two values of y. So, it cannot be a function.

Point to Remember!!!

If any vertical line cuts a curve at at least two different points, then the curve cannot be a function.



Q. **(v)** $y = \sqrt{2x - 1}$

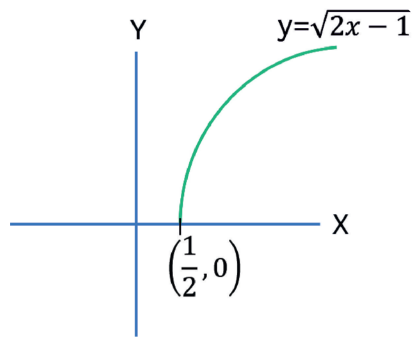
A. $2x - 1 \geq 0 \Rightarrow x \geq \frac{1}{2}$

Domain: $x \in \left[\frac{1}{2}, \infty\right)$

Trick for range: $\sqrt{2\left[\frac{1}{2}, \infty\right) - 1}$

$= \sqrt{[1, \infty) - 1} \;\; = \sqrt{[0, \infty)} = [0, \infty)$

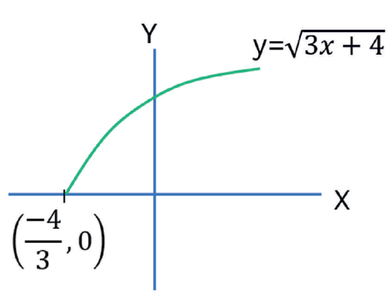
So, range $\equiv [0, \infty)$



Q. **(vi)** $y = \sqrt{3x + 4}$

A. Domain: $x \in \left[\frac{-4}{3}, \infty\right)$

Range: $y \in [0, \infty)$



> **Point to Remember!!!**
>
> $\sqrt{ax+b}$, $a \neq 0$ has its range $[0, \infty)$

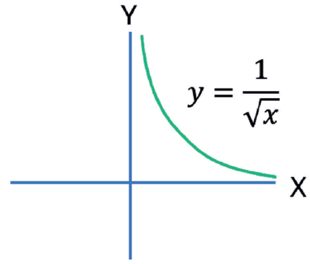
Q. **(vii)** $y = \dfrac{1}{\sqrt{x}}$

A. Here, it is similar to $\dfrac{1}{\sqrt{ax+b}}$.

So $ax + b > 0$

$\therefore$ Domain: $x \in (0, \infty)$

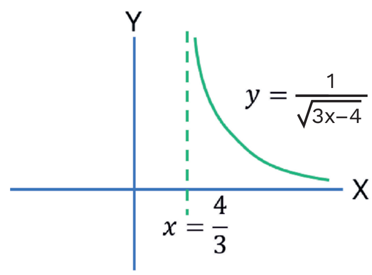
Range: $y \in (0, \infty)$ $\left\{\dfrac{1}{\sqrt{(0,\infty)}} \to \dfrac{1}{(0,\infty)} \to (0,\infty)\right\}$



Q. **(viii)** $y = \dfrac{1}{\sqrt{3x-4}}$

A. Similarly, Domain: $x \in \left(\dfrac{4}{3}, \infty\right)$

Range: $y \in (0, \infty)$ or R^+

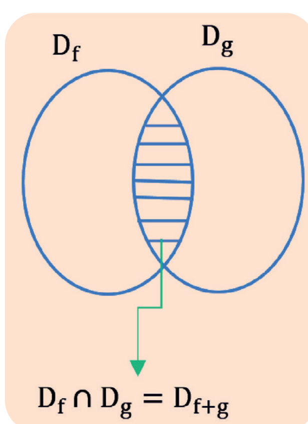


Algebraic operations on functions

(i) Let f and g be functions with domain D_1 and D_2 then the function f + g, is defined as

(f + g) (x) = f(x) + g(x); **Domain: $D_1 \cap D_2$**

In this case, both functions f(x) and g(x) must be real simultaneously. Only then, the overall function will be real.



(ii) Let f and g be function with domain D_1 and D_2, then the function f– g is defined as

(f – g) (x) = f(x) – g(x); **Domain: $D_1 \cap D_2$**

Again, both functions should be real at the same time. So, the domain is set of all the values of x common to both of their domain.

(iii) Let f and g be functions with domain D_1 and D_2, then the function $\left(\frac{f}{g}\right)(x)$ is defined as $\frac{f(x)}{g(x)}$ i.e., $\left(\frac{\mathbf{f}}{\mathbf{g}}\right)(\mathbf{x}) = \frac{\mathbf{f}(\mathbf{x})}{\mathbf{g}(\mathbf{x})}$

Domain:

$\mathbf{D_1 \cap D_2} \ : \{\mathbf{x} \mid \mathbf{g}(\mathbf{x}) \neq 0\}$ or

$\mathbf{D_1 \cap D_2} - \{\mathbf{x} \mid \mathbf{g}(\mathbf{x}) = 0\}$

In this case, Denominator, i.e., g(x) must not be zero.

Ex.

$f(x) = x$; $\qquad g(x) = x^2-1$

$D_f = R$; $\qquad D_g = R$

$\left(\frac{f}{g}\right)(x) = \frac{x}{x^2-1}$

$x^2 - 1 \neq 0 \quad \Rightarrow \quad x \neq 1, -1.$

$\therefore$ Domain of $\left(\frac{f}{g}\right)(x)$ is $x \in R - \{1, -1\}$.

(iv) Let f and g be functions with domain D_1 and D_2, then the function fg is defined as

$(\mathbf{fg})(\mathbf{x}) = \mathbf{f}(\mathbf{x}).\mathbf{g}(\mathbf{x})$; $\qquad$ **Domain:** $\mathbf{D_1 \cap D_2}$

Q. $f(x) = x^3 + 2x^2$ and $g(x) = 3x^2 - 1$. Find domain of $f \pm g$, fg and f/g.

A. $D_f = R$; $\qquad D_g = R$

$D_f \cap D_g = R$

(i) $f \pm g$: $D_f \cap D_g \Rightarrow$ Domain $= R \cap R = R$

(ii) fg : $D_f \cap D_g \Rightarrow$ Domain $= R \cap R = R$

(iii) f/g : $D_f \cap D_g \Rightarrow$ Denominator $\neq 0$

$\Rightarrow x \neq \pm\frac{1}{\sqrt{3}}$

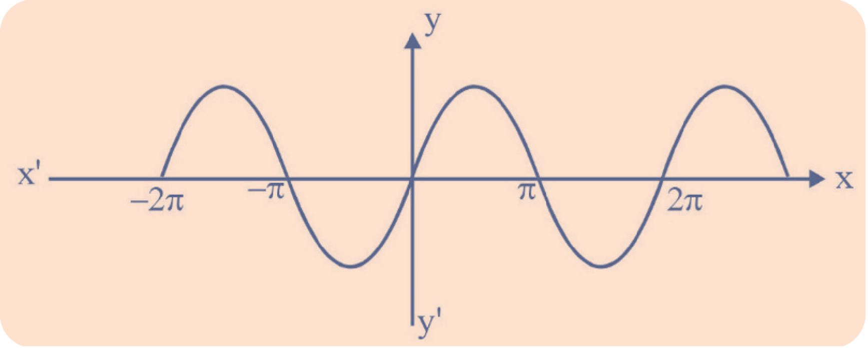
$\therefore$ Domain: $R - \left\{\pm\frac{1}{\sqrt{3}}\right\}$

Continuous Functions

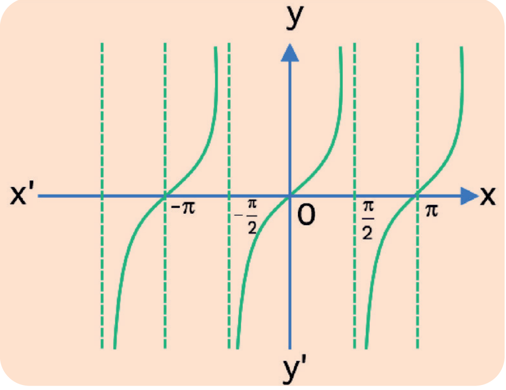
If graph of a function can be drawn without taking up the pen, then function is continuous.

Ex.

(i) Graph of sin x is continuous $\forall\ x \in R$



(ii) Graph of tan x, is discontinuous at $(2n+1)\dfrac{\pi}{2}$, $n \in I$

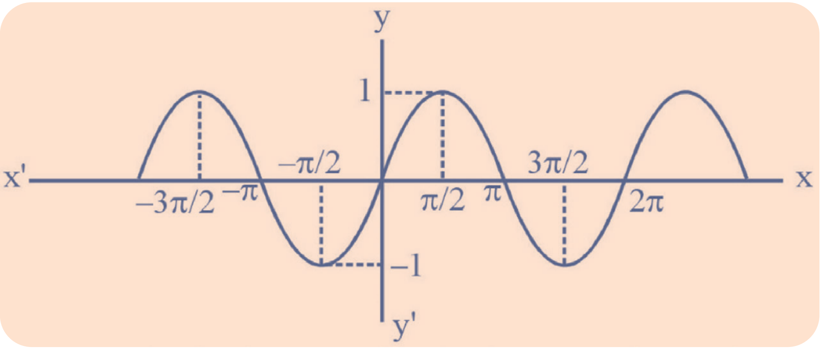


$$\tan x = \frac{\sin x}{\cos x} \Rightarrow \cos x \neq 0 \Rightarrow x \neq (2n+1)\frac{\pi}{2}, \quad n \in I$$

Domain, Range and Graph of Trigonometric Functions

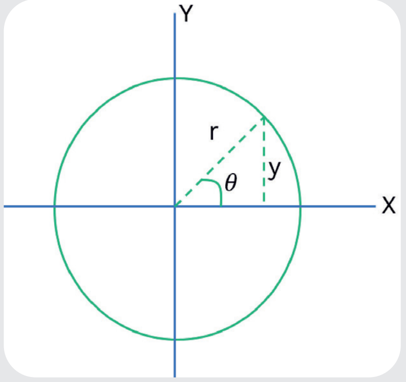
(i) y = sin x

Domain : $x \in R$; Range $\in [-1, 1]$



Point to Remember!!!

$\sin\theta = \frac{y}{r}$



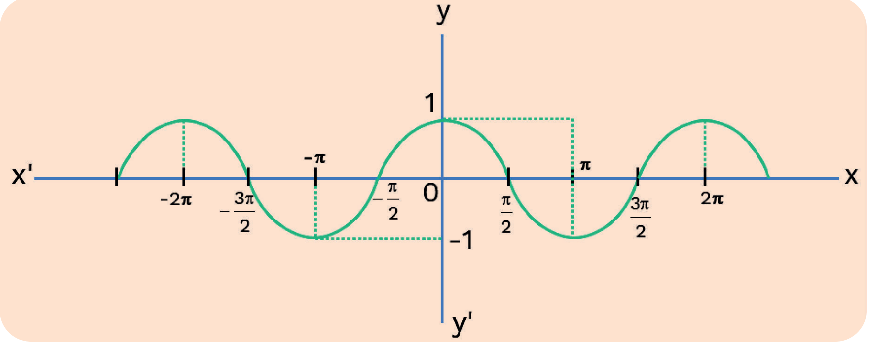
Maximum value of y = sin x is 1 at $x = \frac{\pi}{2}$.

Minimum value of y = sin x is –1 at $x = \frac{3\pi}{2}$.

Domain: $x \in R$

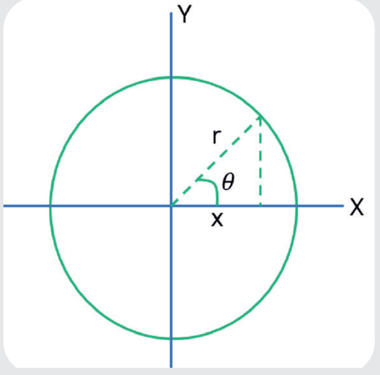
Range: $y \in [-1, 1]$

(ii) y = cos x



Point to Remember!!!

$\cos\theta = \frac{x}{r}$

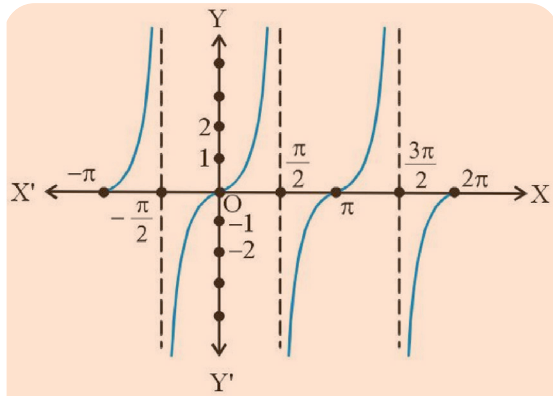


Domain : $x \in R$; Range : $[-1, 1]$

(iii) y= tan x

Domain: $x \in R - (2n+1)\frac{\pi}{2};\ n \in I$

Range: R



y = tan x

(iv) y = cot x

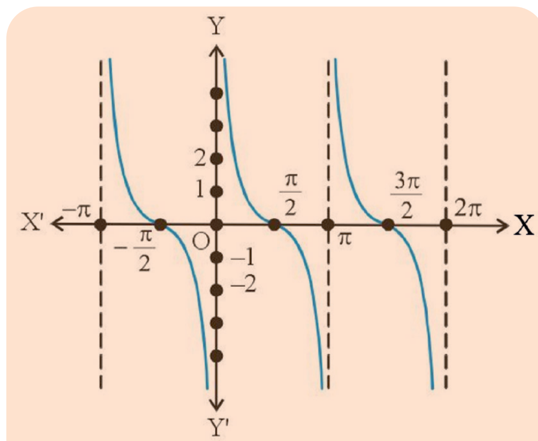
$\cot x = \frac{\cos x}{\sin x}$

$\Rightarrow \sin x \neq 0$

$\Rightarrow x \neq n\pi,\ n \in I$

$\therefore$ Domain: $x \in R - n\pi,\ n \in I$

Range: R



y = cot x

(v) y = cosec x

Graph can be constructed by observing the graph of y = sin x.

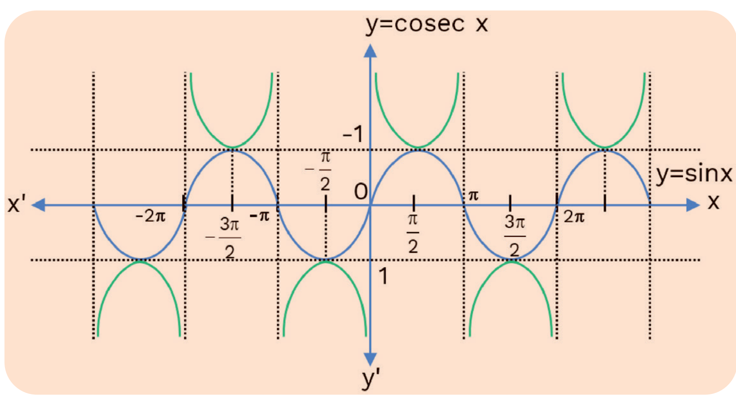
Since $\text{cosec}x = \frac{1}{\sin x}$, $\sin x \neq 0$

$\therefore x \neq n\pi$

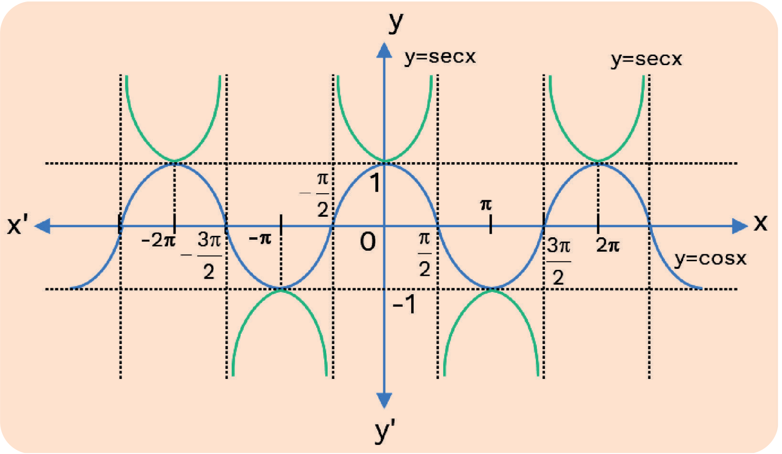
Domain: $R - n\pi$

From graph it can be seen that

range $\in (-\infty, -1] \cup [1, \infty)$



(vi) y = sec x



Similarly, like cosec x, graph of sec x can be drawn by the help from graph of cosx.

Since $\sec x = \frac{1}{\cos x}$, $\cos x \neq 0 \Rightarrow x \neq (2n+1)\frac{\pi}{2}$, $n \in I$

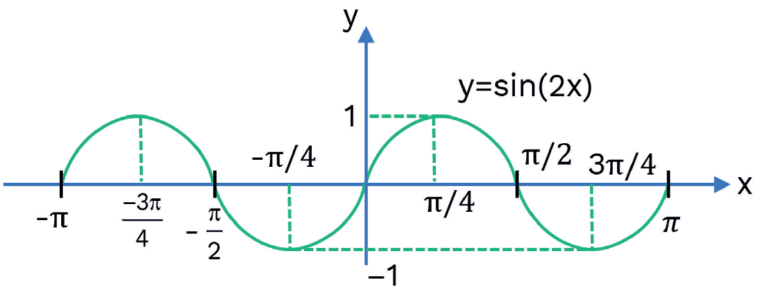
$\therefore$ Domain: $x \in \ R-(2n+1)\frac{\pi}{2}, n\epsilon I$

Range: $(-\infty, -1] \cup [1, \infty)$

Q. **Find range of y :**

(i) y = sin (2x)

A. $x \in (-\infty, \infty)$



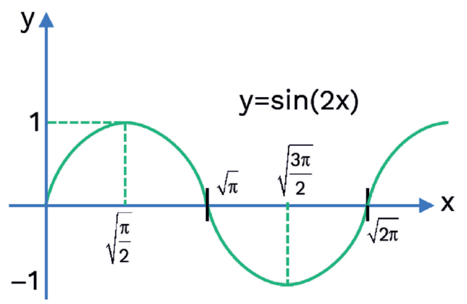
$\Rightarrow$ $2x \in (-\infty, \infty)$

As different values of x are used in sin(2x), input moves from $(-\infty, \infty)$.

So, $\sin(x) \in [-1, 1]$

Q. **(ii) $y = \sin(x^2)$**

A. $x \in (-\infty, \infty)$



$\Rightarrow x^2 \in [0, \infty)$

So, range is [-1, 1]

Q. **(iii)** $y = \sin(\sqrt{x})$

A. $x \in [0, \infty)$

Input for $\sin\sqrt{x}$ is $[0, \infty)$

$\therefore$ Range is $[-1, 1]$

Q. **(iv)** $y = \cos^4 \frac{x}{2} - \sin^4 \frac{x}{2}$

A. $y = \left(\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}\right)\left(\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2}\right)$

$= \left(\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}\right)$

$= \cos\left(2 \cdot \frac{x}{2}\right) = \cos x.$

Range of cos x is $[-1, 1]$

So, $y \in [-1, 1]$

Q. **(v) $y = (\sin x + 2)^2 + 1$**

A. $\sin x \in [-1, 1]$

$\sin x + 2 \in [1, 3]$

$(\sin x + 2)^2 \in [1,9]$

$y = (\sin x + 2)^2 + 1 \in [2, 10]$

Q. **(vi) y = 4 tan x cos x**

A. There is loss of domain because of presence of tanx.
Due to presence of tanx, cos x $\neq$ 0

$y = 4 \tan x \cos x = 4 \dfrac{\sin x}{\cos x}.\cos x$; $(\cos x \neq 0)$

$y = 4 \sin x$ ($\cos x \neq 0$ so $\sin x \neq -1$ and 1)

$\therefore \quad y \in (-4, 4) \qquad (\because \sin x \in (-1, 1))$

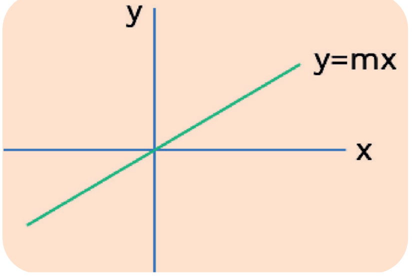
Definition of Polynomial Function

If a function f is defined by $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n$, where n is a non-negative integer and $a_0, a_1, a_2, \dots, a_n$ are real numbers and $a_0 \neq 0$, then f is called a polynomial function of degree n.

