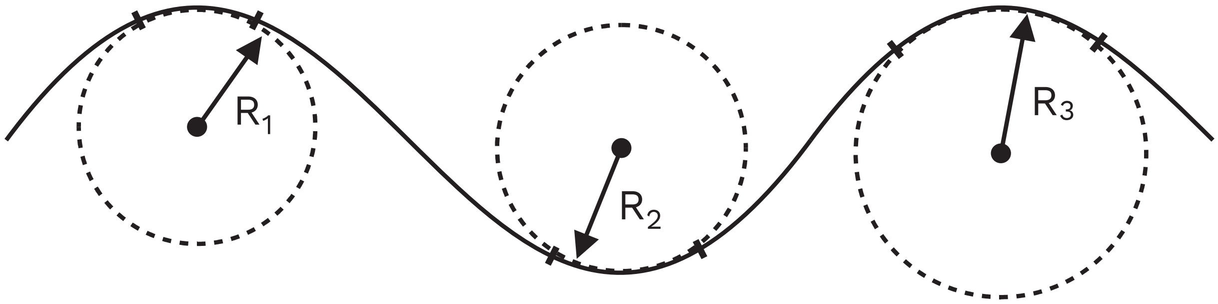
Circular Motion

DISCLAIMER

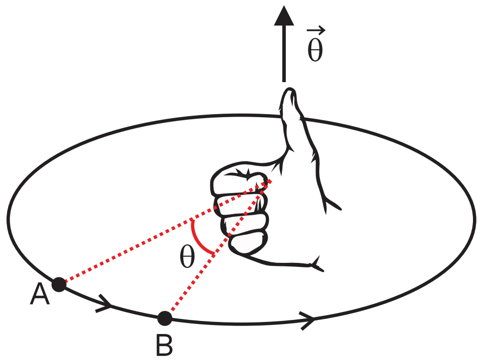
Kinematics of Circular Motion

Motion on a circular path is circular motion. Even a motion on a curved path can be considered a combination of several circular motions.
Also, a straight-line motion can be considered as a circular motion of infinite radius.



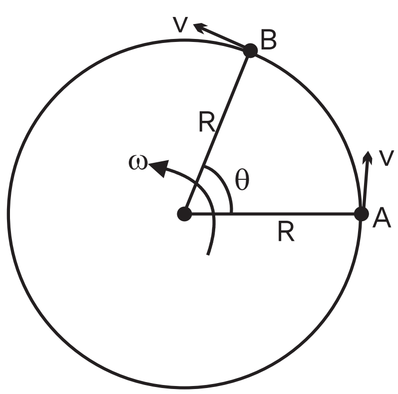
Parameters of Circular Motion

a) Angular displacement: Angular displacement is the angle turned by a particle moving on a circular path in a certain time.
It is a vector quantity and direction of angular displacement is found by using Right Hand thumb rule.
Rotate the curl of fingers of right hand in direction of rotation on circular path, then the direction of thumb of right hand gives the direction of angular displacement vector.
So, angular displacement $\vec{\theta}$ is an axial vector.



b) Angular velocity ($\vec{\omega}$) : The rate with which angular displacement changes with respect to time is called angular velocity.
It is also an axial vector and its direction is same as that of angular displacement vector.

Relation between linear velocity (v) and angular velocity (ω)



We know that,

$$\text{Angle } (\theta) = \frac{\text{Arc length (AB)}}{\text{Radius (R)}}$$

So, D = Rθ
Divide both sides by Δt,

$$\frac{D}{\Delta t} = R \cdot \frac{\theta}{\Delta t}$$

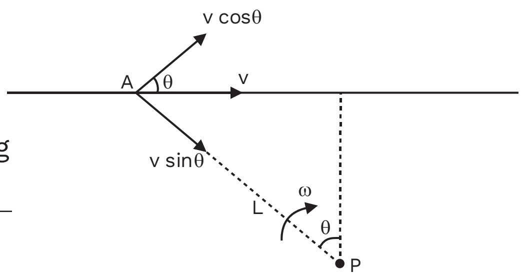
Now, $\frac{D}{\Delta t} = v$ and $\frac{\theta}{\Delta t} = \omega$

So, $\boxed{v = R\omega}$

In vector form, $\boxed{\vec{v} = (\vec{\omega} \times \vec{R})}$

For straight line motion:

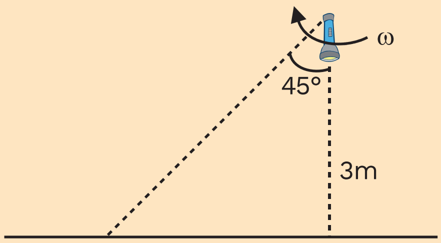
To find angular velocity about point P, use the following formula,



$$\omega_p = \frac{\text{component of velocity} \perp \text{er to the line joining the particle and point P}}{\text{Length of the line joining}}$$

$$\omega_p = \frac{v\cos\theta}{L}$$

Q. **A man is standing 3m from a wall and he lights up a torch on the wall. If he is rotating the torch at an angular speed of 0.1 rad/s, then what is the velocity of spot of light when light ray makes an angle of 45° with the line joining the man and wall.**



Sol. Let v be the light spot's speed as shown.

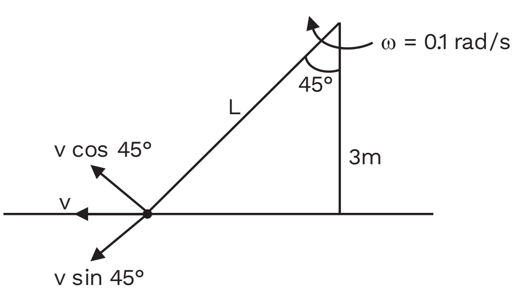
From figure,

$\cos 45° = \frac{3}{L} \Rightarrow \frac{1}{\sqrt{2}} = \frac{3}{L}$ $\boxed{L = 3\sqrt{2}}$

Now, $\omega = \frac{v\cos 45°}{L}$

$\Rightarrow \quad 0.1 = v \times \frac{1}{\sqrt{2} \times 3\sqrt{2}}$

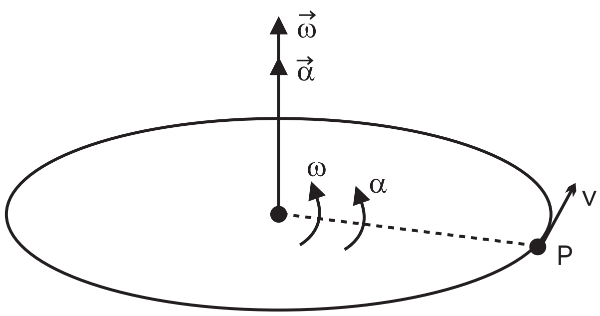
$\Rightarrow \quad v = 0.1 \times 6$ $\boxed{v = 0.6 \text{ m/s}}$



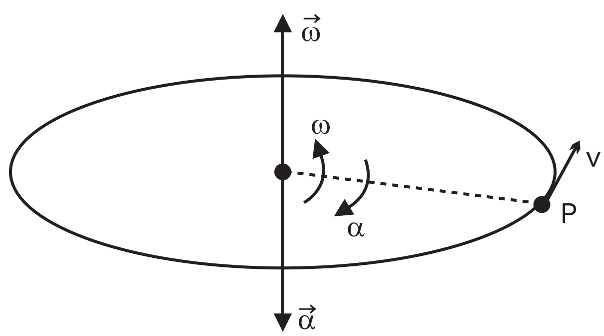
c) **Angular acceleration (α):**
The rate with which angular velocity changes w.r.t time is called angular acceleration.

