Determinants

DISCLAIMER

Determinants

Introduction

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$\Rightarrow$ Number of rows = Number of columns (i.e., defined only corresponding to square matrices)

$\Rightarrow$ The number of rows (or columns) of a determinant is called order of determinant

$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$ is a determinant of order 2 or we can say that it is a second order determinant

Number of rows = 2

Number of columns = 2

Note:

Determinant is a special case of matrix

Expansion of a Determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_3b_2c_1 - b_3c_2a_1 - c_3a_2b_1$$

Minors and Cofactors

Minors:

Minor M_{ij} corresponding to element a_{ij} of determinant $|[a_{ij}]|$ is the determinant corresponding to submatrix obtained by eliminating i^{th} row and j^{th} column.

Note:

Minor of an element of a determinant of order n ($n \geq 2$) is a determinant of order $n - 1$

Cofactor:

Cofactor A_{ij} corresponding to element a_{ij} is defined as

$A_{ij} = (-1)^{i+j} M_{ij}$

where M_{ij} is minor of a_{ij}

Note:

Minors and cofactors of the elements a_{11}, a_{21} in the determinant

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Minor of a_{11}

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$= a_{22}a_{33} - a_{23}a_{32}$

Cofactor of a_{11}

$A_{11} = (-1)^{1+1}M_{11} = a_{22}a_{33} - a_{23}a_{32}$

Minor of a_{21}

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$= a_{12}a_{33} - a_{13}a_{32}$

Cofactor of a_{21}

$A_{21} = (-1)^{2+1}M_{21}$

$= - a_{12}a_{33} + a_{13}a_{32}$

Value of determinant:

Can be evaluated by adding the products of elements of any one row (or column) with their corresponding cofactors hence

$\Delta = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$ (about R_1)

$\Delta = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$ (about C_1)

$\Delta = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32}$ (about C_2)

Q. **Find the minors and cofactors of all the elements of the determinants** $\begin{vmatrix} 1 & -2 \\ 4 & 3 \end{vmatrix}$

Sol. for element "1" $\Rightarrow$ $M_{11} = 3$

$A_{11} = (-1)^{1+1}.3 = 3$

for element "–2" $\Rightarrow$ $M_{12} = 4$

$A_{12} = (-1)^{1+2}.4 = -4$

for element "4" $\Rightarrow$ $M_{21} = -2$

$A_{21} = (-1)^{2+1}.(-2) = 2$

for element "3" $\Rightarrow$ $M_{22} = 1$

$A_{22} = (-1)^{2+2}.1 = 1$

Q. **Evaluate** $\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix}$

Sol. $2(2) - 4(-1)$

$= 4 + 4 = 8$

Q. **Evaluate** $\begin{vmatrix} x & x+1 \\ x-1 & x \end{vmatrix}$

Sol. $= x(x) - (x - 1)(x + 1)$

$= x^2 - (x^2 - 1) = 1$

Q. **Evaluate** $\Delta = \begin{vmatrix} 0 & \sin\alpha & -\cos\alpha \\ -\sin\alpha & 0 & \sin\beta \\ \cos\alpha & -\sin\beta & 0 \end{vmatrix}$

Sol. Expansion about R_1

$\Delta = 0 \begin{vmatrix} 0 & \sin\beta \\ -\sin\beta & 0 \end{vmatrix} - \sin\alpha \begin{vmatrix} -\sin\alpha & \sin\beta \\ \cos\alpha & 0 \end{vmatrix} - \cos\alpha \begin{vmatrix} -\sin\alpha & 0 \\ \cos\alpha & -\sin\beta \end{vmatrix}$

$\Delta = 0 - \sin\alpha(-\cos\alpha \sin\beta) - \cos\alpha (\sin\alpha \sin\beta)$

$\Delta = \sin\alpha \cos\alpha \sin\beta - \sin\alpha \cos\alpha \sin\beta$

$\Delta = 0$

Q. **Find value of x for which** $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$

Sol. $3 - x^2 = 3 - 8 \Rightarrow x^2 = 8 \Rightarrow x = \pm 2\sqrt{2}$

Q. **Evaluate** $\begin{vmatrix} 2 & 3 & -2 \\ 1 & 2 & 3 \\ -2 & 1 & -3 \end{vmatrix}$ **by expanding it along the second row.**

Sol. $= -1\begin{vmatrix} 3 & -2 \\ 1 & -3 \end{vmatrix} + 2\begin{vmatrix} 2 & -2 \\ -2 & -3 \end{vmatrix} - 3\begin{vmatrix} 2 & 3 \\ -2 & 1 \end{vmatrix}$

$= -1(-9 + 2) + 2(-6 - 4) - 3(2 + 6)$

$= 7 - 20 - 24 = -37$

Q. **Show that the value of the determinant** $\begin{vmatrix} \tan(A+P) & \tan(B+P) & \tan(C+P) \\ \tan(A+Q) & \tan(B+Q) & \tan(C+Q) \\ \tan(A+R) & \tan(B+R) & \tan(C+R) \end{vmatrix}$

vanishes for all values of A, B, C, P, Q & R where A + B + C + P + Q + R = 0

Sol. After expanding the determinant

tan(A + P) tan(B + Q) tan(C + R) + tan(B + P) tan(C + Q)
tan(A + R) + tan(C + P) tan(A + Q) tan(B + R) – tan(A + R)
tan(B + Q) tan(C + P) – tan(B + R) tan(C + Q) tan(A + P)
– tan(C + R) tan(A + Q) tan(B + P) ...(1)

Now $\because \alpha + \beta + \gamma = 0$ then

$\tan\alpha\tan\beta\tan\gamma = \tan\alpha + \tan\beta + \tan\gamma$ hence from (1)

{tan(A + P) + tan(B + Q) + tan(C + R)} + {tan(B + P) +
tan(C + Q) + tan(A + R)} + {tan(C + P) + tan(A + Q) +
tan(B + R)} – {tan(A + R) + tan(B + Q) + tan(C + P)}
– {tan(B + R) + tan(C + Q) + tan(A + P)} – {tan(C + R)
+ tan(A + Q) + tan(B + P)}
= 0 **Hence proved.**

Note:

If elements of a row (or column) are multiplied with cofactors of any other row (or column) then their sum is zero.

Q. **Find minors and cofactors of the elements of the determinant $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$ and verify that $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} = 0$**

Sol. for "2" $M_{11} = \begin{vmatrix} 0 & 4 \\ 5 & -7 \end{vmatrix} = -20$

$A_{11} = M_{11} = -20$

for "–3" $M_{12} = \begin{vmatrix} 6 & 4 \\ 1 & -7 \end{vmatrix} = -46$

$A_{12} = -M_{12} = 46$

for "5" $M_{13} = \begin{vmatrix} 6 & 0 \\ 1 & 5 \end{vmatrix} = 30$

$A_{13} = M_{13} = 30$

for "6" $M_{21} = \begin{vmatrix} -3 & 5 \\ 5 & -7 \end{vmatrix} = -4$

$A_{21} = -M_{21} = 4$

for "0" $M_{22} = \begin{vmatrix} 2 & 5 \\ 1 & -7 \end{vmatrix} = -19$

$A_{22} = M_{22} = -19$

for "4" $M_{23} = \begin{vmatrix} 2 & -3 \\ 1 & 5 \end{vmatrix} = 13$

$A_{23} = -M_{23} = -13$

for "1" $M_{31} = \begin{vmatrix} -3 & 5 \\ 0 & 4 \end{vmatrix} = -12$

$A_{31} = M_{31} = -12$

for "5" $M_{32} = \begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix} = -22$

$A_{32} = -M_{32} = 22$

for "–7" $M_{33} = \begin{vmatrix} 2 & -3 \\ 6 & 0 \end{vmatrix} = 18$

$A_{33} = M_{33} = 18$

now

$a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} = 2(-12) + (-3)(22) + 5(18) = 0$

Properties of Determinants

Property-1

The value of a determinant remains unaltered, if the rows and columns are inter-changed

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = D'$$

Note: (i) D & D' are transpose of each other.