Ex. f(x) = 2 is zero degree polynomial

Ex.

$f(x) = ax^2 + bx + c$ ... Quadratic Polynomial
$f(x) = ax^3 + bx^2 + cx + d$ ... Cubic Polynomial
Similarly, polynomial of degree 4 is called 'Biquadratic'.



Point to Remember!!!

a_0 is called leading coefficient.
a_n is called constant term.
$f(0) = a_n$.

Points to Remember!!!

(i) A polynomial function is always continuous.

(ii) A polynomial of degree one is called a linear function.

(iii) A polynomial of degree one with no constant term is called an odd linear function. y = mx is an odd linear function.

(iv) A polynomial of degree odd has its range 'R' but a polynomial of degree even has a range which is always a subset of R.

In case of polynomial, the term with highest degree is always dominant when compared to other terms, whenever $x = \pm\infty$ (other terms are negligible compared to it at $x = \pm\infty$).

So, in case of odd degree polynomial, its range is R.

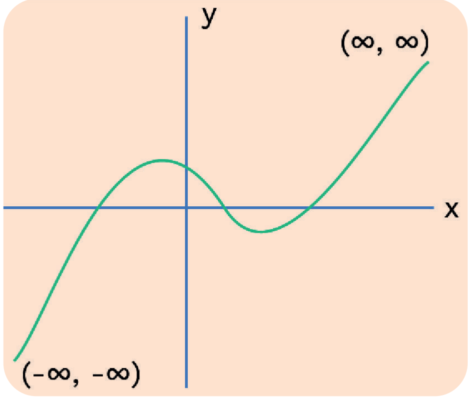
In case of even degree polynomial, the range is only a subset of R.

Ex. $y = x^3 + 2x^2 - 7x + 3$

as $x \to \infty$; $y \to \infty$

as $x \to -\infty$; $y \to -\infty$

Since, **polynomial is continuous, the range is $(-\infty, \infty)$**



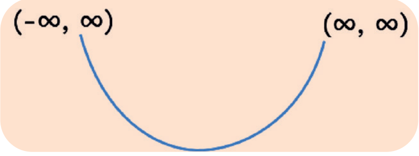
Ex. $y = 2x^{100} + 76$

$x \to +\infty$; $y \to +\infty$

$x \to -\infty$; $y \to +\infty$

Here, since leading coefficient is positive, y can not assume the value of '$-\infty$'. Hence, its graph will be:

This graph has a minima. so, range is subset of R.

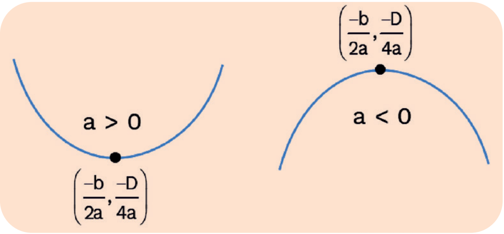


Point to Remember!!!

Range of parabola (a > 0): $\left[-\frac{D}{4a}, \infty\right)$

$y = ax^2 + bx + c$

If a < 0, than range is $\left(-\infty, -\frac{D}{4a}\right]$



Definition of Algebraic functions

A function f is called an algebraic function if it can be constructed using algebraic operations such as addition, subtraction, multiplication or division or taking radical sign (Starting with polynomial).

Here radian sign refers to operations like:

$x^{1/3}; (2x-1)^{-2/3}; \sqrt{x+2}$ etc.

Ex. $f(x) = \sqrt{x^2 + 1}$

Domain : $x^2 + 1 \geq 0$

Since $x^2 \in [0, \infty)$, $x^2 + 1 \in [1, \infty)$ So, $x^2 + 1 \geq 0$ is true for all real x.

$\therefore$ Domain: $(-\infty, \infty)$

Range: $[1, \infty)$

Q. **f(x) =** $\sqrt{x^2 + ax + 4}$

(a) Find 'a' if range is [2, ∞).

(b) Find 'a' if domain is all real.

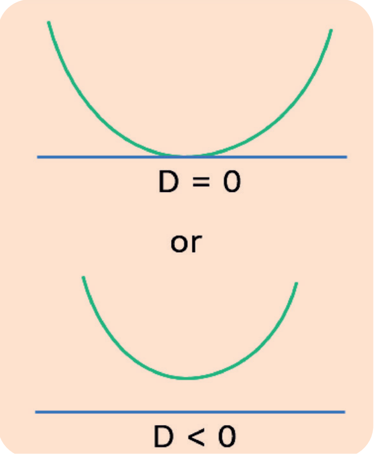
A. **(a)** Since, $\sqrt{x^2 + ax + 4} \in [2, \infty)$.

$x^2 + ax + 4 \in [4, \infty)$.

$\therefore$ Minimum value of $x^2 + ax + 4 = 4$

$\Rightarrow \dfrac{-D}{4a} = 4$

$\dfrac{16 - a^2}{4} = 4 \Rightarrow a = 0$



(b) Domain is all real.

it means $x^2 + ax + 4 \geq 0 \ \forall \ x \in R$.

$\therefore$ $D \leq 0$

$a^2 - 16 \leq 0$

$a \in [-4, 4]$

Definition of Fractional / Rational Function

A fractional function is of the form $y = f(x) = \dfrac{g(x)}{h(x)}$,

where g(x) and h(x) are polynomial and $h(x) \neq 0$.

The domain of f(x) is set of real x such that $h(x) \neq 0$.

Ex.

$f(x) = \dfrac{2x^4 - x^2 + 1}{x^2 - 4}$; **D = {x|x ≠ ± 2}**

Here, denominator ≠ 0

$x^2 - 4 \neq 0$

$x \neq 2, -2$

Definition of Exponential Function

A function $f(x) = a^x$ $(a > 0, a \neq 1, x \in R)$ is called an exponential function.

Ex.

$y = 2^x$

x	$-\infty$	-2	-1	0	1	2	∞
y	0^+	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	∞

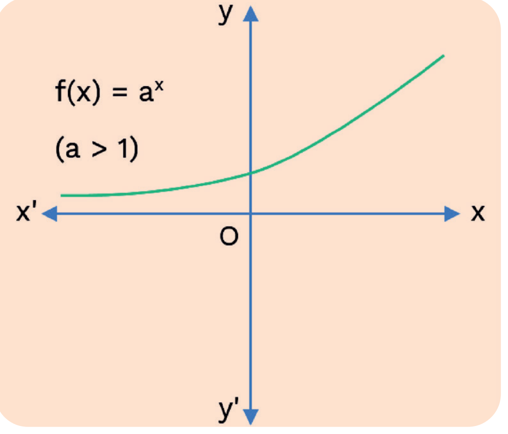
As x increases, y increases.
As x decreases, y decreases.

as $x \rightarrow -\infty$, $y = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times ... = 0^+$

So, if $a^x > a^y \Rightarrow x > y$

Point to Remember!!!

Graph of a^x
base a > 1
Domain: $x \in (-\infty, \infty)$



Ex.

$y = \left(\frac{1}{2}\right)^x$

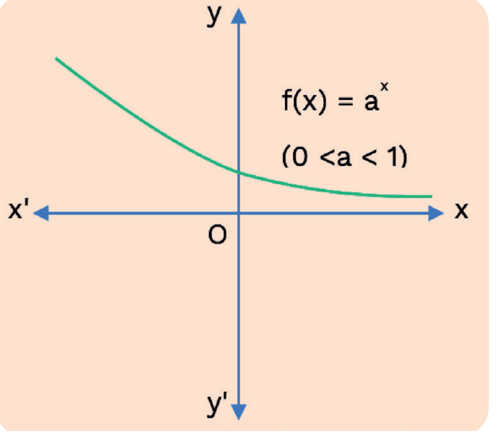
X	$-\infty$	-2	-1	0	2	∞
Y	∞	4	2	1	1/4	0^+

As x increase, y decreases
So, if $a^x > a^y \Rightarrow x < y$
Domain: $(-\infty, \infty)$
Range: $(0, \infty)$

Point to Remember!!!

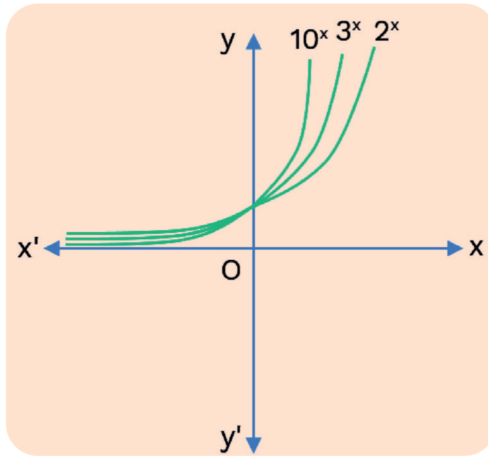
Graph of a^x :

base 0 < a < 1



Ex.

x	-2	-1	0	1	2
2^x	1/4	1/2	1	2	4
3^x	1/9	1/3	1	3	9
10^x	1/100	1/10	1	10	100



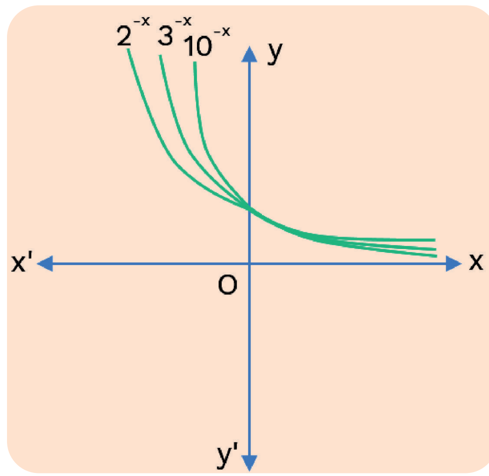
Plotting these points on graph we can see that:

(i) At $x > 0$, if $a > b$, then $a^x > b^x$ ($a, b > 1$)

(ii) At $x < 0$, if $a > b$, then $a^x < b^x$ ($a, b > 1$)

Ex.

x	-2	-1	0	1	2
2^{-x}	4	2	1	1/2	1/4
3^{-x}	9	3	1	1/3	1/9
10^{-x}	100	10	1	1/10	1/100



Plotting these points on graph, it can be seen that:

(i) At $x > 0$, if $a > b$, then $a^x > b^x$ ($a, b \in (0, 1)$)

(ii) At $x < 0$, if $a > b$, then $a^x < b^x$ ($a, b \in (0, 1)$)

Ex.

(i) $y = 2^{\sqrt{x}}$

Domain: $x \geq 0$

$\sqrt{x} \geq 0 \Rightarrow 2^{\sqrt{x}} \geq 1$

Range: $[1, \infty)$

(ii) $y = 2^{x^2+1}$

Domain: $x \in R$

$x^2 + 1 \in [1, \infty)$

$\therefore$ Range: $[2, \infty)$

Definition of Logarithmic function

$y = \log_a x;\ x > 0,\ a > 0,\ a \neq 1$

$y = \log_a x \Rightarrow \underbrace{a^y}_{+ve} = x$

$\therefore$ a > 0 and x > 0

If a = 1 and x = 1, then $\log_1 1$ has more than one value.

So, it will not be a function.

If a = 1 and x ≠ 1, then $\log_a x$ will have no solution.

$\therefore$ a ≠ 1.

Ex.

$y = \log_2 x$

x	0^+	1/4	1/2	1	2	4	∞
y	$-\infty$	-2	-1	0	1	2	∞

For base (a) > 1, the graph is increasing.

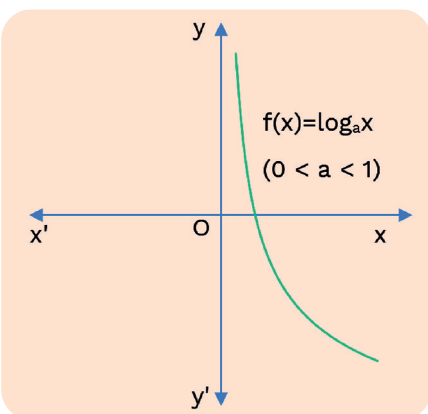
Ex.

y = lnx, it is also written as $y = \log_e x$

$$\frac{dy}{dx} = \frac{1}{x} > 0 \quad \forall\, x \in R^+$$

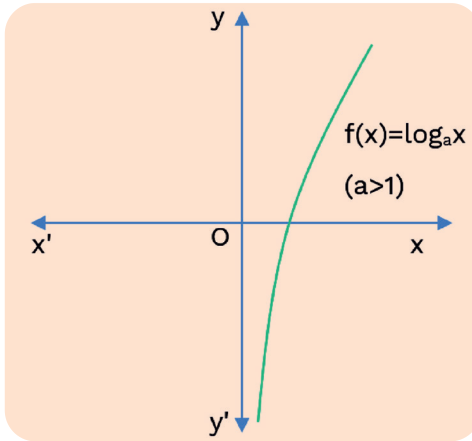
$$\frac{d^2y}{dx^2} = \frac{-1}{x^2} < 0 \quad \forall\, x \in R^+$$

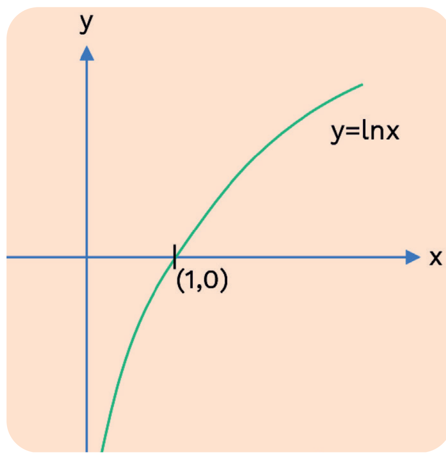
So, graph is always increasing as well as concave down.



Point to Remember!!!

Graph of $y = \log_a x$
base a > 1
Domain: $(0, \infty)$
Range: $(-\infty, +\infty)$





Point to Remember!!!

Graph of $y = \log_a x$
base 0 < a < 1
Domain: $x \in (0, \infty)$

Range: $(-\infty, \infty)$

Q.1 Find the domain and range of $y = \sqrt{\ln x}$

A. $x > 0$ (Rule of $\log_a x$)

$\ln x \geq 0$ (in $\sqrt{f(x)}$, $f(x) \geq 0$)

$\ln x \geq \ln 1$

Since base is greater than 1, graph is increasing.

$\therefore$ $x \geq 1$

$\therefore$ Domain: $[1, \infty)$

Since $\ln x \geq 0$, $\sqrt{\ln x} \geq 0$

$\therefore$ Range: $[0, \infty)$

Q.2 Find domain and range:

(i) $y = \sqrt{\log_{1/3}(x-1)}$

A. Domain conditions:

(i) $x-1 > 0 \Rightarrow x \in (1, \infty)$

(ii) $\log_{1/3}(x-1) \geq 0$

$\Rightarrow$ $\log_{1/3}(x-1) \geq \log_{1/3} 1$

$\Rightarrow$ $x - 1 \leq 1$

(from graph of $\log_a x$, $0 < a < 1$)

$\Rightarrow$ $x \leq 2$

$\therefore$ from (i) and (ii), domain: $(1,2]$

Range: $x \in (1,2] \Rightarrow (x-1) \in (0, 1]$

$\Rightarrow$ $\log_{1/3}(x-1) \in [0, \infty)$

$\therefore$ Range: $[0, \infty)$

Q. (ii) $y = \sqrt{\log_3(\cos(\sin x))}$

A. **Domain:**

$\log_3(\cos(\sin x)) \geq 0$

$\Rightarrow$ $\log_3(\cos(\sin x)) \geq 0 = \log_3 1$

$\Rightarrow$ $\cos(\sin x) \geq 1$

Since $\cos\theta \in [-1, 1]$, there is only one case where above situation is possible, i.e, $\cos(\sin x) = 1$

$\Rightarrow$ $\sin x = 2n\pi$, $n \in I$.

$\sin x = 0, \pm 2\pi, \pm 4\pi, \pm 6\pi, \ldots$

But $\sin x \in [-1, 1]$. So, $\sin x = 0$ is only solution.

$\therefore$ $x = n\pi$, $n \in I$.

Range:

$\sin x = 0 \Rightarrow \cos(\sin x) = 1$

$\Rightarrow$ $\log_3 \cos(\sin x) = \log 1 = 0$

$\therefore$ Range: $\{0\}$

This function can also be called many-one function.

Q. **(iii)** $y = \dfrac{1}{\ln x}$

A. **Domain:**

Condition are:

(i) $x > 0$

(ii) $\ln x \neq 0 \Rightarrow x \neq 1$

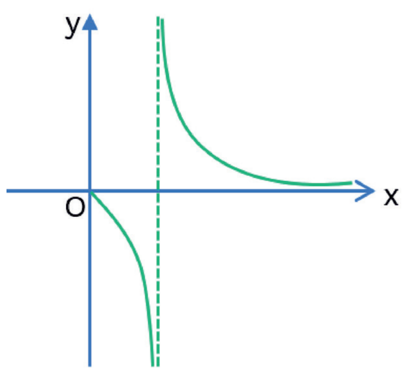
$\therefore$ Domain: $x \in (0, 1) \cup (1, \infty)$.

Range:

$\ln x \in (-\infty, 0) \cup (0, \infty)$.

$\therefore$ $\dfrac{1}{\ln x} \in (-\infty, 0) \cup (0, \infty)$

(Similar to function $y = \dfrac{1}{x}$)



Q. **(iv)** $y = e^{\frac{1}{x}}$

A. **Domain:** $x \neq 0$

$\Rightarrow$ $x \in (-\infty, 0) \cup (0, \infty)$

Range:

$\dfrac{1}{x} \in (-\infty, 0) \cup (0, \infty)$

$\Rightarrow$ $e^{1/x} \in (0, 1) \cup (1, \infty)$

$\therefore$ **Range:** $(0, 1) \cup (1, \infty)$.

or $(0, \infty) - \{1\}$

Q. **(v)** $y = \log_{10}(\log_{10}(1+x^3))$

A. **Domain:**

$$\log_{10}(1+x^3) > 0$$

$\Rightarrow$ $1 + x^3 > 1$

$\Rightarrow$ $x^3 > 0 \Rightarrow x > 0$

$\therefore$ Domain: $x \in (0, \infty)$

Range:

$$\log_{10}\underbrace{\left(\log_{10}\left(1+x^3\right)\right)}_{(0,\ \infty)}$$

$\Rightarrow$ $y \in (-\infty, \infty)$

$(\because \log(0^+) = -\infty \text{ and } \log(\infty) = \infty)$

Q.3 **If f(x) =** $\log_e\left(\frac{1-x}{1+x}\right), |x| < 1$**, then f** $\left(\frac{2x}{1+x^2}\right)$ **is equal to:**

(A) $2f(x)$ (B) $2f(x^2)$ (C) $(f(x))^2$ (D) $-2f(x)$

A. $$f\left(\frac{2x}{1+x^2}\right) = \log_e\left(\frac{1-\frac{2x}{1+x^2}}{1+\frac{2x}{1+x^2}}\right)$$

$$= \log_e\left(\frac{(1-x)^2}{(1+x)^2}\right)$$

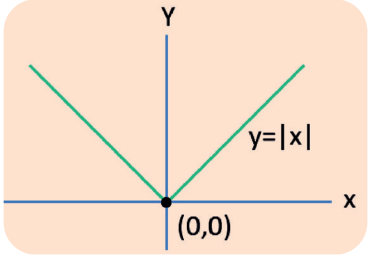
$$= 2\log_e\left(\frac{1-x}{1+x}\right)$$

$= \mathbf{2f(x)}$

Definition of Absolute value function/ Modulus function

$$y = |x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

Domain: R; **Range: [0, ∞)**



Ex. $|-2| = 2$

$|+2| = 2$

$|0| = 0$

Function

Ex.