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

If ω is increasing, then $\vec{\alpha}$ and $\vec{\omega}$ are in same direction.
If ω is decreasing, then $\vec{\alpha}$ and $\vec{\omega}$ are in opposite direction.



(Angular velocity is increasing)



(Angular velocity is decreasing)

Equations of motion for constant angular acceleration:

Just like we have 3 equations of motion for constant acceleration in linear motion, we have 3 equations of motion in circular motion also when angular acceleration is constant.

$$v = u + at \quad \rightarrow \quad \omega = \omega_0 + \alpha t$$

$$s = ut + \frac{1}{2}at^2 \quad \rightarrow \quad \theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$v^2 = u^2 + 2as \quad \rightarrow \quad \omega^2 = \omega_0^2 + 2\alpha\theta$$

Q. **A disc has initial angular speed $\omega_0 = 10\pi$ rad/s. It has angular acceleration of -2π rad/s^2. How many turns are rotated by the disc in 8s?**

Sol. For $\omega_0 = 10\pi$ rad/s, $\alpha = -2\pi$ rad/s^2,

ω becomes zero at t = 5s as $0 = 10\pi - 2\pi t$ $\boxed{t = 5\,s}$

So, direction of rotation reverses at t = 5s.

For 0 < t < 5s,

$$\theta_1 = \omega_0(5) + \frac{1}{2}(-2\pi)5^2$$

$$= 10\pi(5) - 25\pi = 50\pi - 25\pi = 25\pi$$

$$n_1 = \frac{\theta_1}{2\pi} = \frac{25\pi}{2\pi} = 12.5$$

For 5 < t < 8s,

$$\theta_2 = \frac{1}{2}(2\pi)(8-5)^2 = 9\pi$$

$$n_2 = \frac{9\pi}{2\pi} = 4.5$$

Total no. of rotations = 12.5 + 4.5 = 17

Calculus form of equation of motion:

$$\omega = \frac{d\theta}{dt} \quad , \quad v = \frac{dx}{dt}$$

$$\alpha = \frac{d\omega}{dt} \quad , \quad a = \frac{dv}{dt}$$

$$\alpha = \omega \cdot \frac{d\omega}{d\theta} \quad , \quad a = v \cdot \frac{dv}{dx}$$

Q. **A particle moves with constant speed in a circular path. What is the ratio of average velocity to its instantaneous velocity when the particle describes an angle $\theta = \frac{\pi}{2}$.**

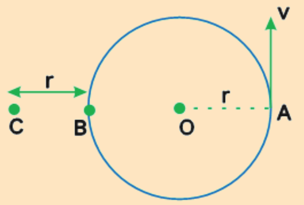
Sol. Time taken to describe angle θ, $t = \frac{\theta}{\omega} = \frac{\theta R}{v} = \frac{\pi R}{2v}$

$$v_{avg} = \frac{\text{Total displacement}}{\text{Total time}} = \frac{\sqrt{2}R}{\pi R / 2v} = \frac{2\sqrt{2}}{\pi} v$$

Instantaneous velocity = v

Required ratio of average velocity to its instantaneous velocity is, $\frac{v_{avg}}{v_{ins}} = \frac{2\sqrt{2}}{\pi}$

Q. **A particle is moving with constant speed in a circle as shown. Find the angular velocity of the particle A with respect to fixed point B and C if angular velocity with respect to O is ω.**



Sol. A particle is moves with constant speed in a circle as shown in the figure. What is the angular velocity of the particle A wrt fixed point B and C if angular velocity wrt point O is ω.

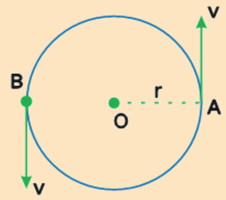
$$\omega_{AO} = \frac{(v_{AO})_{\perp}}{r_{AO}} = \frac{v}{r} = \omega$$

Similarly, we have

$$\omega_{AB} = \frac{(v_{AB})_{\perp}}{r_{AB}} = \frac{v}{2r} = \frac{\omega}{2}$$

and $\omega_{AC} = \frac{(v_{AC})_{\perp}}{r_{AC}} = \frac{v}{3r} = \frac{\omega}{3}$

Q. **Two particles A and B move with constant and equal speeds in a circle as shown in the figure. What is the angular velocity of the particle A wrt B, if angular velocity of particle A w.r.t. O is ω.**



Sol. Angular velocity of A with respect O is

$$\omega_{AO} = \frac{(v_{AO})_\perp}{r_{AO}} = \frac{v}{r} = \omega$$

Now, $$\omega_{AB} = \frac{(v_{AB})_\perp}{r_{AB}}$$

$\Rightarrow$ $v_{AB} = 2v$

Since v_{AB} is perpendicular to r_{AB}

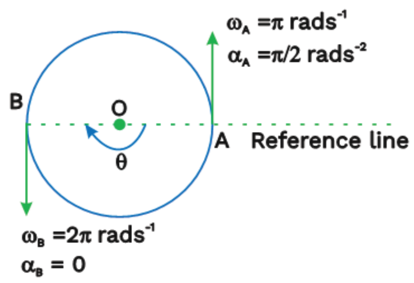
$\Rightarrow$ $(v_{AB})_\perp = v_{AB} = 2v$

$r_{AB} = 2r$

$$\Rightarrow \omega_{AB} = \frac{(v_{AB})_\perp}{r_{AB}} = \frac{2v}{2r} = \omega$$

Q. **Two particles A and B move on a circle. Initially, the particles A and B are diagonally opposite to each other. Particle A moves with angular velocity π rads^{-1}, angular acceleration and particle B moves with constant angular velocity 2π rads^{-1}. Find the time after which both the particles A and B will collide.**

Sol. Suppose angle between OA and OB is θ, then the rate of change of θ is called angular velocity.