(ii) If D' = – D then it is **Skew Symmetric**

(iii) The value of a skew symmetric determinant of odd order is zero.

Property-2

If any two rows (or columns) of a determinant be interchanged, the value of determinant is changed in sign only.

$$\text{Let } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \;\&\; D' = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Then D' = – D

Property-3

If any two rows (or columns) of a determinant are identical (all corresponding elements are same) then value of determinant is zero.

$$\text{Let } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ then it can be verified that } D = 0$$

Property-4

If each element of a row (or a column) of a determinant is multiplied by a constant k, then its value gets multiplied by k.

$$\text{e.g., If } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } D' = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ then } D' = kD$$

Property-5

If all elements of a row or column of a determinant are expressed as sum of two (or more) terms, then the determinant can be expressed as sum of two (or more) determinants.

$$\begin{vmatrix} a_1 + x & b_1 + y & c_1 + z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Property-6

If, to each element of any row (or column) of a determinant, the equimultiples of corresponding elements of other row (or column) are added, then value of determinant remains the same i.e., the value of determinant remains same if we apply the operation

$R_i \to R_i + kR_j$ or $C_i \to C_i + kC_j$

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and

$D' = \begin{vmatrix} a_1 + ma_2 & b_1 + mb_2 & c_1 + mc_2 \\ a_2 & b_2 & c_2 \\ a_3 + na_1 & b_3 + nb_1 & c_3 + nc_1 \end{vmatrix}$ then D' = D

Note:

While applying this property atleast one row (or column) must remains unchanged

Property-7

(i) If $\Delta_r = \begin{vmatrix} f_1(r) & f_2(r) & f_3(r) \\ a & b & c \\ d & e & f \end{vmatrix}$ where $f_1(r)$, $f_2(r)$, $f_3(r)$ are functions of r and a, b, c, d, e, f are constants. Then $\sum_{r=1}^{n} \Delta_r = \begin{vmatrix} \sum_{r=1}^{n} f_1(r) & \sum_{r=1}^{n} f_2(r) & \sum_{r=1}^{n} f_3(r) \\ a & b & c \\ d & e & f \end{vmatrix}$

(ii) Also for $\Delta(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ a & b & c \\ d & e & f \end{vmatrix}$ where $f_1(x)$, $f_2(x)$, $f_3(x)$ are functions of x and a, b, c, d, e, f are constants. We have

$$\int_p^q \Delta(x)\,dx = \begin{vmatrix} \int_p^q f_1(x)\,dx & \int_p^q f_2(x)\,dx & \int_p^q f_3(x)\,dx \\ a & b & c \\ d & e & f \end{vmatrix}$$

Property-8

If by putting x = a ⇒ D = 0 then (x − a) is a factor of D.

Q. **Without expanding prove that the value of the determinant**

$$D = \begin{vmatrix} 0 & b & -c \\ -b & 0 & a \\ c & -a & 0 \end{vmatrix} = 0$$

Sol. $D' = \begin{vmatrix} 0 & -b & c \\ b & 0 & -a \\ -c & a & 0 \end{vmatrix} = -D$

Hence D is skew symmetric determinant of order 3(odd) hence D = 0

Q. **If** $\begin{vmatrix} 1 & 1 & 1 \\ 1 & \beta & 2 \\ 1 & \beta^2 & 4 \end{vmatrix} = 0$**, find possible values of** β

Sol. $R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$

$$\Rightarrow \begin{vmatrix} 0 & 1-\beta & -1 \\ 0 & \beta(1-\beta) & -2 \\ 1 & \beta^2 & 4 \end{vmatrix} = 0$$

Expand about C_1

$\Rightarrow 0 - 0 + 1\{-2(1 - \beta) + \beta(1 - \beta)\} = 0$

$\Rightarrow (1 - \beta)(-2 + \beta) = 0$

$\Rightarrow \beta = 1, 2$

Q. **Evaluate** Δ **=** $\begin{vmatrix} 3 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 2 & 3 \end{vmatrix}$

Sol. Taking 2 common from C_2 and 3 common from C_3

$$\Delta = 2 \times 3 \times \begin{vmatrix} 3 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 1 \end{vmatrix}$$

Now C_2 and C_3 are identical hence $\Delta = 0$

Q. **Evaluate** $\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$

Sol. $R_1 \to R_1 - 6R_3$

$$\begin{vmatrix} 0 & 0 & 0 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix} = 0$$

Q. **Show that** $\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix} = 0$

Sol. In LHS apply

$R_2 \to R_2 - 2R_3$

now $\begin{vmatrix} a & b & c \\ a & b & c \\ x & y & z \end{vmatrix} = 0$

Special Determinants

(i) Symmetric Determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$$

(ii) Skew symmetric Determinant

$$\begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix} = 0$$

(iii) Cyclic Determinants

(a) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

Proof: In LHS $C_1 \to C_1 - C_2, C_2 \to C_2 - C_3$

$$\begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^2-b^2 & b^2-c^2 & c^2 \end{vmatrix}$$

Now $C_1 \to \dfrac{C_1}{(a-b)}, C_2 \to \dfrac{C_2}{(b-c)}$

$$(a-b)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a+b & b+c & c^2 \end{vmatrix}$$

Expansion about R_1

$\Rightarrow (a-b)(b-c)\{0 - 0 + 1(b + c - a - b)\}$

$\Rightarrow (a-b)(b-c)(c-a)$ Hence proved

(b) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

Proof: In LHS $C_1 \to C_1 - C_2, C_2 \to C_2 - C_3$

$$\begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^3-b^3 & b^3-c^3 & c^3 \end{vmatrix}$$

now $C_1 \to \dfrac{C_1}{(a-b)}, C_2 \to \dfrac{C_2}{(b-c)}$

$$(a-b)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a^2+b^2+ab & b^2+c^2+bc & c^3 \end{vmatrix}$$

Expansion about R_1

$\Rightarrow (a-b)(b-c)\{b^2 + c^2 + bc - a^2 - b^2 - ab\}$

$\Rightarrow (a-b)(b-c)\{(c^2 - a^2) + b(c-a)\}$

$\Rightarrow (a-b)(b-c)(c-a)(c+a+b)$ **Hence proved.**

(c) $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

Proof: In LHS $C_1 \to C_1 - C_2, C_2 \to C_2 - C_3$

$$\begin{vmatrix} 0 & 0 & 1 \\ a^2-b^2 & b^2-c^2 & c^2 \\ a^3-b^3 & b^3-c^3 & c^3 \end{vmatrix}$$

Now $C_1 \to \frac{C_1}{(a-b)}$, $C_2 \to \frac{C_2}{(b-c)}$

$$(a-b)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ a+b & b+c & c^2 \\ a^2+b^2+ab & b^2+c^2+bc & c^3 \end{vmatrix}$$

Expansion about R_1

$\Rightarrow (a-b)(b-c)\{(a+b)(b^2+c^2+bc)-(b+c)(a^2+b^2+ab)\}$

$\Rightarrow (a-b)(b-c)\{ab^2+ac^2+abc+b^3+bc^2+b^2c-ba^2-b^3-ab^2-ca^2$
$-cb^2-abc\}$

$\Rightarrow (a-b)(b-c)\{ac(c-a)+b(c^2-a^2)\}$

$\Rightarrow (a-b)(b-c)(c-a)(ac+bc+ba)$ **Hence proved.**

(d) $\begin{vmatrix} x+a & x+b & x+c \\ x+b & x+c & x+a \\ x+c & x+a & x+b \end{vmatrix} = (3x+a+b+c)(ab+bc+ca-a^2-b^2-c^2)$

Proof: $R_1 \to R_1 + R_2 + R_3$

$\begin{vmatrix} 3x+a+b+c & 3x+a+b+c & 3x+a+b+c \\ x+b & x+c & x+a \\ x+c & x+a & x+b \end{vmatrix}$ now $R_1 \to R_1/(3x+a+b+c)$

$$(3x+a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ x+b & x+c & x+a \\ x+c & x+a & x+b \end{vmatrix}$$