$y - 2 = |x - 3|$

$$y - 2 = \begin{cases} +(x-3); & x \geq 3 \\ -(x-3); & x < 3 \end{cases}$$

$$\Rightarrow y = \begin{cases} x - 1 & x \geq 3 \\ 5 - x & x < 3 \end{cases}$$

Now, if graph of both functions,
$y = |x|$ and $y-2=|x-3|$ is compared, then it appears as if graph is shifted from origin to (3, 2).

Know the facts

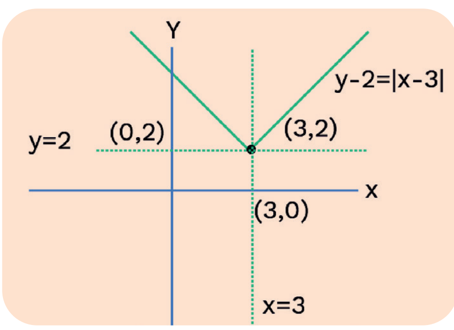
Shifting of origin:
$\mathbf{y - \beta = f(x - \alpha)}$
Take (0, 0) point of graph to (α, β).

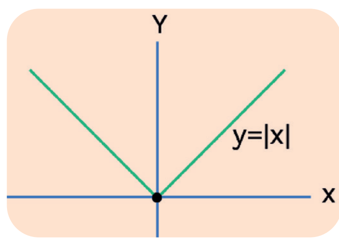
In this case, draw the graph $y = f(x)$ where $X = x - \alpha$ and $Y = y - \beta$

At $x = \alpha$, $X = 0$ and at $y = \beta$, $Y = 0$

So, at (α, β), $X = 0$ and $Y = 0$.

So, draw the graph $Y = f(X)$ at (α, β) as if (α, β) was origin.





Ex.

$y + 2 = |x-1|$

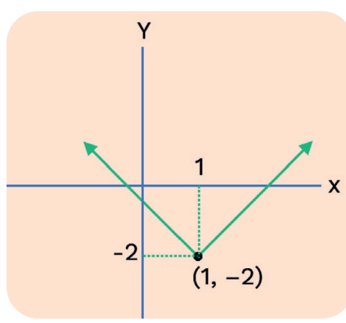
$\beta = -2$

$\alpha = 1$

Sol. $y-(-2) = |x-1|$

$\therefore \alpha = 1$ and $\beta = -2$

So, shift the graph of $y = |x|$ to $(1, -2)$.

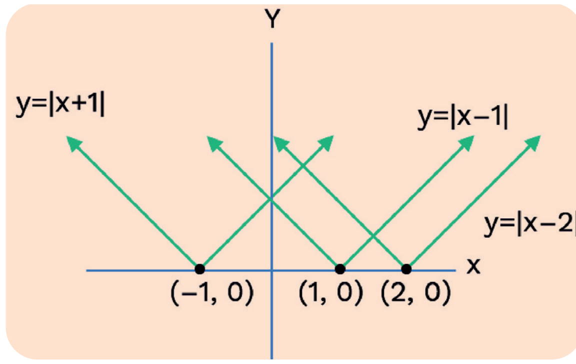


Ex.

$$y = |x-1| = \begin{cases} (x-1), & x \geq 1 \\ -(x-1), & x < 1 \end{cases}$$

($\alpha = 1$; $\beta = 0$)

So, graph shifted to (1, 0)



Ex. Draw the graph of

(i) $y = 1 + |x|$

(ii) $y = 2 + |x|$

(iii) $y = -1 + |x|$

Upshift

(i) $y = 1 + |x|$

$y - 1 = |x|$; $\alpha = 0$ and $\beta = 1$

So, graph shifted to (0, 1).

(ii) Similarly, in case of $y = 2 + |x|$

$y - 2 = |x|$

graph shifted to (0, 2)

(iii) $y = -1 + |x|$

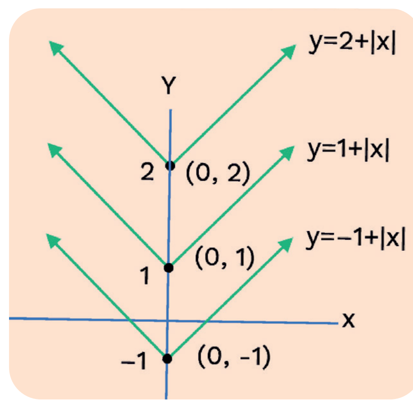
$y - (-1) = |x|$

graph shifted to (0, -1)

Point to Remember!!!

Graph of $f(x-\alpha)$ can be drawn by shifting the graph of f(x) to left or right.

So, there is no difference in the range of f(x) and $f(x-\alpha)$.



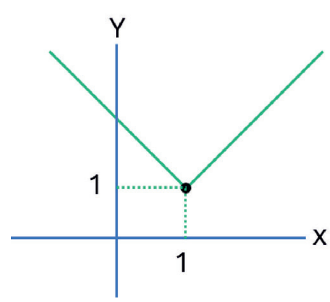
Q. **Find domain, range and graph of y =|x − 1| + 1**

A. $y - 1 = |x - 1|$; $\alpha = 1$ and $\beta = 1$

Graph shifted to (1, 1)

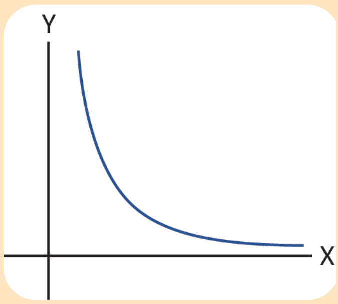
Domain: R

Range: $[1, \infty)$

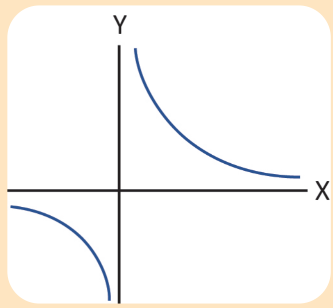


Q. **Which of the following graph represents** $y = \frac{1}{|x|}$ **?**

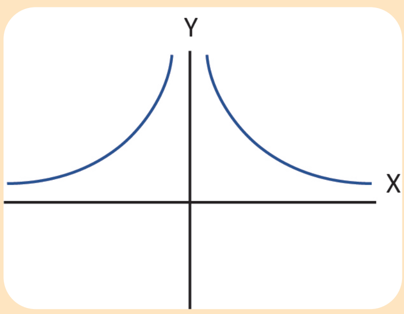
(A)



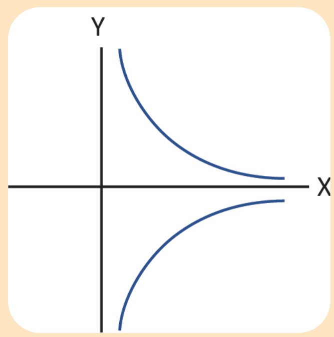
(B)



(C)



(D)



A. Domain: $|x| \neq 0$

$\therefore \quad x \neq 0$

$\therefore$ **Domain:** $R - \{0\}$

Range:

$x = 0^+$	$y = \infty$.
$x = 0^-$	$y = \infty$
$x = -\infty$	$y = 0^+$
$x = +\infty$	$y = 0^+$
$\lvert x\rvert \in (0, \infty)$,	$(\lvert x\rvert \neq 0)$

$\therefore \quad \frac{1}{|x|} \in (0, \infty)$

$\therefore$ **Range:** $(0, \infty)$

So, from domain, graphs (A) and (D) are eliminated.

from Range, graph (B) can be eliminated. **Graph (C)** is the correct answer.

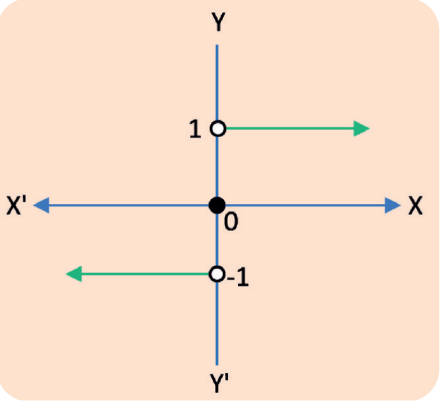
Definition of Signum Function

A function y = f(x) = sgn(x) is defined as follows:

$$y = f(x) = \begin{cases} 1 & \text{for} \quad x > 0 \\ 0 & \text{for} \quad x = 0 \\ -1 & \text{for} \quad x < 0 \end{cases}$$

Domain: R

Range: {–1, 0, 1}



Note:

Hollow circle at (0, 1) and (0, –1) indicates that these points are excluded.

Solid circle at (0, 0) show that this point is included.

Know the facts

$$\text{sgn}(x) = \begin{cases} \dfrac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Q. **Find domain, range of:**

(i) y = sgn(x^2 –1)

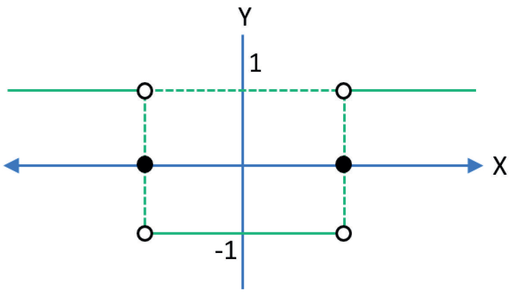
A.

$$y = \begin{cases} 1 & ; \quad x^2 - 1 > 0 \\ 0 & ; \quad x^2 - 1 = 0 \\ -1 & ; \quad x^2 - 1 < 0 \end{cases}$$

$$y = \begin{cases} 1 & ; \quad x \in (-\infty, -1) \cup (1, \infty) \\ 0 & ; \quad x = 1, -1 \\ -1 & ; \quad x \in (-1, 1) \end{cases}$$

Domain: x ∈ R

Range: {-1, 0, 1}



Q. **(ii)** y = $\log_{10}$ sgn(x)

A. sgn(x) > 0 (it is inside logarithm).

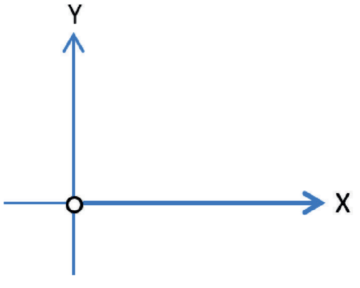
∴ sgnx can only be '1'.

∴ x > 0

∴ **Domain:** x ∈ (0, ∞)

Since sgn(x) = 1, $\log_{10}(\text{sgn}x) = 0$

∴ **Range:** y = {0}



Q. $y = \text{sgn}(\ln(x^2 - x + 2))$

A. Range for quadratic: $\left[\dfrac{-D}{4a}, \infty\right)$

$\therefore\ x^2 - x + 2 \in \left[\dfrac{7}{4}, \infty\right)$

$\therefore$ minimum value of $\ln(x^2 - x + 2)$ is $\ln\dfrac{7}{4}$

$\ln\dfrac{7}{4} > \ln 1$ which is 0.

$\ln\dfrac{7}{4}$ is positive

So, $\text{sgn}(\ln(x^2 - x + 2))$ will always give the value '1'.

$\therefore$ **Range:** $y = \{1\}$

Domain: $x \in R$.

R = I + f

Integral part **Fractional part**

Know the facts

Any real number can be broken into two parts: integral parts and fractional parts.
Integral parts is an integer and fractional parts belongs to [0, 1).

Ex.

(i)

$3.2 = 3 + 0.2$
$3 = 3 + 0$
$0.9 = 0 + (0.9)$
$\pi = 3 + (\pi - 3)$
$e = 2 + (e - 2)$

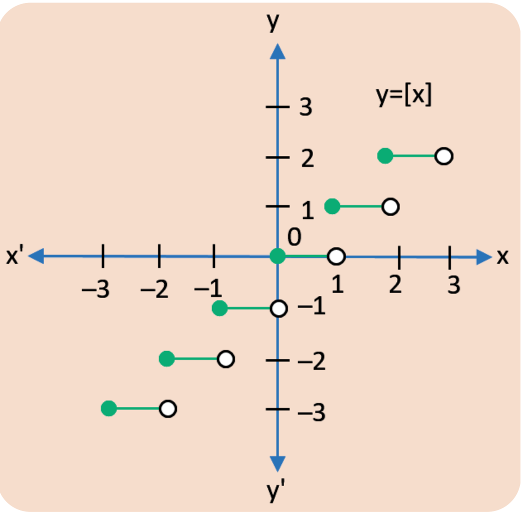
(ii)

$-2.2 = -3 + (0.8)$
$-7.2 = -8 + (0.8)$
$-\pi = -4 + (4 - \quad)$

Definition of Greatest integer function

The function $y = f(x) = [x]$ is called the greatest integer function, where $[x]$ denotes the greatest integer less than or equal to x.

R = I + f, here I is [R].



Ex. $[7.6] = 7$

7.6 is not an integer. The integers less than 7.6 are {7, 6, 5, 4, ...}. Among these integers, greatest integer is 7. So, $[7.6] = 7$.

Ex.

$-1 \le x < 0$	$\Rightarrow$	$[x] = -1$
$0 \le x < 1$	$\Rightarrow$	$[x] = 0$
$1 \le x < 2$	$\Rightarrow$	$[x] = 1$
$2 \le x < 3$	$\Rightarrow$	$[x] = 2$

and so on.

For $f(x) = [x]$, domain is R and range is I.

Here, $[0.1] = 0$

$[0.3] = 0$

$[0.7] = 0$

or $y = [x] = 0 \ \forall \ x \in [0, 1)$

Ex.

For $f(x) = \dfrac{1}{[x]}$, find the domain and range.

(where [.] denotes greatest integer function)

Sol. Domain of [x] is R.

In case of $y = \dfrac{1}{f(x)}$, $f(x) \neq 0$.

So, in above example, $[x] \neq 0$.

$\Rightarrow x \notin [0,1)$

$\therefore$ **Domain:** $x \in R - [0, 1)$

Range: $y = \ldots, \dfrac{1}{-2}, \dfrac{1}{-1}, \dfrac{1}{1}, \dfrac{1}{2}, \dfrac{1}{3}, \ldots$ and so on.

$\therefore$ **Range:** $= \left\{\dfrac{1}{I} : I \text{ is an integer}, \ I \neq 0\right\}$

Property:

(a) $[x] \le x < [x] + 1$

(b) $x - 1 < [x] \le x$

Proof:

(a) $0 \le x - [x] < 1$

$\Rightarrow [x] \le x < [x] + 1$

(b) $x - [x] < 1$

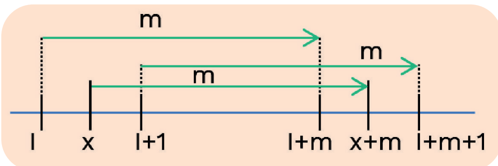
$\Rightarrow x - 1 < [x]$

$x - [x] \ge 0$

$\Rightarrow x \ge [x]$

Property:

(c) **$[x + m] = [x] + m$,** where m is an integer.



From the number line, it can be seen that greatest integer on the left side of x + m is I + m, i.e., [x] + m.

Ex.

$[10.7] = 10$

$[10.7 + 2] = [12.7] = 12 = 10 + 2 = [10.7] + 2.$

Property:

(d) $[x] + [-x] = \begin{cases} 0 & \text{if } x \text{ is an integer} \\ -1 & \text{otherwise} \end{cases}$

Proof:

Case-I: x = integer.

$[x] = x,\ [-x] = -x$

$\therefore$ L.H.S = x − x = 0 = R.H.S.

Case-II: $x = I + f;\ f \in (0, 1)$, I is an integer.

$$\begin{aligned} \text{LHS} &= [I + f] + [-I - f] \\ &= I + [(-I - 1) + (1 - f)],\ 1 - f \in (0, 1) \\ &= I + (-I - 1) \\ &= -1 = \text{RHS} \end{aligned}$$

Ex. $[2.3] + [-2.3] = 2 + (-3) = -1$

Q.1 Let [x] represents the greatest integer less than or equal to x. If all the values of x, such that the product $\left[x-\frac{1}{2}\right]\left[x+\frac{1}{2}\right]$ is prime, belongs to the set $[x_1, x_2) \cup [x_3, x_4)$, find the value of $x_1^2+x_2^2+x_3^2+x_4^2$.

A. Let $x - \frac{1}{2} = R$. Then $x+\frac{1}{2} = R + 1$.

$\therefore \left[x-\frac{1}{2}\right]\left[x+\frac{1}{2}\right] = [R]\,[R+1] = I\,(I+1)$; where $[R] = \left[x-\frac{1}{2}\right] = I$

$I\,(I + 1)$ is prime.

It is only possible in 2 cases.

Case I: $I\,(I + 1) = 1 \cdot 2;\ I = 1$

Case II: $I\,(I + 1) = (-2)\cdot(-1);\ I = -2$

$\therefore \quad 1 \le x - \frac{1}{2} < 2$ or $-2 \le x - \frac{1}{2} < -1$

$\Rightarrow \quad x \in \left[-\frac{3}{2}, -\frac{1}{2}\right) \cup \left[\frac{3}{2}, \frac{5}{2}\right)$

$\therefore \quad x_1 = -\frac{3}{2};\ x_2 = -\frac{1}{2};\ x_3 = \frac{3}{2};\ x_4 = \frac{5}{2}$

So, $x_1^2+x_2^2+x_3^2+x_4^2 = \frac{44}{4} = \mathbf{11}$.

Q.2 Let [t] denotes the greatest integer $\le t$. Then the equation in x,

$[x]^2 + 2[x + 2] - 7 = 0$ has:

(A) exactly four integral solutions.

(B) infinitely many solutions.

(C) no integral solution.

(D) exactly two solutions.

A. Let $[x] = I$.

$\therefore \quad I^2 + 2\,(I + 2) - 7 = 0 \qquad (\because [x + I] = [x] + I)$

$I^2 + 2I - 3 = 0$

$I = 1$ or -3

$\therefore$ $\mathbf{x \in [1, 2) \cup [-3, -2)}$

So, equation has infinite solutions.

So, **(B)** is the correct option.

Definition of Fractional part function

It is defined as:

$$g(x) = \{x\} = x - [x]$$

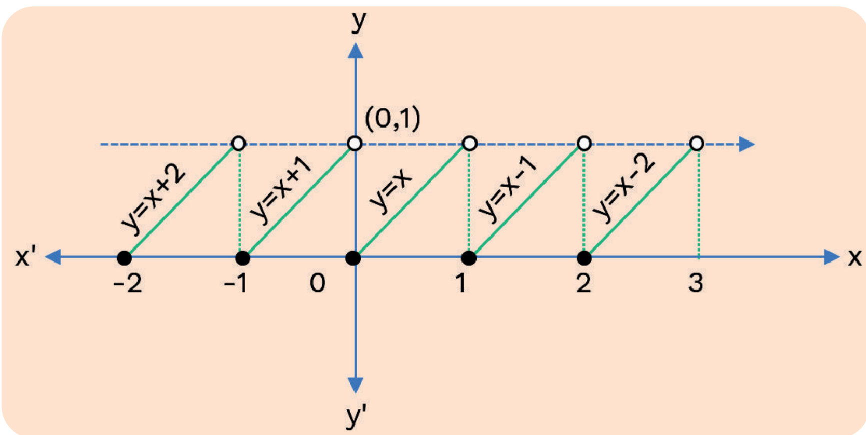
or $\quad x = \underbrace{[x]}_{I} + \underbrace{\{x\}}_{f} \quad$ **y = x – [x] = {x}**

$$y = \begin{Bmatrix} x-0 & ; & x \in [0,1) \\ x-1 & ; & x \in [1,2) \\ x-2 & ; & x \in [2,3) \\ x+1 & ; & x \in [-1,0) \\ \vdots & & \vdots \end{Bmatrix}$$

Know the facts

The period of this function is 1. For f(x) = {x}, domain is R and range is [0, 1).

The graph will have parallel lines and it will be discontinuous at all integers.



Ex.

(i) The fractional part of the number 2.1 is
2.1 – 2 = 0.1

(ii) The fractional part of –3.7 is 0.3.

Know the facts

Graph of y={x} is periodic with length of period 1.