With respect to the point A, we have

$$\omega_{BA} = \omega = \omega_B - \omega_A = 2r - \pi = \pi \text{ rad s}^{-1}$$

$$\alpha_{BA} = \alpha = \alpha_B - \alpha_A = -\frac{\pi}{2} \text{ rad s}^{-2}$$

If angular displacement is $\Delta\theta$, then by equation of motion,

$$\Delta\theta = \omega t + \frac{1}{2}\alpha t^2$$

For collision between A and B, angular displacement is given by

$$\Delta\theta = \pi$$

$$\Rightarrow \pi = \pi t + \frac{1}{2}\left(\frac{-\pi}{2}\right)t^2$$

$$\Rightarrow \quad t^2 - 4t + 4 = 0$$

$$\Rightarrow \quad t = 2\text{sec}$$

Q. **A fan rotating with ω = 100 rads^{-1}, is switched off. After 2n rotation, its angular velocity becomes 50 rads^{-1}. Calculate the angular velocity of the fan after n rotations.**

Sol. $\omega^2 = \omega_0^2 + 2\alpha\theta$

$$\Rightarrow 50^2 = (100)^2 + 2\alpha(2\pi 2n) \qquad \text{...(1)}$$

If angular velocity after n rotation is ω_n, then

$$\omega_n^2 = (100)^2 + 2\alpha(2\pi n) \qquad \text{...(2)}$$

From equation (1) and (2), we get

$$\frac{50^2 - 100^2}{\omega_n^2 - 100^2} = \frac{2\alpha(2\pi 2n)}{2\alpha 2\pi n} = 2$$

$$\Rightarrow \omega_n^2 = \frac{50^2 + 100^2}{2}$$

$$\Rightarrow \omega = 25\sqrt{10} \text{ rads}^{-1}$$

Q. **If angular displacement of a particle is given by θ = a – bt + ct², then find its angular velocity and angular acceleration.**

Sol. Angular velocity, $\omega = \frac{d\theta}{dt} = (-b + 2ct)\,\text{rad s}^{-1}$

Angular acceleration, $\alpha = \frac{d\omega}{dt} = (2c)\,\text{rad s}^{-2}$

Radius of Curvature

To find radius curvature at any point on a curved path, we can use the formula,

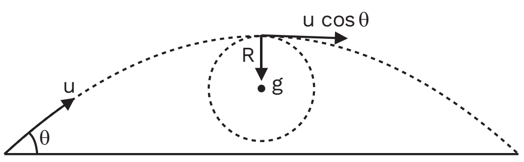
$$R_c = \frac{(\text{Speed})^2}{\text{Component of acceleration} \perp \text{er to velocity}}$$

The radius of cuvature of a curve at a particular point is defined as the radius of the approximately circle at that point.

Q. **A particle is thrown with speed u at an angle θ above horizontal. Find radius of curvature at highest point of trajectory. Also, find radius of curvature at point of projection.**

Sol. Speed at highest point is u cos θ

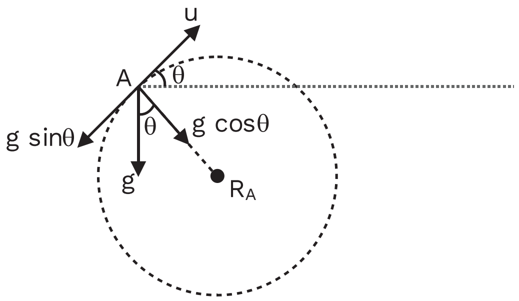
So, $R = \frac{(u\cos\theta)^2}{g} = \frac{u^2\cos^2\theta}{g}$



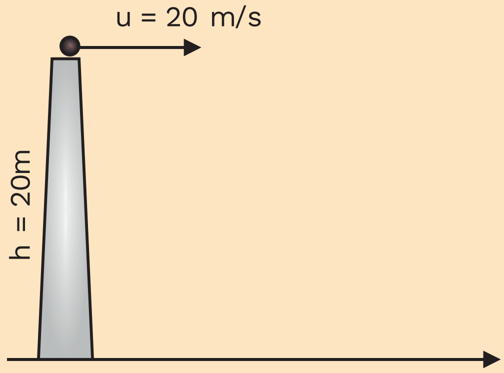
At the point of projection A, $R_A = \frac{u^2}{g\cos\theta}$

If equation of path is given in terms of x and y, then,
Radius of curvature at any point can be found using following formula,

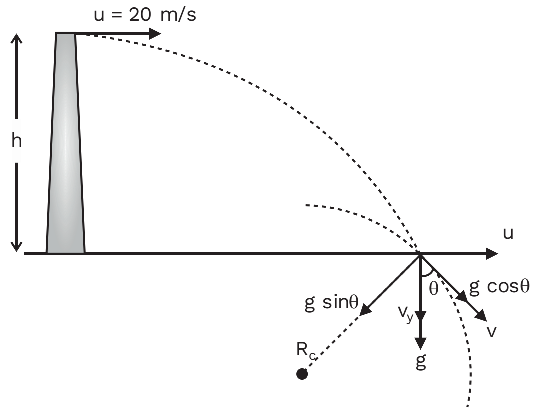
$$R = \left|\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}\right|$$



Q. **A ball is projected horizontally from top of a vertical tower of height 20m with speed 20 m/s. Find the radius of curvature at the point of landing**



Sol. $v_y = \sqrt{2gh} = \sqrt{2 \times 10 \times 20} = 20 \text{ m/s}$



$u = 20 \text{ m/s}$

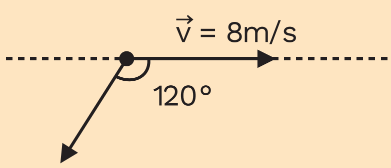
$v = \sqrt{u^2 + v_y^2} = \sqrt{20^2 + 20^2} = 20\sqrt{2} \text{ m/s}$

$\tan\theta = \frac{u}{v} = \frac{20}{20} = 1 \quad \Rightarrow \quad \boxed{\theta = 45^\circ}$

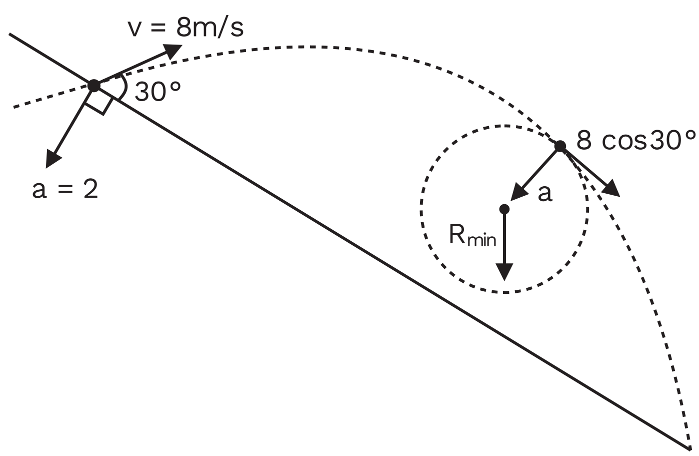
So, radius of curvature at landing point,

$$R_c = \frac{v^2}{g\sin\theta} = \frac{(20\sqrt{2})^2}{10 \times \sin 45^\circ} = \frac{800}{10 \times \frac{1}{\sqrt{2}}} = 80\sqrt{2}\,\text{m}$$

Q. **A particle is moving with velocity and acceleration as shown. What will be the minimum value of radius of curvature?**



Sol.