Now $C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$

$$(3x+a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ b-c & c-a & x+a \\ c-a & a-b & x+b \end{vmatrix}$$

Expansion about R_1

$\Rightarrow (3x+a+b+c)\{(a-b)(b-c)-(c-a)^2\}$

$\Rightarrow (3x+a+b+c)(ab+bc+ca-a^2-b^2-c^2)$ **Hence proved.**

(e) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3+b^3+c^3-3abc) \le 0$ where a, b, c are positive

Proof: $C_1 \to C_1 + C_2 + C_3$

$$\Rightarrow \begin{vmatrix} a+b+c & b & c \\ b+c+a & c & a \\ c+a+b & a & b \end{vmatrix} \text{ now } C_1 \to C_1/(a+b+c)$$

$$(a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$$

Expansion about C_1

$\Rightarrow (a + b + c)\{(cb - a^2) - (b^2 - ac) + (ab - c^2)\}$

$\Rightarrow -(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) \le 0$

$\Rightarrow -(a^3 + b^3 + c^3 - 3abc) \le 0$ **Hence proved.**

Q. **Without expanding evaluate the determinant** $\begin{vmatrix} 41 & 1 & 5 \\ 79 & 7 & 9 \\ 29 & 5 & 3 \end{vmatrix}$

Sol. apply $C_1 \to C_1 - (C_2 + 8C_3)$

$$= \begin{vmatrix} 0 & 1 & 5 \\ 0 & 7 & 9 \\ 0 & 5 & 3 \end{vmatrix} = 0$$

Q. **Without expanding show that** $\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix} = 0$

Sol. $R_1 \to aR_1, R_2 \to bR_2, R_3 \to cR_3$

$$\Rightarrow \frac{1}{abc}\begin{vmatrix} ab^2c^2 & abc & ab+ac \\ bc^2a^2 & abc & bc+ab \\ ca^2b^2 & abc & ac+bc \end{vmatrix}$$

$C_1 \to C_1/(abc)$, $C_2 \to C_2/(abc)$

$$\Rightarrow abc\begin{vmatrix} bc & 1 & ab+ac \\ ca & 1 & bc+ab \\ ab & 1 & ac+bc \end{vmatrix}$$

$C_3 \to C_3 + C_1$

$$\Rightarrow abc\begin{vmatrix} bc & 1 & ab+bc+ca \\ ca & 1 & ab+bc+ca \\ ab & 1 & ab+bc+ca \end{vmatrix}$$

$C_3 \to C_3/(ab + bc + ca)$

$$\Rightarrow abc(ab + bc + ca)\begin{vmatrix} bc & 1 & 1 \\ ca & 1 & 1 \\ ab & 1 & 1 \end{vmatrix}$$

= 0 Hence proved.

Q. **Without expanding evaluate the determinant** $\begin{vmatrix} (a^x+a^{-x})^2 & (a^x-a^{-x})^2 & 1 \\ (a^y+a^{-y})^2 & (a^y-a^{-y})^2 & 1 \\ (a^z+a^{-z})^2 & (a^z-a^{-z})^2 & 1 \end{vmatrix}$ **where a, b, c and x, y, z** $\in$ **R**

Sol.

$$\begin{vmatrix} a^{2x}+a^{-2x}+2 & a^{2x}+a^{-2x}-2 & 1 \\ a^{2y}+a^{-2y}+2 & a^{2y}+a^{-2y}-2 & 1 \\ a^{2z}+a^{-2z}+2 & a^{2z}+a^{-2z}-2 & 1 \end{vmatrix}$$

$C_1 \to C_1 - C_2$

$$\begin{vmatrix} 4 & a^{2x}+a^{-2x}-2 & 1 \\ 4 & a^{2y}+a^{-2y}-2 & 1 \\ 4 & a^{2z}+a^{-2z}-2 & 1 \end{vmatrix}$$

$C_1 \to C_1/4$

$$4\begin{vmatrix} 1 & a^{2x}+a^{-2x}-2 & 1 \\ 1 & a^{2y}+a^{-2y}-2 & 1 \\ 1 & a^{2z}+a^{-2z}-2 & 1 \end{vmatrix} = 0$$

Q. **Prove that** $\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix}$ **= (α − β) (β − γ) (γ − α) (α + β + γ)**

Sol. $C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$

$$\begin{vmatrix} \alpha-\beta & \beta-\gamma & \gamma \\ \alpha^2-\beta^2 & \beta^2-\gamma^2 & \gamma^2 \\ \beta-\alpha & \gamma-\beta & \alpha+\beta \end{vmatrix}$$

$C_1 \to C_1/(\alpha - \beta)$, $C_2 \to C_2/(\beta - \gamma)$

$$\Rightarrow (\alpha-\beta)(\beta-\gamma)\begin{vmatrix} 1 & 1 & \gamma \\ \alpha+\beta & \beta+\gamma & \gamma^2 \\ -1 & -1 & \alpha+\beta \end{vmatrix}$$

$R_1 \to R_1 + R_3$

$$\Rightarrow (\alpha-\beta)(\beta-\gamma)\begin{vmatrix} 0 & 0 & \alpha+\beta+\gamma \\ \alpha+\beta & \beta+\gamma & \gamma^2 \\ -1 & -1 & \alpha+\beta \end{vmatrix}$$

expansion about R_1

$\Rightarrow (\alpha - \beta)(\beta - \gamma)(\alpha + \beta + \gamma)\{-\alpha - \beta + \beta + \gamma\}$

$\Rightarrow (\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)\{\alpha + \beta + \gamma\}$ Hence proved.

Q. **Prove that a ≠ 0,** $\begin{vmatrix} x+1 & x & x \\ x & x+a & x \\ x & x & x+a^2 \end{vmatrix} = 0$ **represents a straight line parallel to the y-axis**

Sol. $C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$,

$$\begin{vmatrix} 1 & 0 & x \\ -a & a & x \\ 0 & -a^2 & x+a^2 \end{vmatrix} = 0$$

Expansion about R_1

$1(ax + a^3 + a^2x) - 0 + x(a^3) = 0$

$a(1 + a + a^2)x = -a^3$

$\Rightarrow x = \dfrac{-a^2}{1+a+a^2}$ (constant) Hence proved.

Q. **Prove that** $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$ **= (a + b + c)³**

Sol. $C_1 \to C_1 - C_2,\ C_2 \to C_2 - C_3$

$$\begin{vmatrix} -(a+b+c) & 0 & 2a \\ (a+b+c) & -(a+b+c) & 2b \\ 0 & (a+b+c) & c-a-b \end{vmatrix}$$

$C_1 \to C_1/(a + b + c),\ C_2 \to C_2/(a + b + c)$

$$(a+b+c)^2 \begin{vmatrix} -1 & 0 & 2a \\ 1 & -1 & 2b \\ 0 & 1 & c-a-b \end{vmatrix}$$

Expansion about R_1

$\Rightarrow (a + b + c)^2\ [-1\ (-c + a + b - 2b) + 2a(1)]$

$\Rightarrow (a + b + c)^3$ Hence proved

Q. **Evaluate** $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$

Sol. $R_1 \to R_1/2b,\ R_2 \to R_2/2a,\ R_3 \to R_3/2$

$$\Rightarrow 8ab \begin{vmatrix} \frac{1+a^2-b^2}{2b} & a & -1 \\ b & \frac{1-a^2+b^2}{2a} & 1 \\ b & -a & \frac{1-a^2-b^2}{2} \end{vmatrix}$$

$C_1 \to 2bC_1,\ C_2 \to 2aC_2,\ C_3 \to 2C_3$

$$\Rightarrow \frac{1}{8ab} \cdot 8ab \begin{vmatrix} 1+a^2-b^2 & 2a^2 & -2 \\ 2b^2 & 1-a^2+b^2 & 2 \\ 2b^2 & -2a^2 & 1-a^2-b^2 \end{vmatrix}$$