Property:

(i) $\{x + n\} = \{x\}$; $n \in I$

Proof:

$\{x + n\} = (x + n) - [x + n]$

$= x + n - ([x] + n)$

$= x - [x] = \{x\}$

Ex. $\{-1.5\} = \{-0.5\} = \{0.5\} = 0.5$

(ii) $\{x\}+\{-x\}=\begin{cases}0 & , \quad x\in I\\ 1 & , \quad x\notin I\end{cases}$

Proof:

Case-I: x is an integer

$\{x\} + \{-x\} = 0 + 0 = 0.$

Case-II: x is not an integer

Let $x = I + f$; $f \neq 0$, $f \in (0, 1)$

$\{x\} + \{-x\} = \{I + f\} + \{-I + (-f)\}$

$= f + \{(-I-1) + (1-f)\}$

$= f + 1 - f$

$= 1$ Hence proved.

Ex.

$\{2.8\} + \{-2.8\}$

$= 0.8 + 0.2 = 1$

(iii) $\{[x]\} = 0$

Proof: $\{[x]\} = \{\text{Integer}\} = 0$

$\{\therefore$ [x] is always an integer$\}$

(iv) $[\{x\}] = 0$

Proof: $\{x\} \in [0, 1)$

$[[0, 1)] = 0$

Know the facts

Greatest integer function of any number in this interval is always zero.

Q.1 Find domain and range of:

(i) $f(x)=\frac{1}{\{x\}}$, ({.} denotes fractional part function)

A. $\{x\} \neq 0$ ($\because$ {x} is in denominator)

So, x can not be an integer.

$\therefore$ **Domain:** $R - \{x \mid x \in I\}$

or

R – I, where I is an integer.

For range, the denominator has the values in interval (0, 1)

$\frac{1}{0^+}=\infty;\frac{1}{1^-}=1^+$

$\therefore$ **Range:** $(1, \infty)$

Q. **(ii)** $f(x) = \log_{10}\{x\}$, ({.} denotes fractional part function)

A. $\{x\} > 0$ as $\{x\}$ is inside logarithm.

$\therefore\ \{x\} \neq 0$

$\Rightarrow\ x \neq I$, where I is an integer.

$\therefore$ **Domain:** $x \in R - I$.

$\{x\} \in (0, 1)$

So, $\log_{10}\{x\} \in \left(\log_{10} 0^{+},\ \log_{10} 1^{-}\right)$

So, **Range** is $(-\infty, 0)$

Q. **(iii)** $y = [\sin\{x\}]$
(where [.] denotes greatest integer function and {.} denotes fractional part function)

A. **Domain:** $x \in R$

$\{x\} \in [0, 1)$

$\therefore\ \sin\{x\} \in [\sin 0, \sin 1) = [0, \sin 1)$

$1^{c} \approx 57^{\circ}$. So, $0 < \sin 1 < 1$.

So, it can be seen that sin{x} has the minimum value 0 and it is always less than 1.
Since, $[x] = 0\ \forall\ x \in [0,1)$, $[\sin\{x\}] = 0\ \forall\ x \in R$.
So, **Range** is {0}, a single element.

Q.2 Find domain, range and graph of: $y = \sqrt{[x]-1} + \sqrt{4-[x]}$
(where [.] denotes greatest integer function)

A. $[x]-1 \geq 0 \quad \Rightarrow [x] \geq 1$

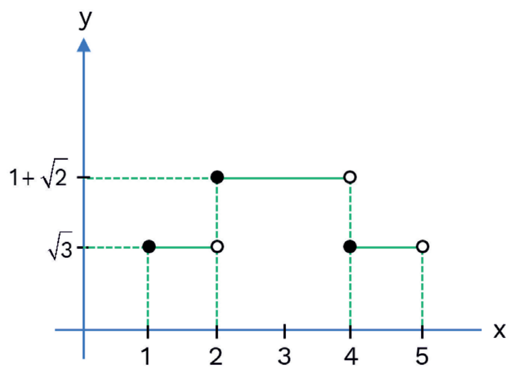
$4-[x] \geq 0 \quad \Rightarrow [x] \leq 4$

So, $1 \leq [x] \leq 4$

So, **domain:** $x \in [1, 5)$

$(\because\ [x] = 4$ for $x \in [4, 5))$

For range, firstly [x] = 1, 2, 3 or 4
Putting these values in y,
we get $\sqrt{3}$, $1+\sqrt{2}$, $1+\sqrt{2}$ and $\sqrt{3}$ respectively.



$\therefore$ Range contains only two values, $\sqrt{3}$ and $1+\sqrt{2}$.

$\mathbf{y \in \{\sqrt{3}, 1+\sqrt{2}\}}$.

Q.3 Find domain of $y=\sqrt{-\left|2[x]-x-\{x\}\right|}$

(where [.] denotes greatest integer function and {.} denotes fractional part function)

A. $|f(x)| \geq 0 \Rightarrow -|f(x)| \leq 0$

$\therefore$ If we have $\sqrt{-|f(x)|}$, it will give real values only

if $|f(x)| = 0$

So, in above question, $2[x] - x - \{x\} = 0$

$\Rightarrow$ **Range** is {0}.

Let $x = I + f$, $[x] = I$ and $\{x\} = f$, $f \in [0,1)$

$\therefore$ $2I - (I + f) - f = 0$

$\Rightarrow I = 2f \Rightarrow f = \frac{I}{2}$

Since $f \in [0, 1)$, f can only be $\frac{0}{2}$ or $\frac{1}{2}$.

$\therefore$ If $I = 0$, $f = 0 \Rightarrow I + f = x = 0$

If $I = 1$, $f = \frac{1}{2} \Rightarrow I + f = x = \frac{3}{2}$

So, $\mathbf{x} = \left\{0, \frac{3}{2}\right\}$ **(Domain)**

Q.4 Find the sum of the given series.

$$\left[\frac{3}{8}\right]+\left[\frac{3}{8}+\frac{1}{100}\right]+\left[\frac{3}{8}+\frac{2}{100}\right]+\ldots\left[\frac{3}{8}+\frac{99}{100}\right]+\ldots\left[\frac{3}{8}+\frac{234}{100}\right]$$

(where [.] denotes greatest integer function)

A. Sum = [0.375] + [0.385] + [0.395] + ... +[2.715]

From observation, $\frac{3}{8}+\frac{63}{100} = 1.005$

$\frac{3}{8}+\frac{163}{100} = 2.005$

$$\therefore \text{ sum } = \underbrace{[0.375]+\ldots+[0.995]}_{63\text{ terms}}+\underbrace{[1.005]+\ldots+[1.995]}_{100\text{ terms}}+\underbrace{[2.005]+\ldots+[2.715]}_{72\text{ terms}}$$

$= 0 \times 63 + 1 \times 100 + 2 \times 72$

$= 0 + 100 + 144 = \mathbf{244}$

Q.5 Find the range:

$$f(x) = [1+\sin x] + \left[2+\sin\left(\frac{x}{2}\right)\right] + \left[3+\sin\left(\frac{x}{3}\right)\right] + \ldots + \left[n+\sin\left(\frac{x}{n}\right)\right];$$

$x \in [0, \pi]$ (where [.] denotes greatest integer function)

A. $(1 + 2 + 3 + 4 + \ldots n) + [\sin x] + \left[\sin\left(\frac{x}{2}\right)\right] + \left[\sin\left(\frac{x}{3}\right)\right] + \ldots + \left[\sin\left(\frac{x}{n}\right)\right]$

Since $x \in [0, \pi]$, $\frac{x}{3} \in \left[0, \frac{\pi}{3}\right]$

So, $\sin\left(\frac{x}{3}\right) \in \left[0, \frac{\sqrt{3}}{2}\right]$

$\therefore$ $\left[\sin\frac{x}{3}\right] = 0$ only.

Similarly, value of $\left[\sin\frac{x}{4}\right], \left[\sin\frac{x}{5}\right]$, ... and $\left[\sin\frac{x}{n}\right]$ will always be 0.

So, $f(x) = n\left(\frac{n+1}{2}\right) + [\sin x] + \left[\sin\frac{x}{2}\right]$

At $x = \pi$, $[\sin x]=0$ and $\left[\sin\frac{x}{2}\right] = 1 \Rightarrow f(x) = n\left(\frac{n+1}{2}\right) + 1$

At $x = \frac{\pi}{2}$, $[\sin x]=1$ and $\left[\sin\frac{x}{2}\right] = 0 \Rightarrow f(x) = n\left(\frac{n+1}{2}\right) + 1$

At all other values in $(0, \pi)$, $[\sin x] = 0$ and $\left[\sin\frac{x}{2}\right] = 0 \Rightarrow f(x) = n\left(\frac{n+1}{2}\right)$

So, f(x) can assume only two values, $\frac{n^2+n+2}{2}$ and $\frac{n^2+n}{2}$.

$\therefore$ **Range** = $\left\{\frac{n^2+n}{2}, \frac{n^2+n+2}{2}\right\}$

Point to Remember!!!

$$[x] + \left[x+\frac{1}{n}\right] + \left[x+\frac{2}{n}\right] + \ldots + \left[x+\frac{n-1}{n}\right] = [nx]$$

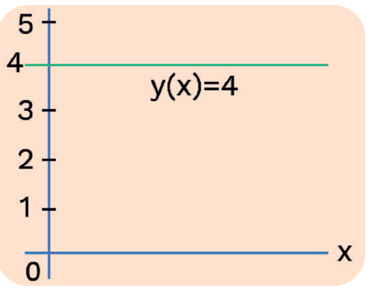
We can also use it to solve the previous question.

Definition of Constant function

Constant function is a function whose (output) value is the same for every input value.

Ex.

For example, the function given is a constant function because the value is 4 regardless of the input value (see diagram)



In this type of function, **domain** is (-∞, ∞), while range contains only a single value. In above example, range is {4}

Definition of Identity function

An identity function, also called an identity relation or identity map or identity transformation, is a function that always returns the same value that was used as its argument.

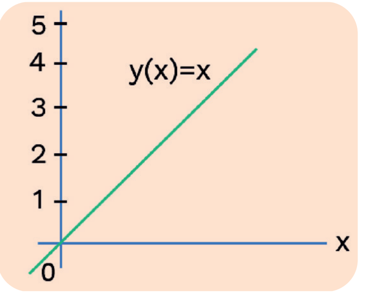
Ex. f(x) = x

Domain = R

Range is $(-\infty, \infty)$

This is an increasing function.

it is also represented by I_x.



Q.1 **Find domain:**

(i) $\sqrt{\log_5\left(\log_3\left(\log_2\left(2x^3+5x^2-14x\right)\right)\right)}$

A. $\log_5\left(\log_3\left(\log_2\left(2x^3+5x^2-14x\right)\right)\right) \geq 0 = \log_5 1$

$\Rightarrow \log_3(\log_2(2x^3 + 5x^2 - 14x)) \geq 1 = \log_3 3$

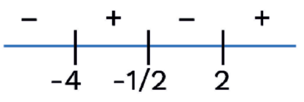
$\Rightarrow \log_2(2x^3 + 5x^2 - 14x) \geq 3 = \log_2 8$

$\Rightarrow 2x^3 + 5x^2 - 14x - 8 \geq 0$

Now, we can guess one root through hit and trial method. Here, it is x = 2.

$\therefore (x - 2)(2x^2 + 9x + 4) \geq 0$

$(x - 2)(x + 4)(2x + 1) \geq 0$



$\Rightarrow \mathbf{x} \in \left[-4, \frac{-1}{2}\right] \cup [2, \infty)$

Q. **(ii)** $\sqrt{(0.625)^{4-3x}-(1.6)^{x(x+8)}}$

A. $\left(\frac{5}{8}\right)^{4-3x} \geq \left(\frac{8}{5}\right)^{x(x+8)}$

$\left(\frac{8}{5}\right)^{3x-4} \geq \left(\frac{8}{5}\right)^{x^2+8x}$

$\Rightarrow 3x - 4 \geq x^2 + 8x$ ($\because$ a^x, $a > 1$ is an increasing function)

$\Rightarrow x^2 + 5x + 4 \leq 0$

$\Rightarrow (x + 1)(x + 4) \leq 0$

$\mathbf{x \in [-4, -1]}$ **(domain)**

Q. **(iii)** $f(x) = \log\left(x-\sqrt{1-x}\right)$

A. $x-\sqrt{1-x} \geq 0$ and $1 - x \geq 0$

$\Rightarrow x \leq 1$ and $x \geq \sqrt{1-x}$

Case I: $x \leq 0$

LHS = Negative, RHS = Positive

$\therefore$ Not possible

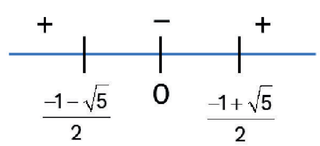
Case II: $x > 0$

Square both sides

$x^2 \geq 1 - x$

$x^2 + x - 1 \geq 0$

Roots are $\frac{-1\pm\sqrt{5}}{2}$



$$\left(x-\left(\frac{-1-\sqrt{5}}{2}\right)\right)\left(x-\left(\frac{-1+\sqrt{5}}{2}\right)\right) \geq 0$$

$$\Rightarrow x \in \left(-\infty, \frac{-1-\sqrt{5}}{2}\right] \cup \left[\frac{-1+\sqrt{5}}{2}, \infty\right)$$

But $x > 0$ and also $x \leq 1$

$$\Rightarrow \mathbf{x} \in \left[\frac{-1+\sqrt{5}}{2}, 1\right]$$

Q. **(iv)** $f(x) = \sqrt{\sin x} + \sqrt{16 - x^2}$

A. $\sin x \geq 0$ and $16 - x^2 \geq 0$

$x^2 \leq 16$

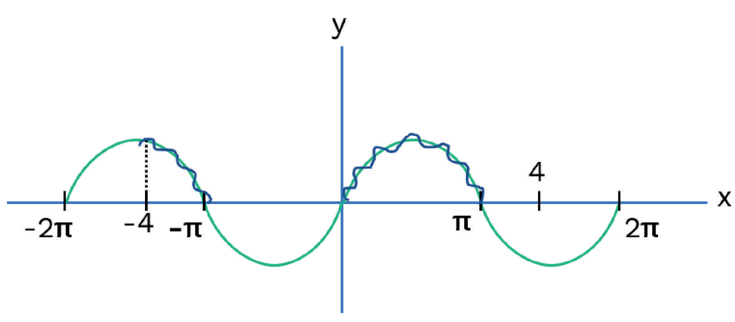
$x \in [-4, 4]$

(from 2nd condition)

by graph, $y = \sin x$

Since $\sin x \geq 0$,

$x \in [-4, -\pi] \cup [0, \pi]$



Q. **(v)** $f(x) = \sqrt{\dfrac{\sqrt{x-2}}{x-3}}$

A. $\dfrac{\sqrt{x-2}}{x-3} \geq 0$ and $x - 2 \geq 0$

Case I: $\sqrt{x-2} = 0$

$x = 2,\ x - 3 = -1$

$\Rightarrow \dfrac{0}{-1} = 0 \geq 0$ So, no problem.

$x = \{2\}$

Case II: $\sqrt{x-2} > 0$

$\Rightarrow x > 2$

in this case, $x - 3$ must be positive.

So, $x > 3$

$\therefore$ $\mathbf{x \in \{2\} \cup (3, \infty)}$

Q. **(vi)** $f(x) = \sqrt{(x^2 - 3x - 10)\ln^2(x-3)}$

A. $(x - 5)(x + 2)\ln^2(x-3) \geq 0$ and $x - 3 > 0$

$\ln^2(x-3) \geq 0$ wherever defined

Case I: $\ln^2(x-3) = 0 \Rightarrow x = 4$

$f(4) = \sqrt{0} = 0$ so, true.

Case II: $x \neq 4,\ x > 3$

$(x - 5)(x + 2) \geq 0$

$x \in (-\infty, -2] \cup [5, \infty)$

But since $x > 3$, $\therefore x \in [5, \infty)$

Case I $\cup$ Case II

$\Rightarrow$ $\mathbf{x \in \{4\} \cup [5, \infty)}$

Q. **(vii)** $f(x)=\frac{1}{[x]}+\log_{1-\{x\}}\left(x^2-3x+10\right)+\frac{1}{\sqrt{2-|x|}}+\frac{1}{\sqrt{\sec(\sin x)}}$, **where [·] denotes greatest integer function and {·} denotes fractional part function.**

A. Domain of $\frac{1}{[x]}$ $\rightarrow$ $x \in R-[0,1)$

Domain of 2nd function $\rightarrow$ $1-\{x\}>0$

$1-\{x\} \neq 1$

$x^2-3x+10>0$

$\therefore$ $x \in R-I$

Domain of 3rd function: $2-|x|>0$

$\Rightarrow x \in (-2,2)$ Domain of 4th function: sec (sinx) > 0

$\Rightarrow \cos(\sin x)>0$

$\sin x \in [-1, 1]$ and $\cos\theta > 0$ for $\theta \in [-1,1]$

So, $x \in R$

Taking intersection of all cases, $\mathbf{x \in (-2,-1) \cup (-1,0) \cup (1, 2)}$

Q. **(viii)** $f(x) = \sqrt{7^x-1}$

A. $7^x - 1 \geq 0$

$\Rightarrow 7^x \geq 1$

$\Rightarrow 7^x \geq 7^0$

$\Rightarrow x \geq 0$ ($\because$ increasing function as base > 1)

Domain: $\mathbf{x \in [0, \infty)}$

Q. **(ix)** $f(x) = \sqrt{7^{x+1}-1}$

A. $7^{x+1} \geq 1$

$x + 1 \geq 0$

$x \geq -1$

Domain: $\mathbf{x \in [-1, \infty)}$

Q. **(x)** $f(x) : \sqrt{\frac{1-5^x}{7^{-x}-7}}$

A. $\frac{1-5^x}{7^{-x}-7} \geq 0$

$\Rightarrow 7^x \frac{\left(5^x-1\right)}{7^{x+1}-1} \geq 0$

7^x is always positive.

$5^x - 1 \geq 0 \quad \forall\ x \geq 0$

$7^{x+1} - 1 \geq 0 \quad \forall\ x + 1 \geq 0$

$\therefore$ Sign scheme of '5^x-1' and 'x' is same. Same is true for '7^{x+1} -1' and 'x+1'

$\therefore$ We can write the inequality as

$\frac{x}{x+1} \geq 0 \quad \Rightarrow$ **$x \in (-\infty, -1) \cup [0, \infty)$**

^nC_r as a function

^nC_r is the number of combinations function, defined as $^nC_r = \frac{n!}{r!(n-r)!}$,

where n and r are non-negative integer. (n ≥ r)

Q.2 **Find range of the following functions:**

(i) $f(x) = \cos^4\frac{x}{5} - \sin^4\frac{x}{5}$

A. Simplifying, we get

$$f(x) = \left(\cos^2\frac{x}{5} - \sin^2\frac{x}{5}\right)\left(\cos^2\frac{x}{5} + \sin^2\frac{x}{5}\right)$$

$$= \cos\frac{2x}{5} \cdot 1$$

$\therefore$ Range $\in$ **[-1, 1]**

Q. **(ii)** $f(x) = \sin\sqrt{x}$

A. Range: **[-1, 1]**

Q. **(iii)** $f(x) = 3 - 2^x$

A. $y = 3 - 2^x$

$2^x \in (0, \infty)$

$3-2^x \in (3 - \infty, 3 - 0)$

$\therefore$ **$y \in (-\infty, 3)$**

Q. **(iv)** $f(x) = \sin(\log_2 x)$

A. $\log_2 x \in (-\infty, \infty)$

$\therefore$ $\mathbf{\sin(\log_2 x) \in [-1, 1]}$

Q. **(v)** $f(x) = \cos 2x - \sin 2x$

A. Range of $a \cos\theta + b \sin\theta$ is $\left[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}\right]$

$\therefore$ Range of $\cos 2x - \sin 2x$ is $\left[-\sqrt{1^2+(-1)^2}, \sqrt{1^2+(-1)^2}\right]$

$\Rightarrow$ Range: $\left[-\sqrt{2}, \sqrt{2}\right]$

Q. **(vi)** $y = 2|\cos x| - |\sin x|$

A. y is continuous function. $|\cos x|$ is maximum at the same time when $|\sin x|$ is minimum.

Also, $|\cos x|$ is minimum when $|\sin x|$ is maximum.