Radius of curvature will be minimum at the point where speed is minimum.

$$R_{min} = \frac{v^2_{min}}{a}$$

$$= \frac{(8\cos 30°)^2}{2}$$

$$= 24 \text{ m}$$

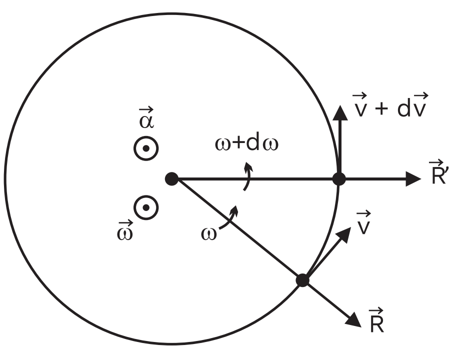
Acceleration in Circular Motion

For any body moving on a curved path, net acceleration of the body has two components.

Net acceleration ($\vec{a}_{net}$) → Radial acceleration (a_R) ; Tangential acceleration (a_T)

So, $\vec{a}_{net} = \vec{a}_R + \vec{a}_T$

We know that, $\vec{v} = \vec{\omega} \times \vec{R}$



Differentiating both sides w.r.t t,

$$\frac{d\vec{v}}{dt} = \vec{\omega} \times \frac{d\vec{R}}{dt} + \frac{d\vec{\omega}}{dt} \times \vec{R}$$

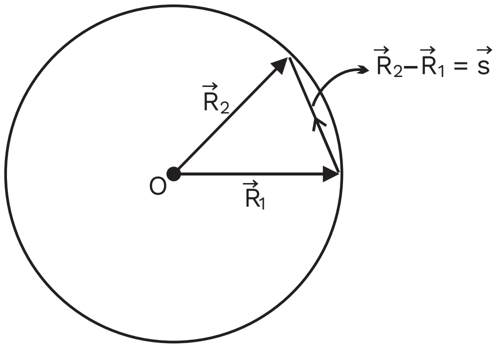
$$\vec{a}_{net} = \left(\vec{\omega} \times \frac{d\vec{R}}{dt}\right) + \left(\frac{d\vec{\omega}}{dt} \times \vec{R}\right)$$

Let's consider a particle is speeding up on a circular path as shown.
Then,

$$\frac{d\vec{\omega}}{dt} = \vec{\alpha}$$

As $\vec{\alpha}$ is outwards the plane of paper and $\vec{R}$ is away from centre,
then, $\vec{\alpha} \times \vec{R}$ will be tangential.
So, tangential acceleration, $\vec{a}_T = (\vec{\alpha} \times \vec{R})$

or $\boxed{a_T = R\alpha}$



Now, the magnitude of $\vec{R}$ is constant but its direction is changing with the motion of particle on circular path,
So, $\vec{R}$ is a not a constant vector.

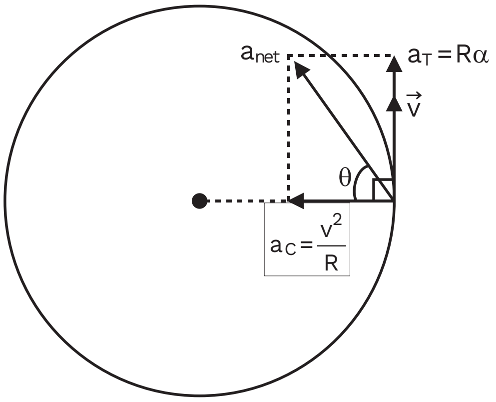
So, $\vec{v} = \frac{\vec{s}}{t} = \frac{\vec{R}_2 - \vec{R}_1}{t} = \frac{d\vec{R}}{dt}$

$\therefore$ Radial acceleration, $\vec{a}_R = \vec{\omega} \times \vec{v}$

Now, $(\vec{\omega} \times \vec{v})$ will be towards the centre.

Thus, radial acceleration is always towards the centre.

$$a_R = \omega v = \left(\frac{v}{R}\right) v = \frac{v^2}{R} = \omega^2 R$$



For uniform circular motion,
v = constant

So, $\frac{dv}{dt} = 0 \Rightarrow \boxed{a_T = R\alpha = 0}$

$\boxed{a_c \neq 0}$

For non-uniform circular motion,

$$a_{net} = \sqrt{a_c^2 + a_T^2}$$

$$\tan\theta = \frac{a_T}{a_c}$$

- When particle is speeding up, $\vec{v}$ and $\vec{a}_T$ are in same direction. Angle between $\vec{v}$ and $\vec{a}_{net}$ is acute.
- When particle is slowing down, $\vec{v}$ and $\vec{a}_T$ are in opposite direction. Angle between $\vec{v}$ and $\vec{a}_{net}$ is obtuse.

Q. **A body moves on a circular path of radius 20 cm at a speed that uniformly increases. If the speed changes from 5 ms^{-1} to 6 ms^{-1} in 2 s, find the angular acceleration.**

Sol. As the speed is increasing uniformly, then the average tangential acceleration is equal to instantaneous tangential acceleration.
The instantaneous tangential acceleration is given by

$$a_T = \frac{dv}{dt} = \frac{v_2 - v_1}{t_2 - t_1}$$

$$\Rightarrow a_T = \frac{6-5}{2} ms^2 = 0.5\,ms^{-2}$$

The angular acceleration, $\alpha = \frac{a_T}{r}$.

$$\Rightarrow \alpha = \frac{0.5\,ms^{-2}}{20\,cm} = \frac{0.5\,ms^{-2}}{0.20\,m} = 2.5\,rads^{-2}$$

Q. **What is the magnitude of the acceleration of a particle which is moving on a circular path of radius 10 cm with uniform speed completing the circle in 4s.**

Sol. The distance covered in one full circle is $2\pi r = 2\pi \times 10$ cm.

The linear speed is $v = \frac{2\pi r}{T} = \frac{2\pi \times 10\,cm}{4s} = 5\pi\,cms^{-1}$

The acceleration is $a_c = \frac{v^2}{r} = \frac{5(\pi\,cms^{-1})^2}{10\,cm} = 2.5\pi^2\,cms^{-2}$

As the speed is uniform, so tangential acceleration is zero. Total acceleration is same as centripetal acceleration.

Q. **A particle is moving in a circle of radius 2 m at a speed given by v = 4t, where v is in ms^{-1} and t is in seconds.**
(a) Calculate the tangential acceleration at t = 1 s.
(b) Find total acceleration at t = 1 s.