$R_1 \to R_1 + R_2,\quad R_2 \to R_2 - R_3$

$$\Rightarrow \begin{vmatrix} 1+a^2+b^2 & 1+a^2+b^2 & 0 \\ 0 & 1+a^2+b^2 & 1+a^2+b^2 \\ 2b^2 & -2a^2 & 1-a^2-b^2 \end{vmatrix}$$

$R_1 \to R_1/(1 + a^2 + b^2)$, $R_2 \to R_2/(1 + a^2 + b^2)$

$$\Rightarrow (1 + a^2 + b^2)^2 \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2b^2 & -2a^2 & 1-a^2-b^2 \end{vmatrix}$$

Expansion about R_1

$\Rightarrow (1 + a^2 + b^2)^2[(1 - a^2 - b^2 + 2a^2) - (-2b^2)]$

$\Rightarrow (1 + a^2 + b^2)^3$

Q. **If** $\Delta_r = \begin{vmatrix} 2^{r-1} & 2.3^{r-1} & 4.5^{r-1} \\ x & y & z \\ 2^n - 1 & 3^n - 1 & 5^n - 1 \end{vmatrix}$ **show that** $\sum_{r=1}^{n} \Delta_r$ **= constant**

Sol. $$\sum_{r=1}^{n} \Delta_r = \begin{vmatrix} \sum_{r=1}^{n} 2^{r-1} & 2\sum_{r=1}^{n} 3^{r-1} & 4\sum_{r=1}^{n} 5^{r-1} \\ x & y & z \\ 2^n - 1 & 3^n - 1 & 5^n - 1 \end{vmatrix}$$

$$\sum_{r=1}^{n} 2^{r-1} = \frac{1(2^n - 1)}{2 - 1} = 2^n - 1$$

$$\sum_{r=1}^{n} 3^{r-1} = \frac{1(3^n - 1)}{3 - 1} = \frac{(3^n - 1)}{2}$$

$$\sum_{r=1}^{n} 5^{r-1} = \frac{1(5^n - 1)}{(5 - 1)} = \frac{5^n - 1}{4}$$

$$\sum_{r=1}^{n} \Delta_r = \begin{vmatrix} 2^n - 1 & 3^n - 1 & 5^n - 1 \\ x & y & z \\ 2^n - 1 & 3^n - 1 & 5^n - 1 \end{vmatrix} = 0$$

Q. If $\Delta_r = \begin{vmatrix} 2r-1 & {}^nC_r & 1 \\ n^2-1 & 2^n & n+1 \\ \cos^2(n^2) & \cos^2 n & \cos^2(n+1) \end{vmatrix}$ where $n \geq r \geq 0$, then evaluate $\sum_{r=0}^{n} \Delta_r$

Sol. $\sum_{r=0}^{n} \Delta_r = \begin{vmatrix} \sum_{r=0}^{n} 2r-1 & \sum_{r=0}^{n} {}^nC_r & \sum_{r=0}^{n} 1 \\ n^2-1 & 2^n & n+1 \\ \cos^2(n^2) & \cos^2 n & \cos^2(n+1) \end{vmatrix}$

$\sum_{r=0}^{n} (2r-1) = 2\frac{n(n+1)}{2} - (n+1) = n^2 - 1$

$\sum_{r=0}^{n} {}^nC_r = 2^n$

$\sum_{r=0}^{n} 1 = (n+1)$

$\sum_{r=0}^{n} \ddot{A}_r = \begin{vmatrix} n^2-1 & 2^n & n+1 \\ n^2-1 & 2^n & n+1 \\ \cos^2(n^2) & \cos^2 n & \cos^2(n+1) \end{vmatrix} = 0$

Q. For a fixed positive integer n if D = $\begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix}$ then show that

$\frac{D}{(n!)^3} - 4$ is divisible by n

Sol. $R_1 \to R_1/n!$, $R_2 \to R_2/(n+1)!$, $R_3 \to R_3/(n+2)!$

$D = n!(n+1)!(n+2)! \begin{vmatrix} 1 & (n+1) & (n+2)(n+1) \\ 1 & (n+2) & (n+3)(n+2) \\ 1 & (n+3) & (n+4)(n+3) \end{vmatrix}$

$R_3 \to R_3 - R_2$, $R_2 \to R_2 - R_1$

$$D = n!(n+1)!\,(n+2)!\begin{vmatrix} 1 & (n+1) & (n+2)(n+1) \\ 0 & 1 & 2(n+2) \\ 0 & 1 & 2(n+3) \end{vmatrix} \quad \text{expansion about } C_1$$

$D = n!(n + 1)!\ (n + 2)!\ \{2(n + 3) - 2(n + 2)\}$

$D = (n!)^3\ (n + 1)\ (n + 2)\ (n + 1) \times 2$

$$\frac{D}{(n!)^3} = (n^3 + 4n^2 + 5n + 2) \times 2$$

$$\frac{D}{(n!)^3} - 4 = 2n(n^2 + 4n + 5) \text{ which is divisible by } n$$

Q. **Without expanding, prove that** $\begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix}$ **= xA + B where A and B are determinants of 3 × 3 square matrices not involving x.**

Sol. Apply $R_2 \to R_2 - (R_1 + R_3)$

$$\begin{vmatrix} x^2+x & x+1 & x-2 \\ -4 & 0 & 0 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix}$$

now $R_1 \to R_1 + \frac{x^2}{4} R_2$, $R_3 \to R_3 + \frac{x^2}{4} R_2$

$$\begin{vmatrix} x & x+1 & x-2 \\ -4 & 0 & 0 \\ 2x+3 & 2x-1 & 2x-1 \end{vmatrix} \quad \text{now } R_3 \to R_3 - 2R_1$$

$$= \begin{vmatrix} x & x+1 & x-2 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} x & x & x \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 1 & -2 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix}$$

$$= x\begin{vmatrix} 1 & 1 & 1 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 1 & -2 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix} = xA + B$$

Q. **If $(x_1 - x_2)^2 + (y_1 - y_2)^2 = a^2$**
$(x_2 - x_3)^2 + (y_2 - y_3)^2 = b^2$ and
$(x_3 - x_1)^2 + (y_3 - y_1)^2 = c^2$ then prove that

$$4\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = (a+b+c)(b+c-a)(c+a-b)(a+b-c)$$

Sol. Let three points in xy plane $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$ hence given that

$PQ^2 = a^2 \Rightarrow PQ = a$

$QR^2 = b^2 \Rightarrow QR = b$

$RP^2 = c^2 \Rightarrow RP = c$

now area of ΔPQR

$\Delta = \sqrt{S(S-a)(S-b)(S-c)}$

$\Rightarrow \Delta^2 = S(S-a)(S-b)(S-c)$

$$\left(\frac{1}{2}\right)^2 \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = \left(\frac{a+b+c}{2}\right)\left(\frac{b+c-a}{2}\right)\left(\frac{c+a-b}{2}\right)\left(\frac{a+b-c}{2}\right)$$

$$4\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = (a+b+c)(b+c-a)(c+a-b)(a+b-c)$$ **Hence proved.**

Q. **If a, b, c are all different and** $\begin{vmatrix} a & a^3 & a^4-1 \\ b & b^3 & b^4-1 \\ c & c^3 & c^4-1 \end{vmatrix} = 0$ **then show that abc(ab + bc + ca) = a + b + c**

Sol. Given determinant is

$$\begin{vmatrix} a & a^3 & a^4 \\ b & b^3 & b^4 \\ c & c^3 & c^4 \end{vmatrix} - \begin{vmatrix} a & a^3 & 1 \\ b & b^3 & 1 \\ c & c^3 & 1 \end{vmatrix} = 0$$

$$abc\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} - \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = 0$$

abc(a–b)(b-c)(c–a)(ab + bc + ca) – (a – b)(b – c) (c – a)(a + b + c) = 0

⇒ abc(ab + bc + ca) – (a + b + c) = 0

⇒ abc(ab + bc + ca) = (a + b + c)