$\therefore$ In such a case, $y_{max} = 2 \times (1) - (0) = 2$

$y_{min} = 2 \times 0 - 1 = -1$

$\therefore$ Range: **[-1, 2]** ($\because$ y is a continuous function)

Q. **(vii)** $f(x) = \dfrac{\tan(\pi[x-\pi])}{x^2 - 3x + 4}$ where $[\cdot]$ denotes greatest integer function.

A. $[x - \pi]$ will always be integer.

$\therefore$ Numerator is always 0.

$\therefore$ Range is **{0}**.

Q. **(viii)** $f(x) = \cot^2\left(x - \dfrac{\pi}{4}\right)$

A. $\cot\left(x - \dfrac{\pi}{4}\right) \in (-\infty, \infty)$

$\therefore$ $\cot^2\left(x - \dfrac{\pi}{4}\right) \in \mathbf{[0, \infty)}$

Q. **(ix)** $f(x) = \dfrac{x^2 - x + 1}{x^2 + x + 1} = y$

A. $\Rightarrow x^2 - x + 1 = x^2y + xy + y$

$\Rightarrow$ $x^2 (1-y) - x (1+y) + 1 - y = 0$

If y is a value in range, then this equation contains solution.

So, $D \geq 0$, $y \neq 1$ (for $y = 1$, check separately)

$\therefore$ $(y + 1)^2 \geq 4(1 - y)^2$

$\Rightarrow$ $(3 - y)(3y - 1) \geq 0$

$\Rightarrow$ $(y-3)\left(y - \dfrac{1}{3}\right) \leq 0$

$\Rightarrow$ $\mathbf{y} \in \left[\dfrac{1}{3}, 3\right]$

Now, for $y = 1$, $x^2 - x + 1 = x^2 + x + 1 \Rightarrow x = 0$, so value of x exist

Hence $y \in \left[\dfrac{1}{3}, 3\right]$

Q. **(x)** $y = \dfrac{x - [x]}{1 + x - [x]}$, where [·] denotes greatest integer function.

A. $y = \dfrac{1 + x - [x]}{1 + x - [x]} - \dfrac{1}{1 + x - [x]}$

$\Rightarrow y = 1 - \dfrac{1}{1 + \{x\}}$

$\{x\} \in [0, 1) \Rightarrow 1 + \{x\} \in [1, 2)$

$\Rightarrow \dfrac{1}{1 + \{x\}} \in \left(\dfrac{1}{2}, 1\right] \Rightarrow \mathbf{y} \in \left[0, \dfrac{1}{2}\right)$

Q. **(xi)** $f(x) = \tan\left(\{x\} \times \dfrac{\pi}{4}\right)$, where {·} denotes fractional part function.

A. $\{x\} \times \dfrac{\pi}{4} \in \left[0, \dfrac{\pi}{4}\right)$

$\therefore$ Range: **[0, 1)**

Q. **(xii)** $f(x) = \tan\left(\dfrac{\pi}{4}\operatorname{sgn}(x^2 - 1)\right)$

A. $\operatorname{sgn}(x^2-1) \in \{-1, 0, 1\}$

$\therefore$ Range includes $\tan\left(-\dfrac{\pi}{4}\right)$, $\tan 0$, $\tan\left(\dfrac{\pi}{4}\right)$

$\therefore$ Range is **{-1, 0, 1}**.

Q. **(xiii)** $f(x)=|x^2 - x - 6|$

A. $x^2 - x - 6 \in \left[\frac{-D}{4a}, \infty\right)$

$\therefore \quad x^2 - x - 6 \in \left[\frac{-25}{4}, \infty\right)$

$\Rightarrow |x^2 - x - 6| \in$ **[0, ∞)**

Q. **(xiv)** $y = |\sin x| + |\cos x|$

A. Squaring both sides.

$y^2 = \sin^2 x + \cos^2 x + 2|\sin x||\cos x| = 1 + |\sin 2x|$

$\therefore \quad y^2 \in [1, 2] \qquad (\because |\sin 2x| \in [0,1])$

$\therefore \quad \mathbf{y \in [1, \sqrt{2}]} \qquad (\because$ y is positive$)$

Q. **(xv)** $y = \frac{1}{x^2 + x + 1}$

A. $x^2 + x + 1 \in \left[\frac{-D}{4a}, \infty\right)$

$x^2 + x + 1 \in \left[\frac{3}{4}, \infty\right)$

$\therefore \quad \frac{1}{x^2 + x + 1} \in \left(\frac{1}{\infty}, \frac{1}{3/4}\right]$

$\Rightarrow \mathbf{y} \in \left(0, \frac{4}{3}\right]$

Q. **(xvi)** $y = \frac{1}{\sin^4 x + \cos^4 x}$

A. $y = \frac{1}{\sin^4 x + \cos^4 x} = \frac{1}{(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x}$

$= \frac{1}{1 - \frac{1}{2}(4\sin^2 x\cos^2 x)} = \frac{1}{1 - \frac{1}{2}(\sin 2x)^2}$

$1 - \frac{1}{2}(\sin 2x)^2 \in \left[1 - \frac{1}{2} \times 1, 1 - \frac{1}{2} \times 0\right]$

$\Rightarrow \quad 1 - \frac{1}{2}(\sin 2x)^2 \in \left[\frac{1}{2}, 1\right]$

$\therefore \quad \mathbf{y \in [1, 2]}$

Q. **(xvii)** $f(x) = [\sin x]$, where $[\cdot]$ denotes greatest integer function.

A. $\sin x \in [-1, 1]$

$\Rightarrow \sin x \in [-1, 0) \cup [0, 1) \cup \{1\}$

$\Rightarrow [\sin x] \in \mathbf{\{-1, 0, 1\}}$

Q. **(xviii)** $f(x) = 2 - [\sin x] - [\sin x]^2 = y$, where $[\cdot]$ denotes greatest integer function.

A. **case I:** $[\sin x] = -1$

$y = 2 - (-1) - (-1)^2 = 2 + 1 - 1 = 2$

Case II: $[\sin x] = 0$, $y = 2 - 0 - 0 = 2$

Case III: $[\sin x] = 1$

$y = 2 - 1 - 1 = 0$

$\therefore$ $\mathbf{y \in \{0, 2\}}$

Remainder theorem

The polynomial remainder theorem states that the remainder of the division of a polynomial $f(x)$ by a linear polynomial $x - \alpha$ is equal to $f(\alpha)$.

In particular, **$x - \alpha$ is a divisor of $f(x)$ if and only if $f(\alpha)=0$, a property known as the factor theorem.**

$$\mathbf{D = (d \times q) + R}$$

where, D = dividend, d = divisor, q = quotient, R = remainder.

Ex. $x^3 + 4x^2 - 7x + 6$ when divided by $x - 1$

Remainder $\rightarrow 1^3 + 4\cdot1^2 - 7\cdot1 + 6 = \mathbf{4}$

Ex. $x^3 + x = (x - 1)\underbrace{(Q(x))}_{\text{Quotient}} + \underbrace{2}_{\text{Remainder}}$

Ex. $x^2 + 2x - 3 = Ax^2 + Bx + c$

$\Rightarrow A = 1, B = 2, C = -3$

Point to Remember!!!

(i) If $f(x) = g(x)$, where $f(x)$ and $g(x)$ are polynomials, then coefficients of all different powers of x are equal on both sides.

(ii) If polynomial is divided by quadratic, the remainder is a linear.

If divided by a cubic, remainder is a quadratic.

If divided by a biquadratic, remainder is a cubic and so on.

Q. $ax^4 + bx^3 - x^2 + 2x + 3$ when divided by $x^2 + x - 2$ gives remainder $4x + 3$. Find a and b.

A. $ax^4 + bx^3 - x^2 + 2x + 3 = (x-1)(x + 2)Q(x) + 4x + 3$

put $x = 1$ both side

$a + b - 1 + 2 + 3 = 0 + 7$

$\Rightarrow a + b = 3$ ...(i)

put $x = -2$

$16a - 8b - 4 - 4 + 3 = 0 - 5$

$\Rightarrow 2a - b = 0$ ...(ii)

from (i) and (ii), **a = 1, b = 2**.

Definition of Equal or Identical functions

Two functions f and g are said to be identical if

1. Domain of f = domain of g.i.e., $D_f = D_g$.

2. Range of f = Range of g.

3. $f(x) = g(x)\ \forall\ x \in D_f$ or $x \in D_g$

Ex.

$f(x) = x$ and $g(x) = \sqrt{x^2}$ are not identical function as $D_f = D_g$ but $R_f = R$, $R_g = [0, \infty)$

Overall, it can be said that graph of f(x) and g(x) must be same everywhere for two functions to be identical.

Ex.

$f(x) = \sin x$; $g(x) = \cos x$

$D_f = D_g = R$

$R_f = R_g = [-1, 1]$

But $\sin x \neq \cos x$ everywhere.

$\therefore$ graph is not same. So, f(x) and g(x) are not identical.

Q. **Check if function are identical or not**

(i) $y = \ln x^2$; $y = 2 \ln x$

A. **Domain** of $\ln x^2 : x^2 > 0$

$\Rightarrow x \neq 0$

Domain of $2\ln x \rightarrow x > 0$

$\therefore$ Domain not same. So **not identical**.

Q. **(ii)** $y = \text{cosec } x$; $y = \dfrac{1}{\sin x}$

A. $\text{cosec } x = \dfrac{1}{\sin x}$

Graph is same

$\therefore$ both functions are **identical**.

Q. **(iii)** $f(x) = \tan x$; $g(x) = \dfrac{1}{\cot x}$

A. $f(x) = \dfrac{\sin x}{\cos x}$, $\cos x \neq 0$

$g(x) = \dfrac{1}{\left(\dfrac{\cos x}{\sin x}\right)}$

$\sin x \neq 0$ because denominator will not be defined.

also, $\cos x \neq 0$

as denominator will become 0.

$\therefore$ f(x) and g(x) are **not identical** because domain are not same.

Q. **(iv)** $f(x) = \sec x$; $g(x) = \dfrac{1}{\cos x}$

A. **Identical**

Q. **(v)** $f(x) = \cot^2(x).\cos^2 x$; $g(x) = \cot^2 x - \cos^2 x$

A. Both f(x) and g(x) contain cotx, so $\sin x \neq 0$.

$\Rightarrow$ Domain is same.

$g(x) = \cot^2 x - \cos^2 x = \cot^2 x\,(1-\sin^2 x) = \cot^2 x \cos^2 x = f(x)$

$\therefore$ graph is same.

$\therefore$ **Identical**

Q. **(vi)** $f(x) = \text{sgn}(x^2 + 1)$; $g(x) = \sin^2 x + \cos^2 x$

A. $x^2 + 1 > 0 \;\forall\; x \in R$

$\therefore$ $f(x) = 1 \;\forall\; x \in R$

$g(x) = 1 \;\forall\; x \in R$

$\therefore$ Both are **identical**.

Q. **(vii)** $f(x) = \tan^2 x . \sin^2 x$; $g(x) = \tan^2 x - \sin^2 x$

A. **Identical**

Q. **(viii)** $f(x) = \sec^2 x - \tan^2 x$; $g(x)=1$

A. $f(x) = 1 \ \forall \ x \in R. - (2n+1)\frac{\pi}{2}$, $n \in I$

$g(x) = 1 \ \forall \ x \in R$

So, domain is not same.

$\therefore$ **Not identical**

Q. **(ix)** $f(x) = \log_x e$; $g(x) = \dfrac{1}{\log_e x}$

A. $D_{f:}$ $(0, 1) \cup (1, \infty)$

$D_{g:}$ $(0, 1) \cup (1, \infty)$

Domain is same.

Also, $f(x) = g(x)$

$\therefore$ **Identical**

Q. **(x)** $f(x) = \log_e x$; $g(x) = \dfrac{1}{\log_x e}$

A. $D_f : x > 0$

$D_g : x > 0, x \neq 1$

$\Rightarrow D_f \neq D_g$

$\therefore$ **Not Identical**

Q. **(xi)** $f(x) = \sqrt{x^2 - 1}$; $g(x) = \sqrt{x-1}\ \sqrt{x+1}$

A. $D_f : x^2 - 1 \geq 0$

$\Rightarrow x \in (-\infty, -1] \cup [1, \infty)$

$D_g : x - 1 \geq 0 \Rightarrow x \geq 1$

$x + 1 \geq 0 \Rightarrow x \geq -1$

$\Rightarrow D_g : x \geq 1$

$\Rightarrow D_f \neq D_g$

$\therefore$ **Not Identical**

Q. **(xii)** $f(x)=\sqrt{1-x^2}$; $g(x)=\sqrt{1-x}.\sqrt{1+x}$

A. $D_f : 1-x^2 \geq 0$

$x \in [-1, 1]$

$D_g : 1- x \geq 0 \Rightarrow x \leq -1$

$1 + x \geq 0 \Rightarrow x \geq -1$

$\Rightarrow D_g : x \in [-1, 1]$

$\Rightarrow D_f = D_g \Rightarrow$ **Identical**

Q. **(xiii)** $f(x)=e^{\ln e^x}; g(x)=e^x$

A. $f(x) = e^x = g(x)$

$(\because a^{(\log_a N)} = N)$

$D_f = D_g$

$\therefore$ **Identical**

Q. **(xiv)** $f(x) = \sqrt{\dfrac{1-\cos^2 x}{2}}; g(x)=\sin x$

A. $f(x) \geq 0$ (Because of square root)

$g(x)$ can be negative.

Range not same.

$\therefore$ **Not Identical**

Q. **(xv)** $f(x) = \log(x + 2) + \log(x - 3)$; $g(x)=\log(x^2 - x - 6)$

A. $D_f : x + 2 > -0 \quad \Rightarrow x > -2$

$x - 3 > 0 \quad \Rightarrow x > 3$

$\therefore D_f : (3\ \infty)$

$D_g : x^2 - x - 6 > 0$

$\Rightarrow (x-3)(x + 2) > 0$

$\Rightarrow x \in (-\infty, -2) \cup (3, \infty)$

$\Rightarrow D_f \neq D_g$

$\therefore$ **Not identical**

Q. **(xvi)** $f(x) = \dfrac{1}{1+\frac{1}{x}}; g(x)=\dfrac{x}{1+x}$

A. $D_f : x \neq 0, -1$

$D_g : x \neq -1$

$\Rightarrow D_f \neq D_g$

$\therefore$ **Not Identical**

Q. **(xvii)** $f(x) = [\{x\}]$; $g(x) = \{[x]\}$

A. $f(x) = 0 = g(x)$

$D_f = D_g = (-\infty, \infty)$

$\therefore$ **Identical.**

Classification of Functions

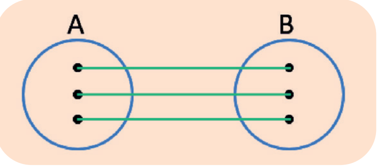
1. Definition of One-one (injective Mapping)

f: A → B such that different elements of A have different f images in B.

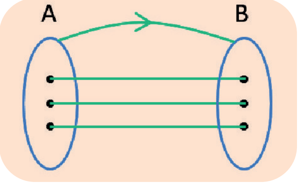
or $x_1, x_2 \in A$ and $f(x_1), f(x_2) \in B$,

$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

or $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

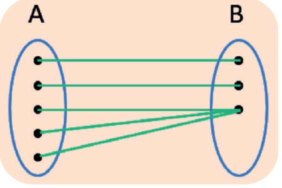


Ex.



(Every input has a different output)

Ex.



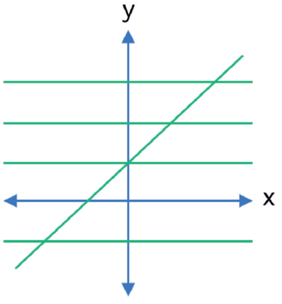
Not one-one (3 inputs have same output)

Q.1 **Find if given functions are one-one or not**

(i) $y = x + 1$

A. All horizontal lines can cut the graph at only 1 point.

$\Rightarrow$ **One-one function**

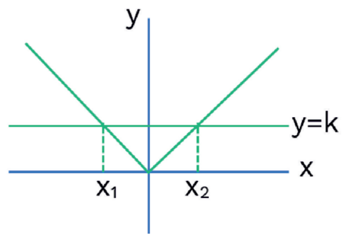


Q. (ii) $y = |x|$

A. A horizontal line cuts this graph at 2 points

$\Rightarrow$ $f(x_1) = f(x_2) = K$

$\Rightarrow$ **Not one-one**



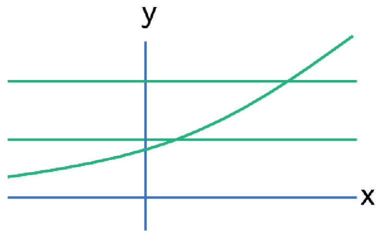
Q. (iii) $y = e^x$

A. $\dfrac{dy}{dx} = e^x > 0$

$\Rightarrow$ Increasing function $\forall\ x \in R$ (continuous also)

$\Rightarrow$ at any two different x, y can't be same

$\Rightarrow$ **One-one function**



Q. (iv) $y = x^3$

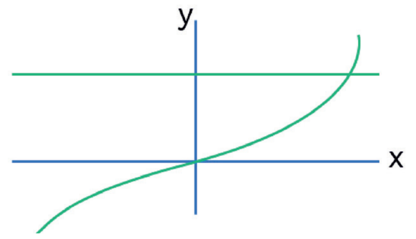
A. $\dfrac{dy}{dx} = 3x^2$

$\therefore\ \dfrac{dy}{dx} \geq 0$

$\Rightarrow$ Continuous and increasing.

$\Rightarrow$ Horizontal line cuts the graph at only one-point.

$\Rightarrow$ **One–one function**



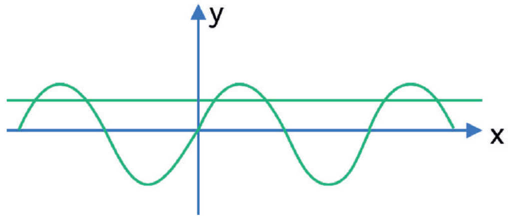
Point to Remember!!!

- If a line parallel to x axis cuts the graph of the function atleast at two points, then f is many one.
- A continuous function whose derivative changes sign is a many one function.

Q. (v) $y = \sin x$

A. **Many-one function**

Horizontal line cuts at more than one point



Q. **Find whether the given function are one one or not:**

$[x], \{x\}, \frac{1}{x}, \frac{1}{x^2}$ cosx, tanx, sgn(x)

A. $[x]$ and $\{x\}$ → different x but same y

→ **Not one-one**

$\Rightarrow$ $y = \frac{1}{x}$ $\Rightarrow$ **one-one**

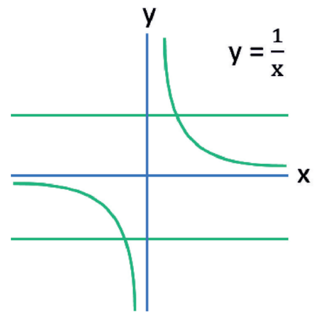
$\Rightarrow$ $y = \frac{1}{x^2}$

$f(2) = f(-2)$

→ **Not one-one**

$\Rightarrow$ cosx and tanx $\Rightarrow$ not one-one (similar to sinx)

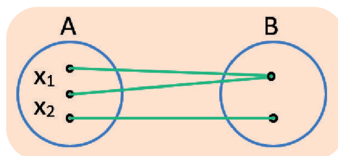
$\Rightarrow$ $(\text{sgn}(1) = \text{sgn}(2))$ $\Rightarrow$ sgn(x) is **not one-one**



2. Definition of Many-one function

f:A → B such that two or more elements of A have the same f-image in B.

or There exist $x_1, x_2 \in A$ such that $f(x_1) = f(x_2)$ but $(x_1) \neq (x_2)$



Ex. $y = (x-1)(x-2)(x-3)$

Sol. $f(1) = f(2) = f(3) = 0$

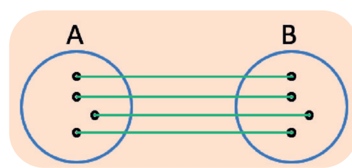
$\Rightarrow$ Many-one function.