Sol. Tangential acceleration

$$a_T = \frac{dv}{dt}$$

$$\Rightarrow a_T = \frac{d}{dt}(4t) = 4\ ms^{-2}$$

$$\Rightarrow a_C = \frac{v^2}{R} = \frac{(4)^2}{2} = 8\,ms^{-2}$$

Total acceleration, $a = \sqrt{a_T^2 + a_c^2} = \sqrt{(4)^2 + (8)^2}$

$$\Rightarrow a = \sqrt{16+64} = \sqrt{80} = 4\sqrt{5}\ ms^{-2}$$

Q. **A particle is moving with a constant angular acceleration of 4 rads^{-2} on a circular path. Initially the particle was at rest. At what time, the magnitudes of centripetal acceleration and tangential acceleration are equal**

Sol. Tangential acceleration, $a_t = \alpha R$

By equation of motion, $v = 0 + \alpha RT$

Centripetal acceleration, $a_c = \frac{v^2}{R} = \frac{\alpha^2 R^2 t^2}{R}$

On equating both the accelerations, $|a_t| = |a_c|$

$$\Rightarrow \alpha R = \frac{\alpha^2 R^2 t^2}{R}$$

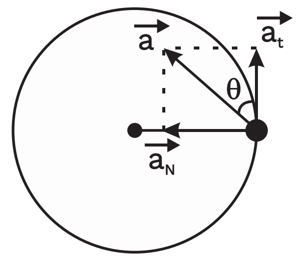
$$\Rightarrow t^2 = \frac{1}{\alpha} = \frac{1}{4}$$

$$\Rightarrow t = \frac{1}{2}s$$

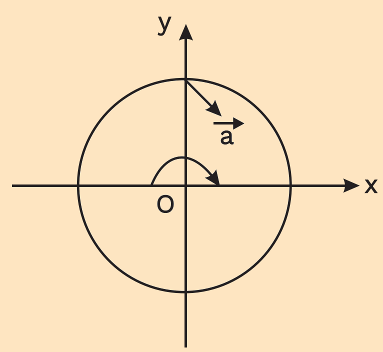
Q. **For a particle moving on a circular path, its acceleration vector is making an angle of 30° with the velocity vector, then the ratio of centripetal acceleration to its tangential acceleration is.**

Sol. $\tan\theta = \dfrac{a_N}{a_t}$

$$\Rightarrow \frac{a_N}{a_t} = \tan(30^\circ) = \frac{1}{\sqrt{3}}$$



Q. **A body is moving is x—y plane as shown in a circular path of radius 2 m. At a certain instant when the body is crossing the positive y-axis its acceleration is $(6\hat{i} - 8\hat{j})\,\text{ms}^{-2}$. Then its angular acceleration and angular velocity at this instant will be.**



Sol. Since $\vec{a} = 6\hat{i} - 8\hat{j}$

$\Rightarrow \quad a_c = 8$ and $a_T = 6$

$\Rightarrow \quad r\omega^2 = 8$ and $r\alpha = 6$

$$\Rightarrow \omega = \sqrt{\frac{8}{2}} = 2\,\text{rads}^{-1} \text{ and } \alpha = \frac{6}{2} = 3\,\text{rads}^{-2}$$

Since the body is rotating in clockwise since, so by using Right Hand Thumb Rule, we get

$\vec{\omega} = -2\hat{k}\,\text{rads}^{-1}$ and $\vec{\alpha} = -3\hat{k}\,\text{rads}^{-2}$.

Q. **A solid body rotates with deceleration about a stationary axis with an angular deceleration $|\alpha| = k\sqrt{\omega}$, where ω is its angular velocity and k is a positive constant. Calculate the average angular velocity of the body averaged over the whole time of rotation if at the initial time instant, its angular velocity was equal to ω_0.**

Sol. $|\alpha| = k\sqrt{\omega}$

$$\Rightarrow -\frac{d\omega}{dt} = k\sqrt{\omega}$$

$$\Rightarrow \int_{\omega_0}^{\omega} \frac{d\omega}{\sqrt{\omega}} = -\int_0^t k\,dt$$

$$\Rightarrow \left[2\sqrt{\omega}\right]_{\omega_0}^{\omega} = -kt$$

$$\Rightarrow \sqrt{\omega_0} - \sqrt{\omega} = \frac{kt}{2}$$

$$\Rightarrow \omega = \left(\sqrt{\omega_0} - \frac{kt}{2}\right)^2$$

The body will stop when $\sqrt{\omega_0} - \dfrac{kt}{2} = 0$

$$\Rightarrow t = \frac{2\sqrt{\omega_0}}{k}$$

Now average angular velocity over this time interval is

$$<\omega> = \frac{\int_0^{\frac{2\sqrt{\omega_0}}{k}} \omega\,dt}{\int_0^{\frac{2\sqrt{\omega_0}}{k}} dt} = \frac{\int_0^{\frac{2\sqrt{\omega_0}}{k}} \left(\sqrt{\omega_0} - \frac{kt}{2}\right)^2 dt}{\int_0^{\frac{2\sqrt{\omega_0}}{k}} dt} = \frac{\omega_0}{3}$$

Q. **The speed (v) of a particle moving in a circle of radius R varies with distance s as v = ks, where k is a positive constant. Calculate the total acceleration of the particle.**

Sol. Speed, v = ks

As the particle moves in a circle, so total acceleration, a is

$$a = \sqrt{a_C^2 + a_T^2}$$

Where $a_C = \frac{v^2}{R}$ and $a_T = \frac{dv}{dt}$

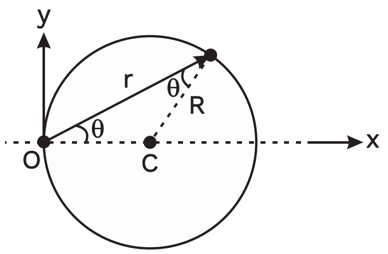
$$\Rightarrow a_C = \frac{k^2s^2}{R} \text{ and } a_T = \frac{d}{dt}(ks) = k\left(\frac{ds}{dt}\right) = kv$$

$$\Rightarrow a_C = \frac{k^2s^2}{R} \text{ and } a_T = k(ks) = k^2s\left\{\because \frac{ds}{dt} = v = ks\right\}$$

So, total acceleration, $a = \sqrt{\frac{k^4s^4}{R^2} + k^4s^2} = k^2s\sqrt{1 + \frac{s^2}{R^2}}$

Q. **A particle P moves along a circle of radius R so that its radius vector $\vec{r}$, relative to the point O at the circumference rotates with constant angular velocity ω. Find the magnitude of the velocity of the particle and the direction of its total acceleration.**

Sol. In triangle OPC, by Sine law,



$$\frac{r}{\sin(\pi - 2\theta)} = \frac{R}{\sin\theta}$$

$$\Rightarrow \frac{r}{2\sin\theta\cos\theta} = \frac{R}{\sin\theta}$$

$\Rightarrow$ r = 2R cos θ

Further we can see that,

$$\vec{r} = (r\cos\theta)\hat{i} + (r\sin\theta)\hat{j}$$

$$\Rightarrow \vec{r} = (2R\cos^2\theta)\hat{i} + (2R\sin\theta\cos\theta)$$

Since $\vec{v} = \frac{d\vec{r}}{dt}$

$$\Rightarrow \vec{v} = -\left(4R\cos\theta\sin\theta\frac{d\theta}{dt}\right)\hat{i} + \left(2R\cos(2\theta)\frac{d\theta}{dt}\right)\hat{j}$$