Q. **Prove that**

$$\begin{vmatrix} (b+c)^2 & ba & ac \\ ba & (c+a)^2 & cb \\ ca & cb & (a+b)^2 \end{vmatrix} = \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3$$

Sol. $C_1 \to C_1/a,\ C_2 \to C_2/b,\ C_3 \to C_3/c$

$$abc\begin{vmatrix} \frac{(b+c)^2}{a} & a & a \\ b & \frac{(c+a)^2}{b} & b \\ c & c & \frac{(a+b)^2}{c} \end{vmatrix}$$

Determinants

now $R_1 \to aR_1$, $R_2 \to bR_2$, $R_3 \to cR_3$

$$\Rightarrow abc \cdot \frac{1}{abc}\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$$

$C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$

$$\begin{vmatrix} (b+c+a)(b+c-a) & 0 & a^2 \\ (b+c+a)(b-c-a) & (c+a+b)(c+a-b) & b^2 \\ 0 & (c+a+b)(c-a-b) & (a+b)^2 \end{vmatrix}$$

$C_1 \to C_1/(a + b + c)$, $C_2 \to C_2/(a + b + c)$

$$(a+b+c)^2\begin{vmatrix} b+c-a & 0 & a^2 \\ b-c-a & c+a-b & b^2 \\ 0 & c-a-b & (a+b)^2 \end{vmatrix}$$

$R_3 \to R_3 - (R_1 + R_2)$

$$(a+b+c)^2\begin{vmatrix} b+c-a & 0 & a^2 \\ b-c-a & c+a-b & b^2 \\ 2(a-b) & -2a & 2ab \end{vmatrix}$$

$C_1 \to C_1 + C_2$

$$(a+b+c)^2\begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix}$$

$C_1 \to aC_1 + C_3$, $C_2 \to bC_2 + C_3$

$$\frac{(a+b+c)^2}{ab}\begin{vmatrix} a(b+c) & a^2 & a^2 \\ b^2 & b(c+a) & b^2 \\ 0 & 0 & 2ab \end{vmatrix}$$

Expansion about R_3

$$\frac{(a+b+c)^2}{ab}\left[2ab\{ab(b+c)(c+a) - a^2b^2\}\right]$$

$2(a + b + c)^2[ab(bc + ab + c^2 + ac - ab)]$

$2abc(a + b + c)^3$ **Hence proved.**

Q. **Let P be a matrix of order 3 × 3 such that all the entries in P are from the set {–1, 0, 1}. Then the maximum possible value of the determinant of P is**

Sol. Let

$$|P| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ where } a_i, b_i, c_i \in \{-1, 0, 1\}$$

$$= \underbrace{a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3}_{x} - \underbrace{\left(a_3b_2c_1 + b_3c_2a_1 + c_3a_2b_1\right)}_{y}$$

now $x \le 3$ and $y \ge -3$

but |P| cannot be 6

$$\left.\begin{array}{l} \text{as if } x = 3 \Rightarrow \text{each term of } x = 1 \\ \text{and } y = -3 \Rightarrow \text{each term of } y = -1 \end{array}\right\} \Rightarrow \text{Contradiction}$$

now next possibility |P| = 4 which is possible when $a_2 = b_3 = -1$ and rest all elements = 1 hence

$|P|_{max.} = 4$

Product of two determinants

1. Method of Multiplication: (Row by column)

eg. Let $\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \Delta_2 = \begin{vmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}$

$\Delta = \Delta_1\Delta_2$

$$\Delta = \begin{vmatrix} a_1\ell_1 + b_1m_1 + c_1n_1 & a_1\ell_2 + b_1m_2 + c_1n_2 & a_1\ell_3 + b_1m_3 + c_1n_3 \\ a_2\ell_1 + b_2m_1 + c_2n_1 & a_2\ell_2 + b_2m_2 + c_2n_2 & a_2\ell_3 + b_2m_3 + c_2n_3 \\ a_3\ell_1 + b_3m_1 + c_3n_1 & a_3\ell_2 + b_3m_2 + c_3n_2 & a_3\ell_3 + b_3m_3 + c_3n_3 \end{vmatrix}$$

2. Method of multiplication: (Row by Row)

$\Delta = \Delta_1\Delta_2$

$$\Delta = \begin{vmatrix} a_1\ell_1 + b_1\ell_2 + c_1\ell_3 & a_1m_1 + b_1m_2 + c_1m_3 & a_1n_1 + b_1n_2 + c_1n_3 \\ a_2\ell_1 + b_2\ell_2 + c_2\ell_3 & a_2m_1 + b_2m_2 + c_2m_3 & a_2n_1 + b_2n_2 + c_2n_3 \\ a_3\ell_1 + b_3\ell_2 + c_3\ell_3 & a_3m_1 + b_3m_2 + c_3m_3 & a_3n_1 + b_3n_2 + c_3n_3 \end{vmatrix}$$

Q. **Prove that**

$$\begin{vmatrix} (a_1-b_1)^2 & (a_1-b_2)^2 & (a_1-b_3)^2 \\ (a_2-b_1)^2 & (a_2-b_2)^2 & (a_2-b_3)^2 \\ (a_3-b_1)^2 & (a_3-b_2)^2 & (a_3-b_3)^2 \end{vmatrix}$$

$= 2(a_1 - a_2)(a_2 - a_3)(a_3 - a_1)(b_1 - b_2)(b_2 - b_3)(b_3 - b_1)$

Sol. LHS is

$$\begin{vmatrix} a_1^2-2a_1b_1+b_1^2 & a_1^2-2a_1b_2+b_2^2 & a_1^2-2a_1b_3+b_3^2 \\ a_2^2-2a_2b_1+b_1^2 & a_2^2-2a_2b_2+b_2^2 & a_2^2-2a_2b_3+b_3^2 \\ a_3^2-2a_3b_1+b_1^2 & a_3^2-2a_3b_2+b_2^2 & a_3^2-2a_3b_3+b_3^2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1^2 & -2a_1 & 1 \\ a_2^2 & -2a_2 & 1 \\ a_3^2 & -2a_3 & 1 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ b_1 & b_2 & b_3 \\ b_1^2 & b_2^2 & b_3^2 \end{vmatrix}$$

now the above determinants

$$2\begin{vmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ b_1 & b_2 & b_3 \\ b_1^2 & b_2^2 & b_3^2 \end{vmatrix}$$

$= 2(a_1 - a_2)(a_2 - a_3)(a_3 - a_1)(b_1 - b_2)(b_2 - b_3)(b_3 - b_1)$

3. Method of multiplication: (Column by column)

$\Delta = \Delta_1\Delta_2$

$$\Delta = \begin{vmatrix} a_1\ell_1+a_2m_1+a_3n_1 & a_1\ell_2+a_2m_2+a_3n_2 & a_1\ell_3+a_2m_3+a_3n_3 \\ b_1\ell_1+b_2m_1+b_3n_1 & b_1\ell_2+b_2m_2+b_3n_2 & b_1\ell_3+b_2m_3+b_3n_3 \\ c_1\ell_1+c_2m_1+c_3n_1 & c_1\ell_2+c_2m_2+c_3n_2 & c_1\ell_3+c_2m_3+c_3n_3 \end{vmatrix}$$

Point to Remember!!!