3. Definition of Onto (surjective mapping)

f: A → B such that each element in B is the f image of at least one element in A.

In case of onto function, codomain is equal to range.

So, to find if the function is onto, find range and match it with codomain.



Ex. (i) $f : R \to R \; ; \; f(x) = 2x + 1$

(ii) $f : R^+ \to R \; ; \; f(x) = \ln x$

(iii) $f : R \to R^+ \; ; \; f(x) = e^x$

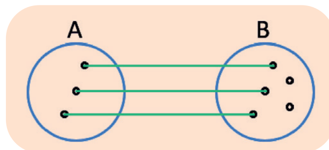
Sol. (i) $y = 2x + 1 \Rightarrow y \in R \Rightarrow$ Codomain = Range $\Rightarrow$ Onto.

(ii) $y = \ln x \Rightarrow y \in R \Rightarrow$ Codomain = Range $\Rightarrow$ Onto.

(iii) $y = e^x \Rightarrow y \in (0, \infty)$ or R^+. $\Rightarrow$ Codomain = Range $\Rightarrow$ Onto function.

4. Definition of Into function

$f: A \to B$ such that at least one element in B (co-domain) is NOT the f image of any element in domain A.



Ex. $f: R \to R$; $f(x) = \{x\}, [x], \text{sgn } x$

Sol. Range: $[0, 1)$, I, $\{-1, 0, 1\}$ respectively

But, codomain is R.

$\Rightarrow$ codomain $\neq$ Range $\Rightarrow$ All are into function.

Know the facts

For into function, codomain≠Range.

Q. **Find whether the given functions are onto or into:**

(i) $f : R \to [0, \infty)$; $f(x) = x^2$

(ii) $f : R \to [0, \infty)$; $f(x) = x^2 + 1$

(iii) $f : R \to [0, \infty)$; $f(x) = x^{100} + 100x^{98} + 3x^2 + |x| + x\,\text{sgn}(x)$

A. (i) $x^2 \in [0, \infty)$ $\Rightarrow$ Codomain = Range $\Rightarrow$ **Onto function**

(ii) $x^2 + 1 \in [1, \infty)$ $\Rightarrow$ Codomain $\neq$ Range $\Rightarrow$ **Into function**

(iii) $y = x^{100} + 100\,x^{98} + 3x^2 + |x| + x\,\text{sgn}(x)$

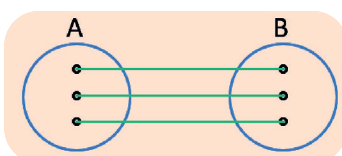
$\geq 0 \quad \geq 0 \quad \geq 0 \quad \geq 0 \quad |x| \geq 0$

$\therefore$ $y \geq 0$ ($f_{min} = 0$ at $x = 0$ and $f_{max} = \infty$)

$\to$ Range = Codomain.

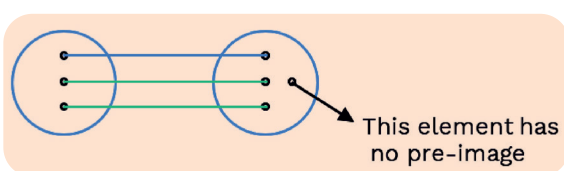
$\to$ **Onto function**.

One-one onto function (Bijective / invertible)



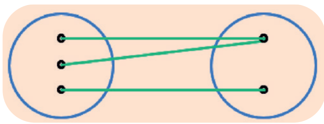
If function is both injective and surjective, then it is called a Bijective function.

One one into function



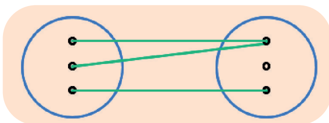
This type of function is one-one but not onto.

Many one onto function



This type of function is both many-one and onto.

Many one into function



This type of function is both many-one and into.

Q. **Classify as one one onto, one one into, many one onto or many one into:**
f:[-1, 1] → [-1, 1] f(x) = sin2x

A. $x \in [-1, 1]$

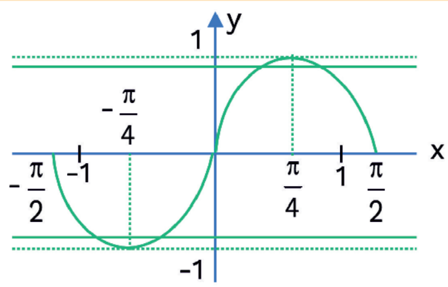
$y \in [-1, 1]$

$\therefore$ Range = codomain $\Rightarrow$ onto function

Horizontal line cuts at 2 points

$\Rightarrow$ Many-one function.

$\therefore$ f(x) = sin 2x is **many one onto function**.



Q. **Classify as one one onto. one one into. many one onto or many one into:**

$$\textbf{f: R} \rightarrow \textbf{R;}\quad f(x) = \frac{2x^2 - x + 5}{7x^2 + 2x + 10}$$

A. **Method-1:**

Find minimum and maximum values by finding critical points where $\frac{dy}{dx} = 0$

Method-2:

Numerator and denominator are both positive $\forall\ x \in R$ as D < O

$\therefore$ f(x) > 0 $\Rightarrow$ Range ≠ Codomain

$\therefore$ function is into

$$f(0) = \frac{5}{10} = \frac{2x^2 - x + 5}{7x^2 + 2x + 10}$$

$\Rightarrow$ $7x^2 + 2x + 10 = 4x^2 - 2x + 10$

$\Rightarrow$ $3x^2 + 4x = 0$

$\Rightarrow$ $x = 0$ or $-\dfrac{4}{3}$ $\Rightarrow$ $f(0) = f\left(\dfrac{-4}{3}\right)$

$\Rightarrow$ Many-one function.

$\therefore$ **f(x) is Many-one into function**.

Q. **The function f $[2, \infty) \to Y$ defined by $f(x) = x^2 - 4x + 5$ is both one and onto if:**

(A) Y = R **(B) Y = $[1, \infty)$**

(C) Y = $[4, \infty)$ **(C) Y = $[5, \infty)$**

A. $f'(x) = 2x - 4 \geq 0 \ \forall \ x \geq 2$

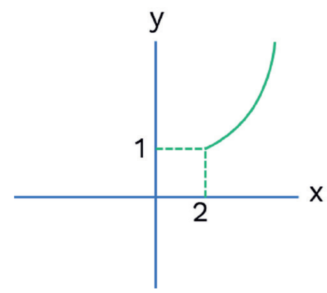
Also, $f(x) = (x-2)^2 + 1$

$\Rightarrow$ Range: $[1, \infty)$

$\Rightarrow$ f(x) is one-one for $x \in [2, \infty)$

$\Rightarrow$ Since f(x) is onto, Range = codomain.

So, option **(B)** is correct answer.

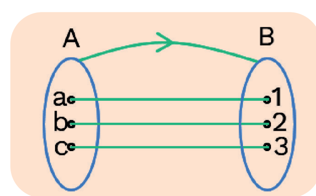


Permutation based problems:

$f: A \to B$

Case-1:

When both the set A and B contain an equal number of elements.



(i) Total number of functions.

Sol. 'a' has 3 options for image.

'b' has 3 options for image.

'c' also has 3 options i.e., 1,2 or 3.

$\therefore$ Total number of functions possible

$= 3 \times 3 \times 3 = 3^3$

Similarly, if no of elements is 5, then number of functions $= 5^5$.

Generalizing it,

Point to Remember!!!

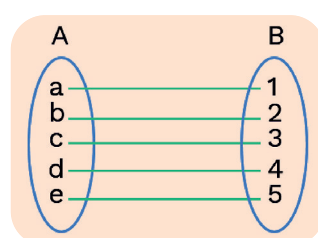
If number of elements is 'n', then number of functions = $\mathbf{n}^n$.

(ii) Total number of one-one functions.

Sol. if number of elements is 5:

'a' has 5 option for image.

After that, 'b' will have 4 options, $c \to 3$, $d \to 2$, $e \to 1$



∴ Total no of one-one function
$= 5 \times 4 \times 3 \times 2 \times 1 = 5!$

Point to Remember!!!

For 'n' elements, number of one-one functions = **n!**

(iii) Number of many one functions.

Sol. Number of many one functions = Total number of functions – number of one-one functions

Point to Remember!!!

Number of many-one functions = $\mathbf{n^n - n!}$

(iv) Number of onto functions.

Sol. In case of equal number of elements in both domain and codomain, all the onto functions are one-one function.

Point to Remember!!!

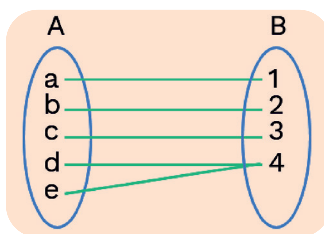
Number of onto functions = **n!**

(v) Number of into functions.

Sol. Number of into functions = Total – Number of onto functions = $\mathbf{n^n - n!}$

Case-2:

When number of elements in A (domain) is more than B



(i) Total number of functions.

Sol. Let number of elements in A and B be 5 and 4 respectively.
Every input has 4 options for output.
So, total number of functions
$= 4 \times 4 \times 4 \times 4 \times 4 = 4^5$

Point to Remember!!!

Total number of functions = $\mathbf{n}^m$ where, m is number of elements in A and n is number of elements in B.

(ii) Number of one-one (injective) functions

Sol. = **0**

(iii) Number of many-one functions

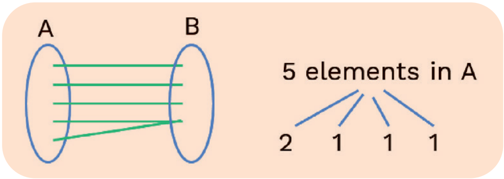
Sol. = total – 0
$= 4^5$
$= \mathbf{n}^m$

(iv) Number of onto functions

Sol. It is calculated using group formation.

Ex.

Number of ways to distribute 5 elements of A among 4 elements of B is $\left(\dfrac{5!}{2!(1!)^3}\times\dfrac{1}{3!}\right)\times 4!$



(v) Number of into functions

Sol. = Total – Number of onto functions.

Case-3:

Number of elements in codomain (B) is more than that in A

(i) Number of total functions

Sol. It is same in this case also and is equal to $\mathbf{n}^m$, where n and m are number of elements in 'B' and 'A' respectively.

(ii) Number of injective mapping

Sol. No of elements in A = 4

No of elements in B = 6

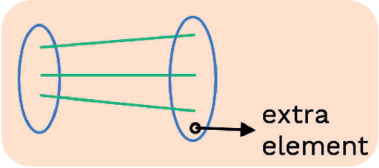
So, number of one-one function

$$= \underbrace{{}^6C_4}_{\text{selection of 4 output}} \times \underbrace{4!}_{\text{distribution}}$$

(iii) Number of many-one functions

Sol. = Total – one-one

$= 6^4 - {}^6C_4 \times 4!$



(iv) Number of onto functions

= 0

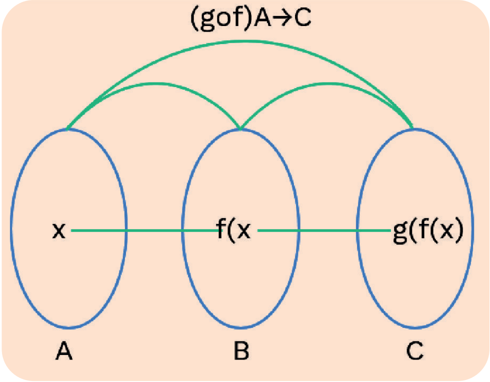
(v) Number of into functions

Sol. Number of into functions = $\mathbf{n}^m$

Composite Functions

Let f: A → B and g : B → C be two functions. Then the function gof : A → C defined by (gof) (x) = g(f(x)) ∀ x ∈A is called the composite of the two functions.

$$g\left(\underbrace{f(x)}_{\text{input}}\right) = h(x)$$



Q.1 **If $f(x) = x^2$ and $g(x) = x-7$. Find: (i) gof (ii) fog**

A. **(i)** gof = g(f(x)) = g(x^2) = $\mathbf{x^2 - 7}$
(ii) fog = f(g(x)) = f(x-7) = $\mathbf{(x-7)^2}$

Q.2 **If $f(x) = x^2$ and $g(x) = x - 7$. Find: (iii) gog (iv) fof**

A. **(iii)** gog = g(g(x)) = g(x-7) = (x-7)-7 = $\mathbf{x - 14}$
(iv) fof = f(f(x)) = f(x^2) = $(x^2)^2$ = $\mathbf{x^4}$

Q.3 **Let f(x) = $\sqrt{x}$; g(x) = $\sqrt{2-x}$. Find: (A) fog (B) gof**

A. **(A)** fog = $f\left(\sqrt{2-x}\right) = \sqrt{\sqrt{2-\mathbf{x}}}$

(B) gof = $g\left(\sqrt{x}\right) = \sqrt{2-\sqrt{\mathbf{x}}}$

Q.4 **Let f(x) = $\sqrt{x}$: $g(x) = \sqrt{2-x}$. Find: (C) fof (D) gog**

A. **(C)** fof = $f\left(\sqrt{x}\right) = \sqrt{\sqrt{x}} = \mathbf{x}^{1/4}$

(D) gog = $g\left(\sqrt{2-x}\right) = \sqrt{2-\sqrt{2-\mathbf{x}}}$

Domain of gog: $2-\sqrt{2-x} \geq 0 \Rightarrow 2 \geq \sqrt{2-x}$

$$\left.\begin{array}{l} \Rightarrow 4 \geq 2 - x \Rightarrow x \geq -2 \\ \text{Also } 2 - x \geq 0 \Rightarrow x \leq 2 \end{array}\right\} \Rightarrow x \in [-2, 2]$$

For **range:** $2-x \in [0, 4] \Rightarrow \sqrt{2-x} \in [0, 2]$

$\Rightarrow 2 - \sqrt{2-x} \in [0, 2] \Rightarrow gog \in \left[0, \sqrt{2}\right]$

Q.5 **Prove that: $f(x) = x^2$ if $\underbrace{fofof\ldots fof}_{n\ times} = x^{2^n}$**

A. Generally it becomes a sequence. So, find fof, fofof, fofofof(x).
Now observe the sequence and hence find the $\underbrace{fofof\ldots fof}_{n\ times}$

fof(x): $\left(x^2\right)^2 = x^4$ $\quad fofof(x) = \left(x^4\right)^2 = x^8$

fofofof = $\left(x^8\right)^2 = x^{16} = x^{2^4} \Rightarrow$ fofof... $fof(x) = x^{(2n)}$

Q.6 $f(x)=\dfrac{x}{\left(1+x^{n}\right)^{1/n}}$ **; g(x)=fofofo...fof(x) (f written 'n' times), g(x) =?**

A. fof(x) = $\dfrac{f}{\left(1+f^{n}\right)^{1/n}}=\dfrac{\dfrac{x}{\left(1+x^{n}\right)^{1/n}}}{\left(1+\dfrac{x^{n}}{1+x^{n}}\right)^{1/n}}=\dfrac{x}{\left(1+2x^{n}\right)^{1/n}}$

f(f(f(x))) = $\dfrac{\dfrac{x}{\left(1+2x^{n}\right)^{1/n}}}{\left(1+\dfrac{x^{n}}{1+2x^{n}}\right)^{1/n}}=\dfrac{x}{\left(1+3x^{n}\right)^{1/n}}$

By observation, it can be said that

fff...f(x) n times = $\dfrac{\mathbf{x}}{\left(1+\mathbf{nx^{n}}\right)^{1/\mathbf{n}}}$

Non-uniformly defined functions

Q.7 $f(x)=\begin{bmatrix}1+x & \text{if} & 0\le x\le 2\\ 3-x & \text{if} & 2<x\le 3\end{bmatrix}$**. Find fof.**

A. f(f(x)) = $\begin{cases}1+f; & 0\le f\le 2\\ 3-f; & 2<f\le 3\end{cases}$

$$=\begin{cases}1+(1+x); & 0\le x\le 2 & 0\le 1+x\le 2\Rightarrow\ x\in[0,1]\\ 1+(3-x); & 2<x\le 3 & 0\le 3-x\le 2\Rightarrow\ x\in(2,3]\\ 3-(1+x); & 0\le x\le 2 & 2<1+x\le 3\Rightarrow x\in(1,2]\\ 3-(3-x); & 2<x\le 3 & 2<3-x\le 3\Rightarrow x\in\phi\end{cases}$$

$\therefore$ **f(f(x)) =** $\begin{cases}2+\mathbf{x}; & \mathbf{x}\in[0,1]\\ 4-\mathbf{x}; & \mathbf{x}\in(2,3]\\ 2-\mathbf{x}; & \mathbf{x}\in(1,2]\end{cases}$

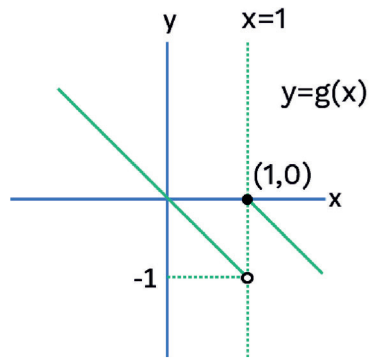
Q.8 **f(x) =** $\begin{cases} 1-x & \text{if } x \le 0 \\ x^2 & \text{if } x > 0 \end{cases}$ **and g(x) =** $\begin{cases} -x & \text{if } x < 1 \\ 1-x & \text{if } x \ge 1 \end{cases}$.

Find (fog)(x)

A. $f(g(x)) = \begin{cases} 1-g(x); & g(x) \le 0 \\ g(x)^2; & g(x) > 0 \end{cases}$

$$= \begin{cases} 1-(-\mathbf{x}); & \mathbf{x} \in [0,1) \\ 1-(1-\mathbf{x}); & \mathbf{x} \in [1,\infty) \\ (-\mathbf{x})^2; & \mathbf{x} \in (-\infty,0) \end{cases}$$

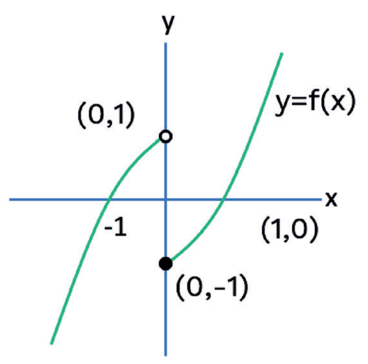
(This time solved directly using the graph of g(x)).



Q.9 **f(x) =** $\begin{cases} 1+x^3 & \text{if } x < 0 \\ x^2-1 & \text{if } x \ge 0 \end{cases}$ **and g(x)=** $\begin{cases} (x-1)^{1/3} & \text{if } x < 0 \\ (x+1)^{1/2} & \text{if } x \ge 0 \end{cases}$. **Find g(f(x)).**

A. $g(f(x)) = \begin{cases} (f-1)^{\frac{1}{3}}; & f < 0 \\ (f+1)^{\frac{1}{2}}; & f \ge 0 \end{cases}$

$$= \begin{cases} \left((1+x^3)-1\right)^{1/3}; & x \in (-\infty,-1) \\ \left((x^2-1)-1\right)^{1/3}; & x \in [0,1) \\ (1+x^3+1)^{1/2}; & x \in [-1,0) \\ (x^2-1+1)^{1/2}; & x \in [1,\infty) \end{cases}$$



$$\mathbf{g(f(x))} = \begin{cases} \mathbf{x} & ;\mathbf{x} \in (-\infty,-1) \\ (\mathbf{x}^2-2)^{1/3} & ;\mathbf{x} \in [0,1) \\ (\mathbf{x}^3+2)^{1/2} & ;\mathbf{x} \in [-1,0) \\ \mathbf{x} & ;\mathbf{x} \in [1,\infty) \end{cases}$$

Q.10 Find number of distinct real c satisfying f(f(f(c))) = 3 where f(x) = $x^2 - 2x$

A. Let f (θ) = 3, θ = f(f(c))

$\theta^2 - 2\theta = 3 \Rightarrow \theta^2 - 2\theta - 3 = 0$

$\theta = 3$ or -1

$\Rightarrow$ f(f(c)) = 3 or -1

$\Rightarrow$ Let f(t) = 3 or -1, t = f(c)

$t^2 - 2t = 3 \Rightarrow t = 3$ or -1

and

$t^2 - 2t = -1$

$(t-1)^2 = 0$

t = 1

$\therefore$ f(c) = 3 or 1 or -1

$$c^2 - 2c - 3 = 0 \begin{cases} 3 \\ -1 \end{cases}$$

$c^2 - 2c + 1 = 0 \Rightarrow c = 1$

$$c^2 - 2c - 1 = 0 \Rightarrow \begin{cases} 1+\sqrt{2} \\ 1-\sqrt{2} \end{cases}$$

$\therefore$ 'c' can be $1 \pm \sqrt{2}$ **, 1, -1, 3**.