Since $\dfrac{d\theta}{dt} = \omega$

$$\Rightarrow \vec{v} = -2R\omega\left[-\sin\left(2\theta\right)\hat{i} + \cos\left(2\theta\right)\hat{j}\right]$$

$$\Rightarrow |\vec{v}| = 2R\omega$$

Now, we know that $\vec{a} = \dfrac{d\vec{v}}{dt}$

$$\Rightarrow \vec{a} = 4R\omega^2(\cos(2\theta)\hat{i} + \text{sub}(2\theta)\hat{j})$$

$$\Rightarrow |\vec{a}| = 4R\omega^2$$

Dynamics of Circular Motion

For motion on a circular path,
Net force towards the centre of circle, $F_C = m.a_R$
where a_R = Radial acceleration
Radial acceleration is also known as centripetal acceleration as it is always directed towards the centre of the circular path.
The force F_C is known as centripetal force.
A centripetal force must be present for motion to be circular.
Net force tangential to the circular path, $F_T = ma_T$
where a_T = Tangential acceleration

- For uniform circular motion,

  $a_T = 0$

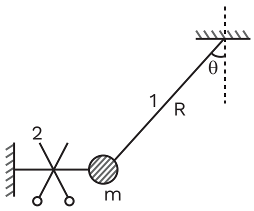
  So, $F_T = 0$

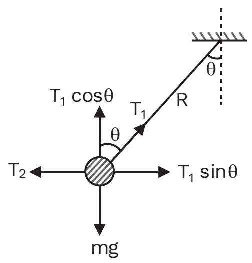
- For non-uniform circular motion,

  $a_T \neq 0$

  So, $F_T \neq 0$

Consider a system as shown in figure below. If the string 2 is cut, find the tangential and centripetal acceleration of the ball just after cutting the string. If after cutting, the ball acquires a velocity v at the bottom-most point, find the value of centripetal and tangential acceleration.





For initial equilibrium,

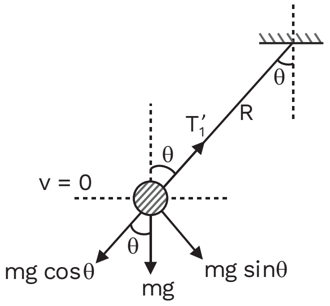
$T_1 \cos\theta = mg$ and, $T_1 \sin\theta = T_2$

Just after cutting the string 2, tension T_2 disappears, and tension T_1 becomes T'_1.
At this instant, v = 0

So, $T_1' = mg\cos\theta$

and, $ma_T = mg\sin\theta$

$\therefore$ $a_C = 0$, $a_T = g\sin\theta$



Now, when the ball reaches its bottom-most point, it has acquired a velocity v.

At this point, $T - mg = ma_C$

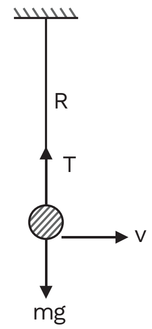
There is no tangential force,

So, $a_T = 0$

$$a_C = \frac{v^2}{R}$$

Then, tension in string at lowest point,

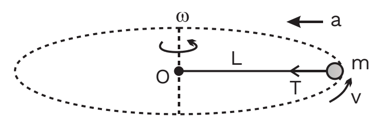
$$T = \left(mg + \frac{mv^2}{R}\right)$$



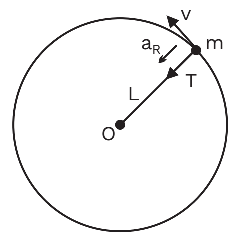
Free-Body Diagram in Circular Motion

Consider a bob of mass m is being rotated with angular velocity ω by attaching it to a string of length L.

For the circular motion, there should be an acceleration towards the centre,



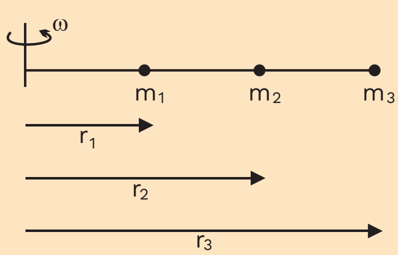
$$a = a_c = \frac{v^2}{L} = \frac{(L\omega)^2}{L} = L\omega^2$$



(Top view)

$$\text{Tension in string, } T = ma = \frac{mv^2}{L} = mL\omega^2$$

Q. **Consider the given system of masses in uniform circular motion. Find tension in the strings.**



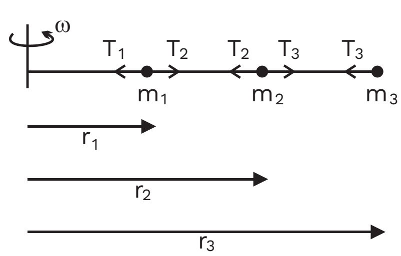
Sol. For mass m_3, $T_3 = m_3\,(r_3\omega^2)$

For mass m_2, $T_2 - T_3 = m_2 r_2\,\omega^2$

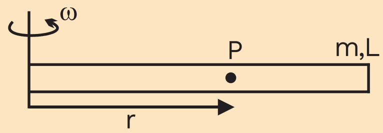
$T_2 = \omega^2\,[m_2 r_2 + m_3 r_3]$

For mass m_1, $T_1 - T_2 = m_1 r_1\,\omega^2$

$T_1 = \omega^2\,[m_1 r_1 + m_2 r_2 + m_3 r_3]$

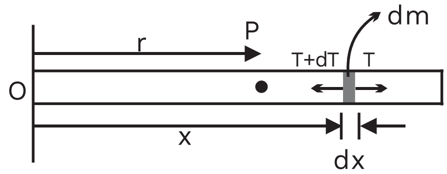


Q. **A straight rod of mass m and length L is rotated about its one end with constant angular velocity ω. Find tension at a point at distance r from the point of rotation**



Sol. Tension at a point during the rotation is due to outer mass only. So, tension at free end will be zero.