If A_1, B_1, C_1, ... are respectively the cofactors of the elements a_1, b_1, c_1, ... of the determinant

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \Delta \neq 0 \text{ then}$$

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \Delta^2$$

Q. **Prove that** $\begin{vmatrix} a_1\alpha_1+b_1\beta_1 & a_1\alpha_2+b_1\beta_2 & a_1\alpha_3+b_1\beta_3 \\ a_2\alpha_1+b_2\beta_1 & a_2\alpha_2+b_2\beta_2 & a_2\alpha_3+b_2\beta_3 \\ a_3\alpha_1+b_3\beta_1 & a_3\alpha_2+b_3\beta_2 & a_3\alpha_3+b_3\beta_3 \end{vmatrix} = 0$

Sol. Given determinant can be written as

$$\begin{vmatrix} a_1\alpha_1+b_1\beta_1+0 & a_1\alpha_2+b_1\beta_2+0 & a_1\alpha_3+b_1\beta_3+0 \\ a_2\alpha_1+b_2\beta_1+0 & a_2\alpha_2+b_2\beta_2+0 & a_2\alpha_3+b_2\beta_3+0 \\ a_3\alpha_1+b_3\beta_1+0 & a_3\alpha_2+b_3\beta_2+0 & a_3\alpha_3+b_3\beta_3+0 \end{vmatrix}$$

Which can be expressed as product of the following two determinants

$$\begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

Hence proved.

Q. **If** $\begin{vmatrix} 1 & x & x^2 \\ x & x^2 & 1 \\ x^2 & 1 & x \end{vmatrix} = 3$ **then find the value of** $\begin{vmatrix} x^3-1 & 0 & x-x^4 \\ 0 & x-x^4 & x^3-1 \\ x-x^4 & x^3-1 & 0 \end{vmatrix}$

Sol. Required determinant is formed by replacing each element with its co-factor in given determinant hence $\begin{vmatrix} x^3-1 & 0 & x-x^4 \\ 0 & x-x^4 & x^3-1 \\ x-x^4 & x^3-1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ x & x^2 & 1 \\ x^2 & 1 & x \end{vmatrix}^2 = 3^2 = 9$

Q. **Express the determinant** $\Delta = \begin{vmatrix} 1 & \cos(\beta-\alpha) & \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) & 1 & \cos(\gamma-\beta) \\ \cos(\alpha-\gamma) & \cos(\beta-\gamma) & 1 \end{vmatrix}$ **as the product of two determinants and hence show that** $\Delta = 0$

Sol. Given determinants is $\Delta = \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 0 \end{vmatrix} \times \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \sin\alpha & \sin\beta & \sin\gamma \\ 0 & 0 & 0 \end{vmatrix}$

$\Rightarrow 0 \times 0 = 0$

Q. **If** α, β **are the roots of the equation** $ax^2 + bx + c = 0$ **and** $S_n = \alpha^n + \beta^n$ **then evaluate**

$$\begin{vmatrix} 3 & 1+S_1 & 1+S_2 \\ 1+S_1 & 1+S_2 & 1+S_3 \\ 1+S_2 & 1+S_3 & 1+S_4 \end{vmatrix}$$

Sol. We know that

$\alpha + \beta = -b/a$

$\alpha\beta = c/a$

now given determinant is $\begin{vmatrix} 1+1+1 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \end{vmatrix} = \{(1-\alpha)(\alpha-\beta)(\beta-1)\}^2$$

$= (-1 + \alpha + \beta - \alpha\beta)^2\{(\alpha+\beta)^2 - 4\alpha\beta\}$

$= \left(-1 - \frac{b}{a} - \frac{c}{a}\right)^2 \left(\frac{b^2}{a^2} - \frac{4c}{a}\right)$

$= \frac{1}{a^4}(a+b+c)^2(b^2 - 4ac)$

Q. **Which of the following values of 'α' satisfy the equation**

$$\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ (2+\alpha)^2 & (2+2\alpha)^2 & (2+3\alpha)^2 \\ (3+\alpha)^2 & (3+2\alpha)^2 & (3+3\alpha)^2 \end{vmatrix} = -648\alpha$$

(A) – 4 **(B) 9** **(C) – 9** **(D) 4**

Ans. **(BC)**

Sol. Given equation

$$\begin{vmatrix} 1+2\alpha+\alpha^2 & 1+4\alpha+4\alpha^2 & 1+6\alpha+9\alpha^2 \\ 4+4\alpha+\alpha^2 & 4+8\alpha+4\alpha^2 & 4+12\alpha+9\alpha^2 \\ 9+6\alpha+\alpha^2 & 9+12\alpha+4\alpha^2 & 9+18\alpha+9\alpha^2 \end{vmatrix} = -648\alpha$$

$$\begin{vmatrix} 1 & 2\alpha & \alpha^2 \\ 4 & 4\alpha & \alpha^2 \\ 9 & 6\alpha & \alpha^2 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = -\ 648\alpha$$

$$2\alpha^3 \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = -648\alpha$$

$-\ 2\alpha^3\{(1-2)\ (2-3)\ (3-1)\}^2 = -\ 648\alpha$

$-\ 8\alpha^3 = -\ 648\alpha \Rightarrow \alpha = 0,\ \alpha^2 = 81 \Rightarrow \alpha = \pm\ 9$

Q. **Evaluate** $\begin{vmatrix} 2 & \alpha+\beta+\gamma+\delta & \alpha\beta+\gamma\delta \\ \alpha+\beta+\gamma+\delta & 2(\alpha+\beta)(\gamma+\delta) & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) \\ \alpha\beta+\gamma\delta & \alpha\beta(\gamma+\delta)+(\alpha+\beta)\gamma\delta & 2\alpha\beta\gamma\delta \end{vmatrix}$

Sol. $$\begin{vmatrix} 1 & 1 & 0 \\ (\gamma+\delta) & (\alpha+\beta) & 0 \\ \gamma\delta & \alpha\beta & 0 \end{vmatrix} \times \begin{vmatrix} 1 & (\alpha+\beta) & \alpha\beta \\ 1 & (\gamma+\delta) & \gamma\delta \\ 0 & 0 & 0 \end{vmatrix} = 0$$

System of linear equations

$a_1x + b_1y + c_1z = d_1$
$a_2x + b_2y + c_2z = d_2$
$a_3x + b_3y + c_3z = d_3$

For example x = 3, y = 3 and z = 6 is the solution of the system of equations

$5x - 6y + 3z = 15$
$7x + 4y - 2z = 21$
$2x + y + 6z = 45$

Solution of a system of linear equations (theorem: cramer's rule)

The solution of the system of linear equations

$a_1x + b_1y + c_1z = d_1$ ... (i)
$a_2x + b_2y + c_2z = d_2$ ... (ii)
$a_3x + b_3y + c_3z = d_3$ ... (iii)

is given by $x = \frac{D_1}{D}, y = \frac{D_2}{D}$ and $z = \frac{D_3}{D}$ provided that $D \neq 0$

where $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ $\quad D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$

$D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ $\quad D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

System of homogeneous linear equations

$a_1x + b_1y + c_1z = 0$ ...(i)
$a_2x + b_2y + c_2z = 0$ ...(ii)
$a_3x + b_3y + c_3z = 0$ ...(iii)

Summary

(i) If $D \neq 0$, then given system of equations has only **trivial solution** and the number of solutions in this case is one.

(ii) If D = 0, then given system of equations has **nontrivial solution** as well as **trivial solution** and the number of solutions in this case is infinite

Special case

When $D = D_1 = D_2 = D_3 = 0 \Rightarrow$ **no solution** or **infinite many solution**

eg. x + y + z = 1, x + y + z = 2 and x + y + z = 3 has no solution since all are parallel planes.

eg. x + y + z = 1, 2x + 2y + 2z = 2 and 3x + 3y + 3z = 3 has infinite solutions since all plane coincides

Q. **Solve by cramer's rule**
x + y + z = 6
x – y + z = 2
3x + 2y – 4z = – 5

Sol.

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 3 & 2 & -4 \end{vmatrix} = 14, \qquad D_1 = \begin{vmatrix} 6 & 1 & 1 \\ 2 & -1 & 1 \\ -5 & 2 & -4 \end{vmatrix} = 14$$

$$D_2 = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 2 & 1 \\ 3 & -5 & -4 \end{vmatrix} = 28, \qquad D_3 = \begin{vmatrix} 1 & 1 & 6 \\ 1 & -1 & 2 \\ 3 & 2 & -5 \end{vmatrix} = 42$$

$$x = \frac{D_1}{D} = 1, y = \frac{D_2}{D} = 2, z = \frac{D_3}{D} = 3$$

Q. **For what value of p and q, the system of equation**
2x + py + 6z = 8
x + 2y + qz = 5
x + y + 3z = 4 has
(i) no solution
(ii) a unique solution
(iii) infinitely many solutions.