Total **5 solutions**.

Q.11 y = f(x); Domain $\in$ [-3, 2]
Then domain of y = f(|[x]|) is? ([·] denotes greatest integer function)

A. |[x]| $\in$ [-3,2]

$\Rightarrow$ |[x]| $\in$ [0, 2] ($\because$ |x| is always non-negative)

$\Rightarrow$ [x] $\in$ [-2, 2]

$\Rightarrow$ **x $\in$ [-2, 3)**

Functional Equations

Q.1 **$2f(x) + f(1-x) = x^2$; f (4) = ??**

A. $2f(x) + f(1 - x) = x^2$ ...(i)

In functional equations, if we replace 'x' by some 'y' in L.H.S, then we also must replace 'x' by 'y' in R.H.S.
Applying this concept here, replace 'x' by '1-x'.
We get,

$2f(1-x) + f(1-(1-x)) = (1-x)^2$

$2f(1-x) + f(x) = 1-2x+x^2$ ...(ii)

Multiplying 1st equation by 2, we get

$4f(x) + 2f(1-x) = 2x^2$ ...(iii)

subtracting (ii) from (iii), we get

$3f(x) = x^2 + 2x - 1$

$\Rightarrow \quad f(x) = \dfrac{x^2+2x-1}{3} \quad \Rightarrow \quad f(4) = \dfrac{23}{3}$

Q.2 $f(x) + 3x\, f\left(\dfrac{1}{x}\right) = 2(x+1)$ $\forall$ **x > 0. Find f(10099).**

A. Replace x by $\dfrac{1}{x}$

$\Rightarrow \left\{f\left(\dfrac{1}{x}\right) + \dfrac{3}{x} f(x) = 2\left(\dfrac{1}{x}+1\right)\right\} \times (3x)$ and subtract the first equation.

$\Rightarrow$ 9f(x)-f(x)=6+6x – 2(x+1)

$\Rightarrow \quad f(x)= \dfrac{x+1}{2} \Rightarrow f(10099) = \mathbf{5050}$

Q.3 **Find natural number values of a for which** $\sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$

f(x) satisfies f(x + y) = f(x) f(y) $\forall$ x, y $\in$ N; f(1) = 2

A. $x = y = 1 \Rightarrow f(2) = f(1) \cdot f(1) = 2 \cdot 2 = 2^2$

$x = 2, y = 1 \Rightarrow f(3) = f(2) \cdot f(1) = 2^2 \cdot 2 = 2^3$

$x = 3, y = 1 \Rightarrow f(4) = f(3) \cdot f(1) = 2^3 \cdot 2 = 2^4$

By observation, $f(n) = 2^n$

$\sum_{k=1}^{n} 2^{a+k} = 16(2^n - 1) \qquad \Rightarrow \quad 2^a \sum_{k=1}^{n} 2^k = 16(2^n - 1)$

$\Rightarrow \quad 2^a.2(2^n - 1) = 16(2^n - 1) \qquad \Rightarrow \quad 2^{a+1} = 2^4$

$\Rightarrow \quad a = \mathbf{3}$

Q.4 $f(x)=\dfrac{a^x}{a^x+\sqrt{a}}, a>0$. Find $\displaystyle\sum_{r=1}^{2n-1} f\left(\frac{r}{2n}\right)$.

A. **(i)** $f\left(\dfrac{1}{2n}\right)+f\left(\dfrac{2}{2n}\right)+f\left(\dfrac{3}{2n}\right)+\ldots+f\left(\dfrac{2n-1}{2n}\right)=S$

(ii) $\underbrace{f\left(\dfrac{2n-1}{2n}\right)}_{1-\frac{1}{2n}}+\underbrace{f\left(\dfrac{2n-2}{2n}\right)}_{1-\frac{2}{2n}}+\ldots+f\left(\dfrac{1}{2n}\right)=S$

$$f(\alpha)+f(1-\alpha)=\frac{a^\alpha}{a^\alpha+\sqrt{a}}+\frac{a^{1-\alpha}}{a^{1-\alpha}+\sqrt{a}}$$

$$=\frac{a^\alpha}{a^\alpha+\sqrt{a}}+\frac{a}{a+\sqrt{a}\,a^\alpha}$$

$$=\frac{a^\alpha}{a^\alpha+\sqrt{a}}+\frac{\sqrt{a}}{\sqrt{a}+a^\alpha}=1$$

So, sum of terms equidistant from beginning and end is 1.

$\therefore$ $1+1+1+\ldots 1(2n-1 \text{ times})=2S$

$\therefore$ $S=\dfrac{2\mathbf{n}-1}{2}$

Q.5 f(x) is a polynomial of degree 6 and leading coefficient 4.
$f(1)=6, f(2)=5, f(3)=4, f(4)=3, f(5)=2, f(6)=1$. Find f(7).

A. $T_1=6, T_2=5, T_3=4\ldots., T_6=1$

From observation = $f(x)=7-x$ for $x=1, 2, 3, 4, 5$ and 6

or $f(x)-(7-x)=0\ \forall\ x\in\{1, 2, 3, 4, 5, 6\}$

$\Rightarrow$ $f(x)-7+x=4(x-1)(x-2)(x-3)(x-4)(x-5)(x-6)$

$\Rightarrow$ $f(x)=4(x-1)(x-2)(x-3)(x-4)(x-5)(x-6)+7-x$

$\Rightarrow$ $f(7)=4\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1+7-7=\mathbf{2880}$

Inverse of function for bijective function

Let $f: A\to B$ be a one-one and onto function, then there exists a unique function $g: B\to A$ such that $f(x)=y \Leftrightarrow g(y)=x,\ \forall\ x\in A$ and $y\in B$.

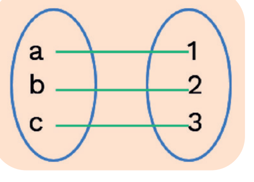
Then g is said to be inverse of f.

Thus $\mathbf{g=f^{-1}: B\to A\ \{(f(x), x)\ |(x, f(x))\in f|\}}$

$g=f^{-1}\neq\dfrac{1}{f}$

$f\equiv\{(a, 1), (b, 2), (c, 3)\}$

$\Rightarrow$ $g\equiv\{(1, a), (2, b), (3, c)\}$



Know the facts

Condition for a function to have a inverse:

(1) f must be bijective

(2) g is also bijective

Q. Find the inverse of the following:

(i) $y = 2x$ **(ii) $y = x - 1$** **(iii) $y = 4-x$**

(iv) $y = \frac{x}{4}$ **(v) $y = \frac{1}{x}$**

A. **(i)** $y = 2x$; $x \to y$ and $y \to x$

$\Rightarrow x = 2y$

$\Rightarrow y = \frac{x}{2}$

$\Rightarrow \mathbf{f^{-1}(x) = \frac{x}{2}}$

(ii) $y = x - 1$

$\Rightarrow x = y - 1 \Rightarrow y = x + 1 \Rightarrow f^{-1}(x) = \mathbf{x + 1}$

(iii) $y = 4-x$

$\Rightarrow x = 4-y \Rightarrow y = 4-x \Rightarrow f^{-1}(x) = \mathbf{4 - x}$

(iv) $y = \frac{x}{4}$

$\Rightarrow x = \frac{y}{4} \Rightarrow y = 4x \Rightarrow f^{-1}(x) = \mathbf{4x}$

(v) $y = \frac{1}{x}$

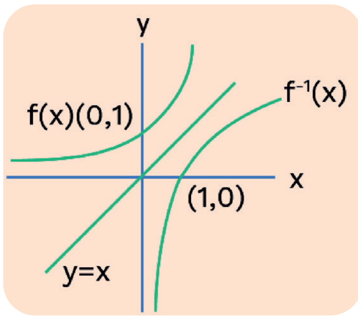
$\Rightarrow x = \frac{1}{y} \Rightarrow y = \frac{1}{x} \Rightarrow f^{-1}(x) = \frac{1}{\mathbf{x}} = f(x)$

Point to Remember!!!

Function given as $y = f(x)$, try to separate all terms of y from x and make the form $x = g(y)$.

Point to Remember!!!

f and f^{-1} are mirror image of each other about line $y = x$



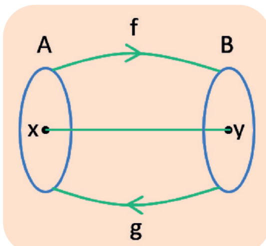
Ex. $y = f(x) = e^x$

$f^{-1}(x) = \ln x$

Graph can be understood by concept that (b, a) is image of (a, b) about line $y = x$

Know the facts

Since, image is unique, inverse of a function is unique.



Note:

If $y = f(x)$ and $g(y) = x$

$\Rightarrow$ $g(f(x)) = x$ and $f(g(y)) = y$

$\Rightarrow$ $f^{-1}(f(x)) = x$

$\Rightarrow$ If $f(g(x)) = x$

$\Rightarrow$ f and g are inverse of each other.

Q.1 Compute the inverse: f: R → R^+, $f(x) = 10^{x+1}$

A.

$y = 10^{x+1}$

$x = 10^{y+1}$

$\Rightarrow \log_{10} x = y + 1$

$\Rightarrow \log_{10} x - 1 = y$

$\Rightarrow f^{-1}(x) = \log_{10} \mathbf{x} - 1$

Q.2 Compute the inverse: f: (-2, ∞) → R, f(x) = 1 + ln (x +2)

A.

$y = 1 + \ln(x + 2)$

$\Rightarrow x = 1 + \ln(y + 2)$

$\Rightarrow x - 1 = \ln(y + 2)$

$\Rightarrow y + 2 = e^{x-1}$

$\Rightarrow y = f^{-1}(x) = \mathbf{e^{x-1} - 2}$

Q.3 Compute the inverse: f: R → (0, 1), $f(x) = \dfrac{2^x}{1+2^x}$

A.

$$y = \frac{2^x}{1+2^x}$$

$$\Rightarrow x = \frac{2^y}{1+2^y} = 1 - \frac{1}{1+2^y}$$

$$\Rightarrow \frac{1}{1+2^y} = 1 - x$$

$$\Rightarrow 1 + 2^y = \frac{1}{1-x}$$

$$\Rightarrow 2^y = \frac{1}{1-x} - 1 = \frac{x}{1-x}$$

$$\Rightarrow y = f^{-1}(x) = \mathbf{\log_2}\left(\frac{\mathbf{x}}{1-\mathbf{x}}\right)$$

Q.4 Compute the inverse: $\ln\left(x + \sqrt{x^2 + 1}\right)$

A.

$$y = \ln\left(x + \sqrt{x^2 + 1}\right)$$

$$\Rightarrow x = \ln\left(y + \sqrt{y^2 + 1}\right)$$

$$\Rightarrow y + \sqrt{y^2 + 1} = e^x$$

$$\Rightarrow \sqrt{y^2+1} = e^x - y$$

$$\Rightarrow y^2 + 1 = y^2 + e^{2x} - 2y\,e^x$$

$$\Rightarrow y = \frac{e^{2x}-1}{2e^x} = \mathbf{f^{-1}(x)}$$

Q.5 If f: R → R ; f(x) = x³ + (a+2)x² + 3ax + 5 is an invertible mapping. Find 'a'.

A. f is one-one and onto

$\Rightarrow$ $f'(x) = 3x^2 + 2(a+2)x + 3a \geq 0$ or ≤ 0 always (Here $f'(x) \leq 0$ is not possible)

$\Rightarrow$ $3x^2 + 2(a+2)\,x + 3a \geq 0$

$\Rightarrow$ $D \leq 0$

$\Rightarrow$ $4(a+2)^2 - 4 \times 9a \leq 0$

$\Rightarrow$ $a^2 - 5a + 4 \leq 0$

$\Rightarrow$ $\mathbf{a \in [1, 4]}$

Q.6 f: [0, ∞) → [1, ∞); f(x) = $\frac{e^x + e^{-x}}{2}$. Find $f^{-1}(x)$.

A.

$$x = \frac{e^y + e^{-y}}{2}$$

$$2xe^y = e^{2y} + 1$$

$$e^{2y} - 2xe^y + 1 = 0$$

$$\Rightarrow e^y = \frac{2x \pm \sqrt{4x^2-4}}{2}$$

$$\Rightarrow y = \ln\left(x \pm \sqrt{x^2-1}\right)$$

Range of f^{-1} will be $[0, \infty)$

$$\therefore f^{-1}(x) = \ln\left(\mathbf{x} + \sqrt{\mathbf{x}^2 - 1}\right)$$

Q.7 $\mathbf{f(x)} = \begin{cases} x & \text{if} \quad x < 1 \\ x^2 & \text{if} \quad 1 \leq x \leq 4 \\ 8\sqrt{x} & \text{if} \quad x > 4 \end{cases}$ **. Find $f^{-1}(x)$**

A. When

$$\left.\begin{array}{c} x < 1,\ f(x) \in (-\infty, 1) \\ 1 \leq x \leq 4,\quad f(x) \in [1, 16] \\ x > 4, f(x) \in (16, \infty) \end{array}\right\} \text{Range}$$

Range become domain for $f^{-1}(x)$

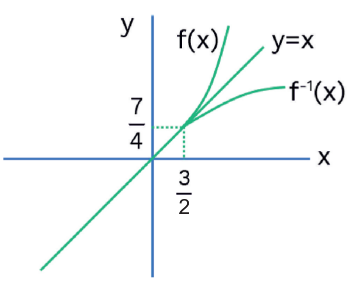
So, $f^{-1}(x) = \begin{cases} x, & x \in (-\infty, 1) \\ \sqrt{x}, & x \in [1, 16] \\ \dfrac{x^2}{64}, & x \in (16, \infty) \end{cases}$

Q.8 **A function** $f : \left[\frac{3}{2}, \infty\right) \to \left[\frac{7}{4}, \infty\right)$ **defined as** $f(x) = x^2 - 3x + 4$**.**

Solve the equation $f(x) = f^{-1}(x)$**.**

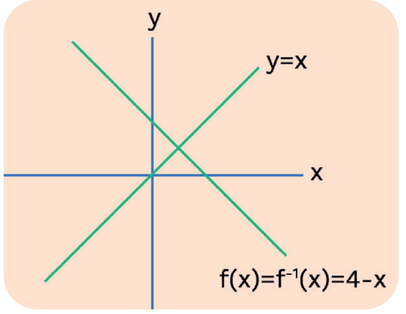
A. Solution of $f(x) = x$ will also be solution of $f(x) = f^{-1}(x)$.
But there may be extra solution.
So, always check the graph before solving $f(x)=x$
Now, since there is only 1 solution on $y = x$ (by graph)

$f(x) = x \Rightarrow x^2 - 3x + 4 = x$
$\Rightarrow (x - 2)^2 = 0$
$\Rightarrow x = \mathbf{2}$



Ex.

$y = f(x) = 4 - x$
$f^{-1}(x) = 4 - x$
$\Rightarrow f(x) = f^{-1}(x)$ have infinite solution here.
While $f(x) = x$ has only 1 solution.
So, always check the graph.



Properties of inverse function

(i) The inverse of a bijection is unique.
(ii) Domain Range interchange
(iii) Inverse of bijection is also a bijection
(iv) $(gof)^{-1} = f^{-1}(g^{-1}(x))$. (Reversal law)

Definition of Homogeneous Functions

A function is said to be homogeneous with respect to any set of variable when each of its terms is of the same degree with respect to those variables.

$f(tx, ty) = t^n f(x, y)$ then $f(x, y)$ is homogeneous function of degree n.

Ex. $f(x,y) = ax^2 + 2hxy + by^2$

$f(tx, ty) = a(tx)^2 + 2h\,(tx)\,(ty) + b(ty)^2$

$= t^2\,(ax^2 + 2hxy + by^2)$

$\Rightarrow$ **Homogeneous equation of degree 2.**

Q. Find if $f(x,y) = \frac{x}{y}\ln\left(\frac{x}{y}\right)$ is homogeneous or not?

A. $f(tx,\ ty) = \frac{tx}{ty}\ln\left(\frac{tx}{ty}\right)$

$= t^0\,\frac{x}{y}\ln\left(\frac{x}{y}\right) = t^0 f(x,y)$

$\Rightarrow$ **Homogeneous function of degree 0.**

Definition of Implicit / Explicit function

A function defined by an equation not solved for the dependent variable (y) is called an **Implicit function**.

If y has been expressed in terms of x alone then it is called an **Explicit function**.

Ex. $y = e^x + \sin x + \tan x + x^2 \rightarrow$ **Explicit**

Ex. $2xy + x^2 + \sin y + \frac{\tan y}{x^2+1} = 0 \rightarrow$ **Implicit**

(It is not solved for y)

Q. $x = y^2 - 2y$. Find domain of expicit form.

A. $y^2 - 2y = x$

$\Rightarrow y^2 - 2y + 1 = 1 + x$

$\Rightarrow (y - 1)^2 = 1 + x$

$\Rightarrow y - 1 = \pm\sqrt{1+x}$

$\Rightarrow y = 1 \pm \sqrt{1+x}$ (2 branches $\Rightarrow$ 2 functions)

$\Rightarrow 1 + x \geq 0 \Rightarrow x \geq -1$

$\Rightarrow$ Domain: $\mathbf{x \in [-1, \infty)}$

Note:

$y^2 = x$ is not a function

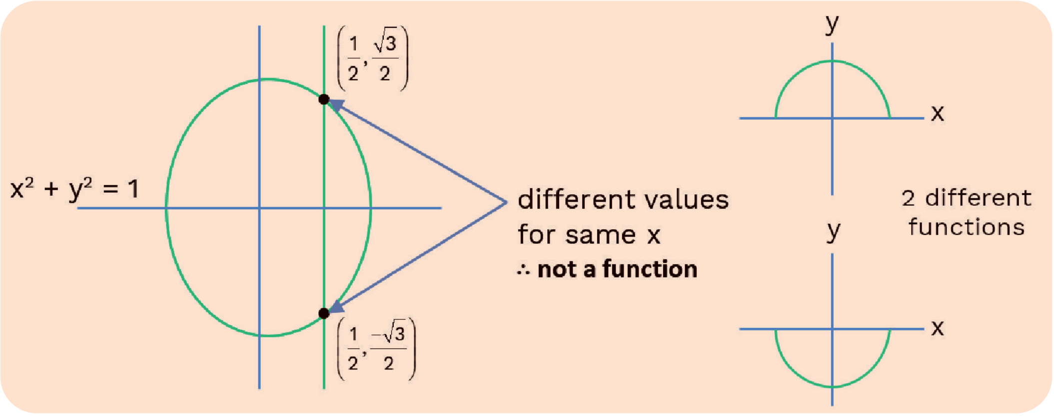
$\Rightarrow \left.\begin{matrix} y = x \\ y = -x \end{matrix}\right\}$ it represents 2 separate function.

Same is the case for $x^2 + y^2 = 1$

$$y^2 = 1 - x^2$$

$$\Rightarrow \quad y = \sqrt{1-x^2}$$

$$= -\sqrt{1-x^2}$$



Definition of Bounded Function

A function is said to be bounded if $|f(x)| \leq M$, where M is a finite quantity.

Ex.

(i) $y = |\sin x| \leq 1 \rightarrow$ Bounded

(ii) $y = |[x]| \rightarrow$ No maximum or minimum value. So unbounded.

(iii) $y = e^x \rightarrow$ unbounded

(iv) $y = \ln x \rightarrow$ unbounded

(v) $y = \{x\} \rightarrow$ bounded

(vi) $y = \sec x \rightarrow$ unbounded.