Consider an infinitesimal element of mass dm and length dx at a distance x as shown.



$$dm = \left(\frac{m}{L}\right)dx$$

For the dm element,

$$T + dT - T = dm \cdot x \cdot \omega^2$$

$$dT = \left(\frac{m}{L}\right)dx \cdot x \cdot \omega^2$$

$$T_r = \int_r^L dT = \frac{m\omega^2}{L}\int_r^L x \cdot dx$$

$$= \frac{m\omega^2}{L}\left[\frac{x^2}{2}\right]_r^L$$

$$\boxed{T_r = \frac{m\omega^2}{2L}\left(L^2 - r^2\right)}$$

Tension will be maximum at point O,

At O, r = 0

$$T_O = \frac{m\omega^2 L^2}{2L}$$

Tension will be minimum at free end,

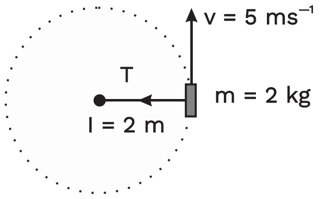
At free end, r = L

$$T_e = \frac{m\omega^2}{2L}\left(L^2 - L^2\right) = 0$$

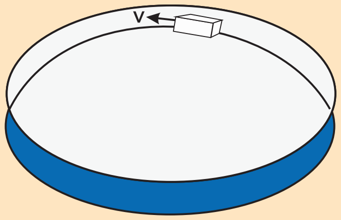
Q. **A block of mass 2 kg is attached to a string having a length of 2m and the other end of the string is fixed. The block is being moved on a smooth horizontal table with constant speed 5 ms^{-1}. Find the tension in the string.**

Sol. In this case, the centripetal force necessary for rotation is provided by tension, so,

$$T = \frac{mv^2}{r} = \frac{2 \times 5^2}{2} = 25\,N$$



Q. **A block of mass m is moving with speed v against a smooth, fixed vertical circular groove having radius r kept on smooth horizontal surface.**



Find the
(a) Normal reaction applied by the floor on the block..
(b) Normal reaction applied by the vertical wall on the block.

Sol. Here, normal reaction from the vertical wall provides the necessary centripetal force.

(a) Normal reaction due to the floor, $N_F = mg$

(b) Normal reaction due to the vertical wall $N_W = \frac{mv^2}{r}$

Q. **A horizontal turn table rotating with a constant angular velocity ω has a block of mass m kept on its edge. It is rotating along with the block about its axis. If the coefficient of friction is μ, find the friction force between block and table if the block is at rest with respect to table.**

Sol. The centripetal force is provided by friction force in this case.

Friction force = centripetal force = $mR\omega^2$

Q. **A simple pendulum is made constructed by attaching a bob of mass m to a string of length L fixed at its upper end. The bob oscillates in a vertical circle. If is found that the speed of the bob is v when the string makes an angle θ with the vertical, then what is the tension in the string and the magnitude of net force on the bob at the instant.**

Sol. (a) The forces acting on the bob are

(i) the tension T

(ii) the weight mg

As the bob moves in a circle of radius L with centre of O,

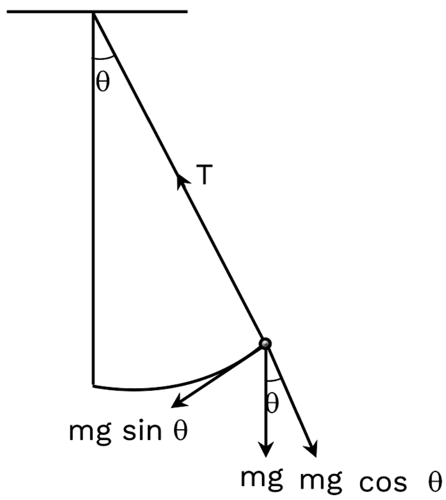
a centripetal force of magnitude $\frac{mv^2}{L}$ is required towards O.

This force will be provided by the resultant of T and

mg $\cos\theta$. Thus,

$$T - mg\cos\theta = \frac{mv^2}{L}$$

$$\Rightarrow T = m\left(g\cos\theta + \frac{v^2}{L}\right)$$



(b) Since a_c is provided by (T — mg cos θ) acting radially inwards, so

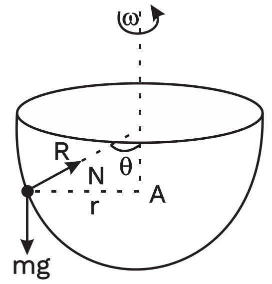
$$a_c = \frac{T - mg\cos\theta}{m} = \frac{v^2}{l}.$$

$$a_{net} = \sqrt{a_T^2 + a_c^2} = \sqrt{(g\sin\alpha)^2 + \left(\frac{v^2}{l}\right)^2}$$

$$|\vec{F}_{net}| = ma_{net} = m\sqrt{g^2\sin^2\alpha + \frac{v^4}{L^2}}$$

Q. **A hemispherical bowl of radius R is rotating about its axis of symmetry which is kept vertical. A small ball kept in the bowl rotates with the bowl without slipping on its surface. It the surface of the bowl is smooth and the angle made by the radius through the ball with the vertical is θ. Find the angular speed at which the bowl is rotating.**

Sol. Let ω be the angular speed of rotation of the bowl. Two forces are acting on the ball.



(a) Normal reaction (N)

(b) Weight (mg)

The ball is rotating in a circle of radius $r(= R\sin\theta)$ with centre at A at an angular speed ω. Thus,

$N\sin\theta = mr\omega^2 = mR\omega^2 \sin\theta$

$\Rightarrow N = mR\omega^2$

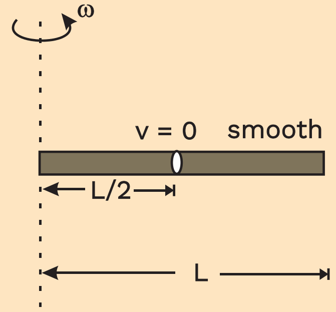
and $N\cos\theta = mg$

Dividing (1) by (2), we get

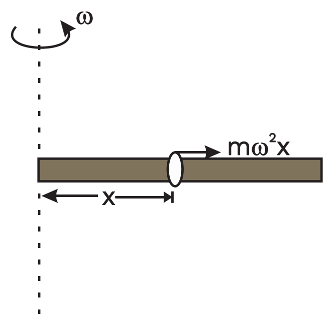
$$\frac{1}{\cos\theta} = \frac{\omega^2 R}{g}$$

$$\Rightarrow \omega = \sqrt{\frac{g}{R\cos\theta}}$$

Q. **A ring which can slide along the rod is kept at midpoint of a smooth rod of length L. The rod is rotated with constant angular velocity ω about vertical axis passing through its one end. Ring is released from mid-point. Find the velocity of the ring when it just leaves the rod.**



Sol. Rearranging and integrating, we get



$$\int_{L/2}^{L} \omega^2 x\,dx = \int_0^v v\,dv$$

$$\Rightarrow \omega^2\left(\frac{x^2}{2}\right)_{L/2}^{L} = \left(\frac{v^2}{2}\right)_0^v$$

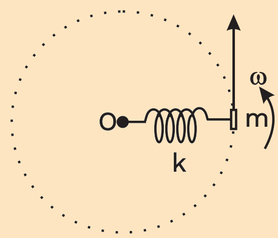
$$\Rightarrow \omega^2\left(\frac{L^2}{2}-\frac{L^2}{8}\right) = \frac{v^2}{2}$$

$$\Rightarrow v = \frac{\sqrt{3}}{2}\omega L$$

Velocity at time of leaving the rod is the resultant of tangential speed and the radial speed of the particle. So,

$$v' = \sqrt{(\omega L)^2 + \left(\frac{\sqrt{3}}{2}\omega L\right)^2} = \frac{\sqrt{7}}{2}\omega L$$

Q. **A block of mass m is tied to a spring of spring constant k, natural length l and the other end of spring is fixed at O. If the block moves in a circular path on a smooth horizontal surface with constant angular velocity ω, find tension in the spring.**



Sol. Assume extension in the spring is x.
Here centripetal force is provided by spring force.

$$kx = m\omega^2(l + x)$$

$$\Rightarrow \quad x = \frac{m\omega^2 l}{k - m\omega^2}$$

$$\Rightarrow \quad \text{Tension } = kx = \frac{km\omega^2 l}{k - m\omega^2}$$

Q. **A block of mass m is kept on rough horizontal turn table at a distance r from centre of table. Coefficient of friction between turn table and block is μ. Now turn table starts rotating with uniform angular acceleration α.**
(a) Find the time after which slipping occurs between block and turn table.
(b) Find angle made by friction force with velocity at the point of slipping.