Sol.

$$D = \begin{vmatrix} 2 & p & 6 \\ 1 & 2 & q \\ 1 & 1 & 3 \end{vmatrix} = (2 - p)(3 - q)$$

$$D_1 = \begin{vmatrix} 8 & p & 6 \\ 5 & 2 & q \\ 4 & 1 & 3 \end{vmatrix} = (p - 2)(4q - 15)$$

$$D_2 = \begin{vmatrix} 2 & 8 & 6 \\ 1 & 5 & q \\ 1 & 4 & 3 \end{vmatrix} = 0$$

$$D_3 = \begin{vmatrix} 2 & p & 8 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{vmatrix} = (p - 2)$$

(i) $D = 0$, atleast one $D_i \neq 0 \Rightarrow p \neq 2, q = 3$
(ii) $D \neq 0 \Rightarrow p \neq 2, q \neq 3$
(iii) $D = 0$ & $D_i = 0 \Rightarrow p = 2, q \in R$ ($\because$ all planes are not parallel)

Q. **If x, y, z are not all zero such that**
$ax + y + z = 0$
$x + by + z = 0$
$x + y + cz = 0$
then prove that $\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}=1$

Sol. $D = \begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$

$R_1 \to R_1 - R_2,\ R_2 \to R_2 - R_3$

$$\begin{vmatrix} a-1 & 1-b & 0 \\ 0 & b-1 & 1-c \\ 1 & 1 & c \end{vmatrix} = 0$$

expand about R_1
$\Rightarrow (a - 1)\{c(b - 1) - (1 - c)\} - (1 - b)\{-(1 - c)\} = 0$
$\Rightarrow c(1 - a)(1 - b) + (1 - a)(1 - c) + (1 - b)(1 - c) = 0$
$\Rightarrow \frac{c}{1-c}+\frac{1}{1-b}+\frac{1}{1-a}=0$
$\Rightarrow \frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}=1$

Q. **If sinq $\neq$ cosq and x, y, z satisfy the equations**
xcosp – y sinp + z = cosq + 1
xsinp + y cosp + z = 1 – sinq
xcos(p + q) – ysin(p + q) + z = 2 then find the value of $x^2 + y^2 + z^2$

Sol. x cosp – y sinp + z = cosq + 1 ...(i)
x sinp + y cosp + z = 1 – sinq ...(ii)
xcos(p + q) – y sin(p + q) + z = 2 ...(iii)
(i) × cosq – (ii) × sinq
x cos(p + q) – ysin(p + q) + z(cosq – sinq) = 1 + (cosq – sinq) ...(iv)

(iv) − (iii)

z(cosq − sinq − 1) = (cosq − sinq − 1)

⇒ z = 1

from (i) x cosp − y sinp = cosq ...(v)

from (ii) x sinp + y cosp = − sinq ...(vi)

$(v)^2 + (vi)^2 \Rightarrow x^2 + y^2 = 1$

Hence $x^2 + y^2 + z^2 = 2$

Q. Investigate for what values of λ, μ the simultaneous equations
x + y + z = 6
x + 2y + 3z = 10
x + 2y + λz = μ have
(a) A unique solution
(b) An infinite number of solutions
(c) No solution

Sol.

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{vmatrix} = (\lambda - 3)$$

$$D_1 = \begin{vmatrix} 6 & 1 & 1 \\ 10 & 2 & 3 \\ \mu & 2 & \lambda \end{vmatrix} = 2\lambda + \mu - 16$$

$$D_2 = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 10 & 3 \\ 1 & \mu & \lambda \end{vmatrix} = 2(2\lambda - \mu + 4)$$

$$D_3 = \begin{vmatrix} 1 & 1 & 6 \\ 1 & 2 & 10 \\ 1 & 2 & \mu \end{vmatrix} = (\mu - 10)$$

(i) $D \neq 0 \Rightarrow \lambda \neq 3$

(ii) $D = 0$ & $D_i = 0 \Rightarrow \lambda = 3, \mu = 10$

(iii) $D = 0$ & atleast one $D_i \neq 0 \Rightarrow \lambda = 3, \mu \neq 10$

($\because$ all planes are not parallel)

Q. **For what values of p, the equations:**
x + y + z = 1
x + 2y + 4z = p and
x + 4y + 10z = p² have a solution?
Solve them completely in each case.

Sol.

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{vmatrix} = 0$$

$$D_1 = \begin{vmatrix} 1 & 1 & 1 \\ p & 2 & 4 \\ p^2 & 4 & 10 \end{vmatrix} = 2(p-1)(p-2)$$

$$D_2 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & p & 4 \\ 1 & p^2 & 10 \end{vmatrix} = -3(p-1)(p-2)$$

$$D_3 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & p \\ 1 & 4 & p^2 \end{vmatrix} = (p-1)(p-2)$$

system have infinite solution for p = 1, 2

for $p = 1 \Rightarrow \quad x + y + z = 1 \quad ...(1)$

$x + 2y + 4z = 1 \quad ...(2)$

$(2) - (1) \Rightarrow y + 3z = 0 \Rightarrow y = -3z$

$\Rightarrow x = 1 + 2z$

$(x, y, z) \equiv (1 + 2z, -3z, z);\ Z \in R$

for $p = 2 \Rightarrow \quad x + y + z = 1 \quad ...(1)$

$x + 2y + 4z = 2 \quad ...(2)$

$(2) - (1) \Rightarrow y + 3z = 1 \Rightarrow y = 1 - 3z$

$x = 2z$

$(x, y, z) \equiv (2z, 1 - 3z, z);\ Z \in R$

Q. **Solve the equations:**
kx + 2y – 2z = 1
4x + 2ky – z = 2
6x + 6y + kz = 3
Considering specially the case when k = 2

Sol.

$$D = \begin{vmatrix} k & 2 & -2 \\ 4 & 2k & -1 \\ 6 & 6 & k \end{vmatrix}$$

$$= 2\left(k^3 + 11k - 30\right)$$

$$= 2\left(k - 2\right)\left(k^2 + 2k + 15\right)$$

$$D_1 = \begin{vmatrix} 1 & 2 & -2 \\ 2 & 2k & -1 \\ 3 & 6 & k \end{vmatrix}$$

$$= 2\left(k^2 + 4k - 12\right)$$

$$= 2\left(k - 2\right)\left(k + 6\right)$$

$$D_2 = \begin{vmatrix} k & 1 & -2 \\ 4 & 2 & -1 \\ 6 & 3 & k \end{vmatrix}$$

$$= (2k + 3)(k - 2)$$

$$D_3 = \begin{vmatrix} k & 2 & 1 \\ 4 & 2k & 2 \\ 6 & 6 & 3 \end{vmatrix}$$

$$= 6(k - 2)^2$$

when $k \neq 2$ then

$$x = \frac{k+6}{k^2 + 2k + 15}, y = \frac{2k+3}{2\left(k^2 + 2k + 15\right)}, z = \frac{3\left(k-2\right)}{\left(k^2 + 2k + 15\right)}$$

when k = 2 then $2x + 2y - 2z = 1$ ...(1)
$6x + 6y + 2z = 3$ ...(2)

(2) – (1) × 3 $\Rightarrow z = 0, y = \dfrac{1-2x}{2}$

$$(x, y, z) \equiv \left(x, \frac{1-2x}{2}, 0\right) \quad \forall x \in R$$

Q. **The system of equations**

$\alpha x + y + z = \alpha - 1$

$x + \alpha y + z = \alpha - 1$

$x + y + \alpha z = \alpha - 1$

has infinite solutions, if α is

(A) – 2 **(B) either – 2 or 1** **(C) not 1** **(D) 1**

Ans. **(D)**

Sol.
$$D = \begin{vmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{vmatrix} = (\alpha - 1)^2 (\alpha + 2)$$

for infinite solution $D = 0 \Rightarrow \alpha = 1, -2$

for $\alpha = 1$ all equations are identical $\Rightarrow$ infinite solutions

for $\alpha = -2$ equations are

$-2x + y + z = -3$ ...(1)