(vii) $y = x^3 - 10x^2 + 3x + 1 = f(x)$, f; $[-100, 26] \rightarrow R$

$\rightarrow$ bounded

(Since domain is finite, range will also be finite. So, it is bounded)

Know the facts

If graph of function can be bounded between 2 horizontal lines, then function is bounded.

If range of function contain ∞ or -∞ $\Rightarrow$ function is not bounded.

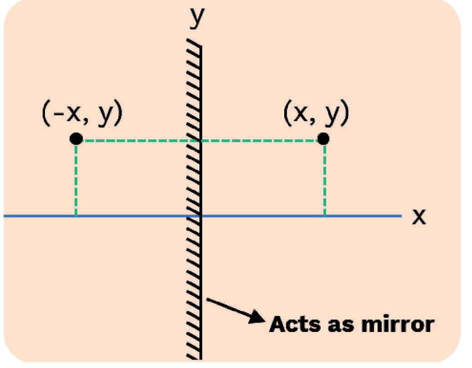
(viii) $y = \text{sgn}x \rightarrow$ bounded
as range is finite {-1, 0, 1}

Definition of ODD and EVEN Functions

A function f(x) defined on the symmetric interval (–a, a)

If f(-x) = f(x) for all x in the domain of 'f' then f is said to be an even function.

If f(-x) = -f(x) for all x in the domain of 'f' then f is said to be an odd function.

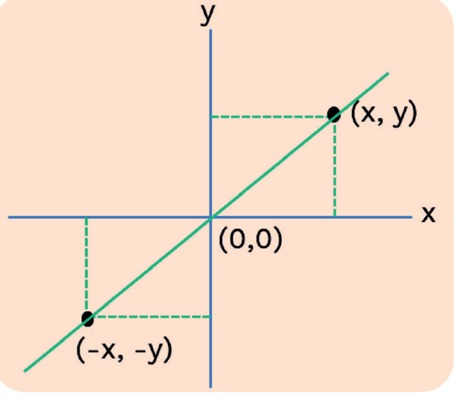


Geometrical interpretation of odd and even function

(i) $y = f(x) \Rightarrow (x, y)$

For even function, $f(-x) = y = f(x)$

$\Rightarrow$ **Graph will be symmetric about y-axis**.



(ii) For odd function $f(-x) = -f(x)$

We can see that, for odd function,
Graph is symmetric about origin.

Ex. Common odd functions: f = sinx, tanx, x, x^3
Ex. Common even functions: f = cos x, $\sin^2 x$, x^2, |x|, |sinx|

Ex.
(i) $f(x) = x^2 + x \Rightarrow f(-x) = x^2 - x \neq f(x)$ or $-f(x)$
$\therefore$ $x^2 + x$ is neither odd nor even.

(ii) (a) Let $h(x) = \dfrac{f(x)+f(-x)}{2}$

$h(-x) = \dfrac{f(-x)+f(x)}{2} = h(x)$ so,

it is even function.

Know the facts

(i) A function may neither be odd nor even. **eg.** $x^2 + x$, sinx + cosx.
(ii) Every function can be expressed as the sum of an even and an odd function.

$$\mathbf{f(x)} = \underbrace{\frac{\mathbf{f(x)+f(-x)}}{2}}_{\text{even}} + \underbrace{\frac{\mathbf{f(x)-f(-x)}}{2}}_{\text{odd}}$$

(iii) The only function which is defined on the entire number line and is even and odd at the same time is f(x) = 0.

(b) Let $h(x) = f(x) - f(-x)$

$h(-x) = f(-x) - f(x) = -h(x)$.

So, it is odd function.

Sum of both of above functions is f(x). Hence f(x) can be divided into sum of even and odd function.

Q.1 Express e^x as sum of an odd and even function

A.

$$e^x = \underbrace{\left(\frac{e^x + e^{-x}}{2}\right)}_{\text{even function}} + \underbrace{\left(\frac{e^x - e^{-x}}{2}\right)}_{\text{odd function}}$$

Q.2 Identify given below functions as odd, even or neither odd nor even.

(i) $f(x) = \ln\left(\frac{1-x}{1+x}\right) = \ln(1-x) - \ln(1+x)$

A. $f(-x) = \ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = -f(x)$

So, it is **odd** function.

Q. **(ii)** $f(x) = \ln\left(x + \sqrt{1+x^2}\right)$

A. $f(-x) = \ln(-x + \sqrt{1+x^2})$

$f(x) + f(-x) = \ln((1+x^2) - x^2) = \ln 1 = 0$

$\Rightarrow$ $f(-x) = -f(x)$, so f(x) is **odd** function.

Q. **(iii)** $f(x) = \sqrt{1+x+x^2} - \sqrt{1-x+x^2}$

A. $f(-x) = \sqrt{1-x+x^2} - \sqrt{1+x+x^2} = -f(x) \rightarrow$ **odd**

Q. **(iv)** $f(x) = x\dfrac{2^x+1}{2^x-1}$

A.

$$f(-x) = -x\left(\frac{2^{-x}+1}{2^{-x}-1}\right) = -x\left(\frac{\frac{1+2^x}{2^x}}{\frac{1-2^x}{2^x}}\right)$$

$$= -x\left(\frac{1+2^x}{1-2^x}\right) = x\left(\frac{2^x+1}{2^x-1}\right) = f(x) \rightarrow \textbf{even}$$

Q. **(v)** $f(x) = \dfrac{(1+2^x)^2}{2^x}$

A. $f(-x) = \dfrac{(1+2^{-x})^2}{2^{-x}} = \dfrac{1+2.2^{-x}+2^{-2x}}{2^{-x}}$

$= 2^x + 2 + 2^{-x} = \dfrac{2^{2x}+2.2^x+1}{2^x} = f(x)$

So, it is **even** function.

Q. **(vi)** $f(x) = \dfrac{1+2^x}{1-2^x}$

A. $f(-x) = \dfrac{1+2^{-x}}{1-2^{-x}} = \dfrac{2^x+1}{2^x-1} = -f(x)$ → **odd**

Q. f(x) = ([a]²-5[a] + 4) x³ + (6{a}² – 5{a} +1) x + x tan x is even function.
Find $\sum a_i$ (a_i are values of a for which function is even)
([.] is the greatest integer function and {.} is fractional part function)

A. $f(x) = \underbrace{\left([a]^2-5[a]+4\right)x^3+\left(6\{a\}^2-5\{a\}+1\right)x}_{\text{odd part}} + \underbrace{x\tan x}_{\text{even part}}$

For f(x) to be even, make odd part = 0

∴ $[a]^2 - 5[a] + 4 = 0$

∴ [a] = 1 and 4

$6\{a\}^2 - 5\{a\} + 1 = 0 \Rightarrow \{a\} = \dfrac{1}{2}$ and $\dfrac{1}{3}$

So possible values of a= [a] + {a} are $1+\dfrac{1}{2}, 1+\dfrac{1}{3}, 4+\dfrac{1}{2}, 4+\dfrac{1}{3}$

∴ $\sum a_i = 10 + 1 + \dfrac{2}{3} = 11\dfrac{2}{3} = \dfrac{35}{3}$

Note:

f(x)	g(x)	f(x) +g(x)	f(x)-g(x)	f(x)·g(x)	f(x)/g(x)	(gof)(x)
odd	odd	odd	odd	even	even	odd
even	even	even	even	even	even	even
odd	even	neither odd nor even	neither odd nor even	odd	odd	even
even	odd	neither odd nor even	neither odd nor even	odd	odd	even

Proof:

Let's prove the first:

h(x) = f(x) + g(x), f(x) and g(x) are odd

h(-x)= f(-x) + g(-x)=-f(x)-g(x)=-h(x)

∴ h(x) is odd

Let h(x) = g(f(x))

h(–x) = g(f(-x)) = g(-f(x)) = -g(f(x))

= – h(x) → odd.

Similarly, rest can be proved.

Definition of Periodic Function

A function f(x) is called periodic if there exists a positive number **T** (T>0) called the period of the function such that **f(x + T) = f(x)**, for all values of x within the domain of x.

f(x) = f(x + T) = f(x + 2T) = f(x + nT); n∈ I

Ex.

f(x) = sinx → T_1 = 2π → T can be 2π, 4π, 6π, 8π

g(x) = {x} → T_2 = 1 → T can be 1, 2, 3, 4, 5

∴ No common T. ∴ f(x) + g(x) is **not periodic**.

Ex. (i) y = cos x → Period = **2π**

(ii) y = a+ b cos x → Period = **2π**

Know the facts

(a) f(T) = f(0) = f(–T), where T is the period.

(b) Inverse of a periodic function does not exist.

(c) Every constant function is always periodic, with no fundamental period.

(d) If f(x) has a period T and g(x) also has a period T then it does not mean that f(x) + g(x) must have a period T. **e.g.,** f(x) = |sin x| + |cos x|.

(e) If f(x) is periodic and g(x) is also periodic then it does not mean that f(x) + g(x) must be periodic.

(f) If f(x) has a period p, then $\frac{1}{f(x)}$, and $\sqrt{f(x)}$ also has a period p

(g) If f(x) has a period T then f(ax + b) has a period T/|a|.

Point to Remember!!!

Common periodic functions	sinx,	cosx,	tanx,	{x} ,	|sinx|,	|cosx|,	$\sin^2x$
Period →	2π	2π	π	1	π	π	π

Ex. f(x) = sinx. Find period.

Sol. f(x + T) = f(x)

$\Rightarrow$ sin (x + T) – sin(x) = 0

$\Rightarrow 2\sin\frac{T}{2}\cos\left(x+\frac{T}{2}\right) = 0$

$\Rightarrow \sin\frac{T}{2} = 0 \Rightarrow \frac{T}{2} = n\pi$, $n \in I$

$\Rightarrow$ T = 2nπ but T should be smallest positive value

$\Rightarrow$ T = **2π**

Know the facts

LCM of rational and irrational number is not defined.

LCM of rational numbers $\frac{p}{q}, \frac{\ell}{m}$ and $\frac{r}{s} = \frac{\text{LCM of } p, \ell \text{ and } r}{\text{HCF of } q, m \text{ and } s}$

Know the facts

In case of f(ax + b), let period of f(x) is T and f(ax + b) be T′.

f(a(x + T′) + b) = f(ax + b)

$\Rightarrow$ f(ax + b + aT′) = f(ax + b)

$\Rightarrow aT' = T \Rightarrow T' = \frac{T}{a}$.

But since period must be positive, $\mathbf{T'} = \frac{\mathbf{T}}{|\mathbf{a}|}$

Examples on Periodic Functions

Q.1 **Find period of:**

(i) $f(x) = \cos\frac{2x}{3} - \sin\frac{4x}{5}$

A. Period of cosx = 2π

Period of $\cos\left(\frac{2}{3}x\right) = \frac{2\pi}{2/3} = 3\pi$

Similarly, period of $\sin\left(\frac{4}{5}x\right) = \frac{2\pi}{4/5} = \frac{5\pi}{2}$

LCM of 3π and $\frac{5\pi}{2}$ is 15π

So, period is **15π**.

Q. **(ii)** f(x) = cos (sinx)

A. Period of sinx = 2π

But f(x) is a composite in trigonometric function.

So, period may be $\frac{T}{2}, \frac{T}{4}, \frac{T}{8}$ .i.e., also.

So, we must check.

$\frac{T}{2} = \pi \Rightarrow f(x+\pi) = \cos(\sin(x+\pi)) = \cos(-\sin x) = \cos(\sin x)$

$\therefore$ π can be period.

Now check for $\frac{T}{4} = \frac{\pi}{2}$

$\cos\left(\sin\left(x+\frac{\pi}{2}\right)\right) = \cos(\cos x) \neq f(x)$

so, $\frac{\pi}{2}$ is not the period.

$\therefore$ $\boldsymbol{\pi}$ is the period.

Q. **(iii)** f(x) = sin (cosx)

A. **Sol.** Period of cosx = 2π = T

now check for $\frac{T}{2} = \pi$

$f(x+\pi) = \sin(\cos(x+\pi)) = \sin(-\cos x) = -\sin(\cos x) \neq f(x)$

$\therefore$ π is not the period.

So, T = $\boldsymbol{2\pi}$

Q. **(iv)** $f(x) = \sin^4 x + \cos^4 x$

A. $f(x) = 1 - \frac{1}{2}(\sin 2x)^2$

$f(x) = 1 - \frac{1}{2}\left(\frac{1-\cos 4x}{2}\right)$

Period of f(x) = period of cos (4x) = $\frac{2\pi}{4} = \frac{\pi}{2}$

Q. **(v)** If period of $\sin(\pi kx)$ is 2, find k.

A. Period of sinx = 2π

$\therefore$ Period of $\sin(\pi kx) = \dfrac{2\pi}{|\pi k|} = 2$

$\Rightarrow$ $|k| = 1$

$\Rightarrow$ k = **1, -1**

Q. **(vi)** f(x) = x − [x]

A. f(x) = {x}

$\therefore$ Period = **1**

Q.2 f(x) = sinx + cos ax is a periodic function. Then prove that 'a' must be rational.

A. LCM of two numbers will only exist if either both numbers are rational, or they are same type of irrational.

Period of sin x = 2π

Period of cos x = $\dfrac{2\pi}{a}$

LCM of 2π and $\dfrac{2\pi}{a}$ is possible only when 'a' is rational.

Note:

(i) f(x) = cos $\sqrt{x}$; and sinx + {x} are aperiodic.

(ii) f(x) = xsinx is aperiodic

(iii) f(x) = sin (x + sinx) is periodic

Proof: $x \to x + 2\pi$

$f(x + 2\pi) = \sin((x + 2\pi) + \sin(x + 2\pi))$

$= \sin(2\pi + x + \sin x) = \sin(x + \sin x) = f(x)$

$\therefore$ f(x) is periodic with period 2π.

Q.3 **Find period of:**

(i) f(x) = {x} + {2x} + {3x}

A. $T_1 = 1$ $T_2 = \dfrac{1}{2}$ $T_3 = \dfrac{1}{3}$

T = LCM of 1, $\dfrac{1}{2}$ and $\dfrac{1}{3}$

T = **1**

Q. **(ii)** $f(x) = \{x\} + \left\{\frac{x}{2}\right\} + \left\{\frac{x}{3}\right\}$

A. $T_1 = 1$; $T_2 = 2$; $T_3 = 3$

T = LCM of 1, 2 and 3

T = **6**

Q. **(iii)** $f(x) = [x] + [2x] + [3x] + ...+[nx] - \frac{(n)(n+1)}{2}x$

A. $= [x] + [2x] + [3x] + ...+ [nx] - (x + 2x + 3x + ...+nx)$

$= -((x-[x]) + (2x-[2x]) + (3x-[3x]) +...+ (nx-[nx]))$

$= -(\{x\} + \{2x\} + \{3x\} +...\{nx\})$

Period of these small functions are $1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}$

Period of their sum = LCM of $1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}$ = **1.**

Q.4 f(x) + f(1 + x) = 0. Find T.

A. By functional Rule, $x \to x + 1$

$\Rightarrow$ f(1 + x) + f(2 + x) = 0

Subtracting given equation

f(2 + x) – f(x) = 0 $\Rightarrow$ f(x + 2) = f(x) $\Rightarrow$ **T = 2**

Q.5 f(x + 2) + f(x – 2) = f(x) $\forall$ x $\in$ D_f. Find T.

A. $x \to x + 2$

$\Rightarrow$ f (x + 4) + f(x) = f(x + 2)

Add both equations.

$\Rightarrow$ f (x + 4) + f(x – 2) = 0

$\Rightarrow$ $x \to x + 2$

$\Rightarrow$ f(x + 6) = –f(x)

Now $x \to x + 6$

$\Rightarrow$ f(x + 12) = -f(x+6)

$\Rightarrow$ f(x+12) = – (–f(x)) (from previous equation)

$\Rightarrow$ f(x) = f(x+12)

$\Rightarrow$ **T = 12**

Q.6 $f(x+\lambda) = 1 + \sqrt{2f - f^2}$ $\forall$ x $\in$ R. Prove that f is periodic.

A. $(f(x + \lambda)-1)^2 = 2f - f^2 = 1 - (f(x) - 1)^2$

$x \to x + \lambda$

$(f(x+2\lambda)-1)^2 = 1- (f(x + \lambda)-1)^2 = (f(x)-1)^2$

$\Rightarrow$ $f(x + 2\lambda) - 1 = f(x) - 1$ or $-(f(x)-1)$

In 1^{st} case, $f(x+2\lambda) = f(x)$

$\Rightarrow$ f is periodic with $T = 2\lambda$

In 2^{nd} case, $f(x + 2\lambda) + f(x) = 2$

But from original equation $f(x) \geq 1$

$\therefore$ above equation can only be true, if $f(x) = 1$ always but $f(x) = 1$ does not satisfy the original equation.

so, second case is rejected.

$\therefore$ $f(x+2\lambda) = f(x)$ and $\mathbf{T = 2\lambda}$

Q.7 $f(x) = x\ \forall\ x \in [0,2]$ and f(x) is even with period = 4.
Find f(5), f(7.1), f(-1), f(-7), f(2019).

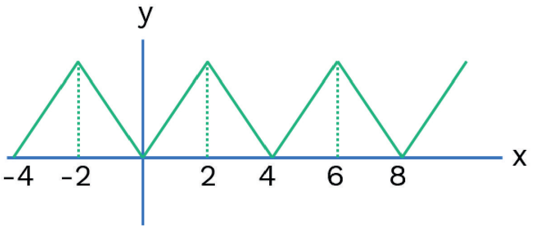
A. $f(x + 4) = f(x)$

$f(5) = f(1) = \mathbf{1}$

$f(7.1) = f(3.1) = f(-0.9) = f(0.9) = \mathbf{0.9}$

$f(-1) = f(1) = \mathbf{1}$; $f(-7) = f(-3) = f(1) = \mathbf{1}$

$f(2019) = f(-1 + 505 \times 4) = f(-1) = \mathbf{1}$



Q.8 $f: R \to R$, $f(x) = x^3 - 2x^2 + 5x + 3$ is

(A) one one, onto (B) one one, into
(C) many one, into (D) many one, onto

A. $f'(x) = 3x^2 - 4x + 5$

$a > 0$, $D < 0$

$\Rightarrow$ $f'(x) > 0\ \forall\ x \in R \Rightarrow f(x)$ is one-one.

Since f(x) is odd degree polynomial, its range is R which is codomain.

$\Rightarrow$ f is also onto. so **(A)** is correct.

Q.9 $f : R \to R$, $f(x) = 2x^3 - 6x^2 - 18x + 17$ is:

(A) one-one, onto (B) one-one, into
(C) many-one, into (D) many-one, onto

A. $f'x = 6x^2 - 12x - 18 = 6(x^2 - 2x - 3)$

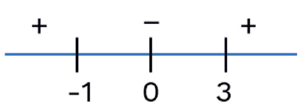
$f'(x) = 0$ for $x = -1$ and 3.

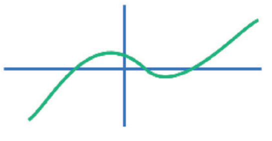
graph will be like:

so, f is many one.

Range is R. So, f is onto.

$\Rightarrow$ **(D)** is correct answer.





Q.10 f: $[3, \infty) \to [a, \infty)$, $f(x) = 2x^3 - 6x^2 - 18x + 80$ is an onto function, find a.

A. $f'(x) = 6x^2 - 12x - 18 = 6(x^2 - 2x - 3) = 6(x-3)(x+1) > 0 \ \forall \ x > 3$

$\therefore$ f_{min} at x = 3. So a = f(3)

$\Rightarrow$ $a = 2 \times 3^3 - 6 \times 3^2 - 18 \times 3 + 80$

$\Rightarrow$ a = **26**.

Q.11 $f(x) = x^2 + bx + 3$ is not injective for $x \in [0, 1]$, then the set of b is:

(A) $(0, \infty)$ (B) $(-2, 0)$ (C) $(0, 2)$ (D) $(2, \infty)$

A. f is upward facing parabola. It will only be true if minima lies in interval (0, 1)

x-coordinate of minima = $\frac{-b}{2}$

$\Rightarrow$ $0 < \frac{-b}{2} < 1 \Rightarrow \mathbf{b} \in \mathbf{(-2, 0)}$