$x - 2y + z = -3$ ...(2)

$x + y - 2z = -3$ ...(3)

(1) + (2) + (3) $\Rightarrow 0 = -9$ No solution

Q. **Let** $A = \begin{vmatrix} 5 & 5\alpha & \alpha \\ 0 & \alpha & 5\alpha \\ 0 & 0 & 5 \end{vmatrix}$ **If $|A^2| = 25$ then $|\alpha|$ equals.**

(A) $\frac{1}{5}$ **(B) 5** **(C) 25** **(D) 1**

Ans. **(A)**

Sol. $|A| = 25\alpha$ also

$|A^2| = 25 \Rightarrow |A|^2 = 25$

$(25\alpha)^2 = 25 \Rightarrow 25\alpha^2 = 1$

$|\alpha| = \frac{1}{5}$

Q. **Let a, b, c be any real numbers. Suppose that there are real numbers x, y, z not all zero such that $x = cy + bz$, $y = az + cx$ and $z = bx + ay$ then $a^2 + b^2 + c^2 + 2abc$ is equal to**

(A) 2 **(B) 1** **(C) Zero** **(D) 3**

Ans. **(B)**

Sol. Given equations are

$x - cy - bz = 0$

$cx - y + az = 0$

$bx + ay - z = 0$

the system have non-trivial solution hence

$D = 0$

$$\begin{vmatrix} 1 & -c & -b \\ c & -1 & a \\ b & a & -1 \end{vmatrix} = 0$$

$1(1 - a^2) + c(-c - ab) - b(ac + b) = 0$

$\Rightarrow - a^2 - b^2 - c^2 - 2abc + 1 = 0$

$\Rightarrow a^2 + b^2 + c^2 + 2abc = 1$

Q. **The number of values of k for which the linear equation**

$4x + ky + 2z = 0$

$kx + 4y + z = 0$

and $2x + 2y + z = 0$ posses a non-zero solution is

(A) 2 **(B) 1** **(C) Zero** **(D) 3**

Ans. **(A)**

Sol. for non-zero solution D = 0

$$\begin{vmatrix} 4 & k & 2 \\ k & 4 & 1 \\ 2 & 2 & 1 \end{vmatrix} = 0$$

$4(2) - k(k - 2) + 2(2k - 8) = 0$

$\Rightarrow 8 - k^2 + 2k + 4k - 16 = 0$

$\Rightarrow k^2 - 6k + 8 = 0$

$(k - 2)(k - 4) = 0 \Rightarrow k = 2, 4$ (two values)

Q. **If the system of linear equations**
$x - 4y + 7z = g$
$3y - 5z = h$
$-2x + 5y - 9z = k$ is consistent then
(A) $g + h + k = 0$ **(B) $2g + h + k = 0$**
(C) $g + h + 2k = 0$ **(D) $g + 2h + k = 0$**

Ans. (B)

Sol. $D = \begin{vmatrix} 1 & -4 & 7 \\ 0 & 3 & -5 \\ -2 & 5 & -9 \end{vmatrix} = 0$

so for the system to be consistent $D_1 = D_2 = D_3 = 0$

$D_1 = \begin{vmatrix} g & -4 & 7 \\ h & 3 & -5 \\ k & 5 & -9 \end{vmatrix} = g(-2) - h(1) - k(1) = 0$

$\Rightarrow 2g + h + k = 0$

$D_2 = \begin{vmatrix} 1 & g & 7 \\ 0 & h & -5 \\ -2 & k & -9 \end{vmatrix} = 1(-9h + 5k) - 2(-5g - 7h) = 0$

$\Rightarrow 2g + h + k = 0$

$D_3 = \begin{vmatrix} 1 & -4 & g \\ 0 & 3 & h \\ -2 & 5 & k \end{vmatrix} = 1\,(3k - 5h) - 2(-4h - 3g) = 0$

$\Rightarrow 2g + h + k = 0$

Q. **The values of λ and μ for which the system of linear equations**
$x + y + z = 2$
$x + 2y + 3z = 5$
$x + 3y + \lambda z = \mu$
has infinitely many solutions are, respectively
(A) 6 to 8 **(B) 5 and 8** **(C) 5 and 7** **(D) 4 and 9**

Ans. (B)

Sol. for infinite solutions $D = D_1 = D_2 = D_3 = 0$

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow (2\lambda - 9) - (\lambda - 3) + (1) = 0$$

$\Rightarrow \lambda = 5$

$$D_3 = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 1 & 3 & \mu \end{vmatrix} = 0$$

$$\Rightarrow (2\mu - 15) - (\mu - 5) + 2(1) = 0$$

$\Rightarrow \mu = 8$

Q. **The system of linear equations**
$x + \lambda y - z = 0$
$\lambda x - y - z = 0$
$X + y - \lambda z = 0$ has a non-trivial solution for:
(A) Infinitely many value of λ
(B) Exactly one value of λ
(C) Exactly two values of λ
(D) Exactly three values of λ

Ans. **(D)**

Sol. For non-trivial solution
D = 0

$$\begin{vmatrix} 1 & \lambda & -1 \\ \lambda & -1 & -1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$$

$1(\lambda + 1) - \lambda(-\lambda^2 + 1) - 1(\lambda + 1) = 0$
$\Rightarrow \lambda + 1 + \lambda^3 - \lambda - \lambda - 1 = 0$
$\Rightarrow \lambda^3 - \lambda = 0 \Rightarrow \lambda = -1, 0, 1$

Q.

Let S be the set of all column matrices $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ **such that** $b_1, b_2, b_3 \in R$ **and the system of equations (in real variables)**

$-x + 2y + 5z = b_1$
$2x - 4y + 3z = b_2$
$x - 2y + 2z = b_3$

has atleast one solution. Then which of the following system(s) (in real variables) has (have) atleast one solution of each $\begin{bmatrix} b_1 \\ b_2 \\ b_2 \end{bmatrix} \in S$ **?**

(A) $x + 2y + 3z = b_1$, $4y + 5z = b_2$ **and** $x + 2y + 6z = b_3$
(B) $x + y + 3z = b_1$, $5x + 2y + 6z = b_2$ **and** $-2x - y - 3z = b_3$
(C) $-x + 2y - 5z = b_1$, $2x - 4y + 10z = b_2$ **and** $x - 2y + 5z = b_3$
(D) $x + 2y + 5z = b_1$, $2x + 3z = b_2$ **and** $x + 4y - 5z = b_3$

Ans. (ACD)

Sol. for given system D = 0 hence for atleast one solution

$D_1 = D_2 = D_3 = 0 \Rightarrow b_1 + 7b_2 = 13b_3$ ...(1)

option (A) $D \neq 0 \Rightarrow$ unique solution

option (D) $D \neq 0 \Rightarrow$ unique solution

option (C) D = 0 now equations are

$x - 2y + 5z = -b_1$

$x - 2y + 5z = b_2/2$

$x - 2y + 5z = b_3$

which can be consistent only if

$-b_1 = \frac{b_2}{2} = b_3 = k(\text{let})$

$\Rightarrow b_1 = -k, b_2 = 2k, b_3 = k$

put in (1)

$(-k) + 7(2k) = 13(k)$ which is true $\forall\ k \in R$ hence for every element $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ this system will have atleast one solution

option (D)

$D = 0$, $D_1 = 0$ but $D_2 = 3(b_1 + b_2 + 3b_3)$ from (1) we can say that one of the set of values

(b_1, b_2, b_3) can be $(7, -1, 0)$ but for this set $D_2 \neq 0$ hence system will not have any solution