Sol. (a) $a_t = \alpha r$

$$\frac{dv}{dt} = \alpha r$$

Speed after t time, $v = 0 + \alpha rt$

Centripetal acceleration

$$a_c = \frac{v^2}{r} = \alpha^2 rt^2$$

Net acceleration $a_{net} = \sqrt{a_t^2 + a_c^2}$

$$\Rightarrow a_{net} = \sqrt{\alpha^2 r^2 + \alpha^2 r^2 t^4}$$

When the block just starts slipping, $\mu mg = ma_{net} = m\sqrt{\alpha^2 r^2 + \alpha^4 r^2 t^4}$

$$\Rightarrow t = \left(\frac{\mu^2 g^2 - \alpha^2 r^2}{\alpha^4 r^2}\right)^{1/4}$$

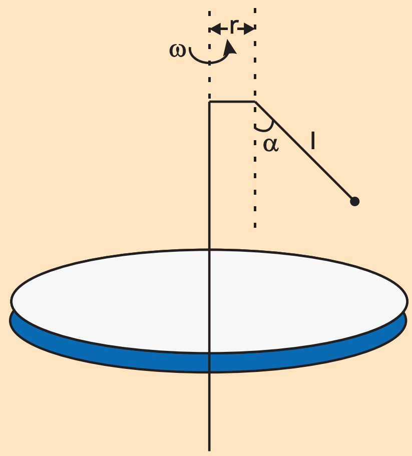
$$\Rightarrow t = \left[\left(\frac{\mu g}{\alpha^4 r}\right)^2 - \left(\frac{1}{\alpha}\right)^2\right]^{1/4}$$

(b) $\tan\theta = \frac{a_c}{a_t}$

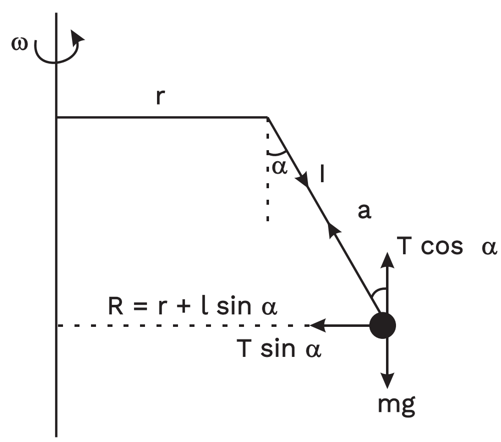
$$\Rightarrow \tan\theta = \frac{\alpha^2 rt^2}{\alpha r}$$

$$\Rightarrow \theta = \tan^{-1}(\alpha t^2)$$

Q. **A plumb-line is set up on rotating disk and makes an angle α with the vertical, as in figure. The distance r from the point of suspension to the axis of rotation is known, and so is the length l of the thread. The angular velocity of rotation is.**



Sol.



$$T\cos\alpha = mg$$

$$T\sin\alpha = m(r + l\sin\alpha)\omega^2$$

$$\Rightarrow \omega^2 = \frac{g\tan\alpha}{r + l\sin\alpha}$$

$$\Rightarrow \omega = \sqrt{\frac{g\tan\alpha}{r + l\sin\alpha}}$$

Centrifugal Force

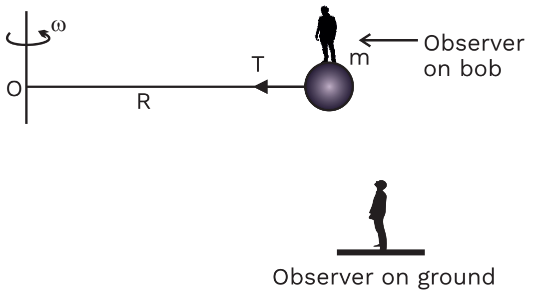
Then, tension in string, T = ma_c

Centrifugal force is pseudo force of circular motion.

Consider a bob of mass m rotating in a circle of radius r with angular speed ω.

For an observer on ground, bob is rotating in circular path. So, it has centripetal acceleration towards centre O.

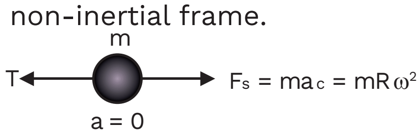
Then, tension in string, T = ma_c



$$\boxed{T = mR\omega^2}$$

For an observer sitting on the bob, the acceleration of bob is zero, but the ground observer says that bob has acceleration a_c.

So, as per the observer on bob, there must be another force balancing the tension. This force is the Pseudo force as the observer on bob is sitting in non-inertial frame.



So, for the observer on bob,

Then, $T - F_s = m(0)$

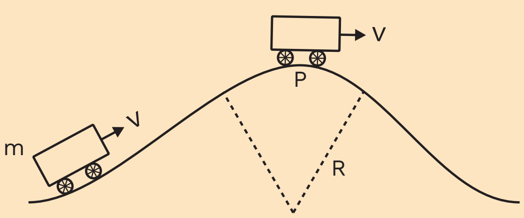
$T = F_s$

$$\boxed{T = mR\omega^2}$$

This force $F_s = mR\omega^2$ is the Pseudo force which is also known as "Centrifugal force".

Centripetal force and centrifugal force cannot be used together at same time, as centripetal force is from ground frame and centrifugal force is from reference frame of rotating body.

Q. **A cart of mass m is moving on the given path. Find the normal reaction at point P.**



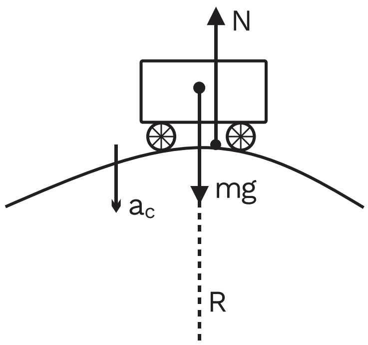
Sol.

- From ground frame

  Net force towards centre = m × a_c

$$mg - N = \frac{mv^2}{R}$$

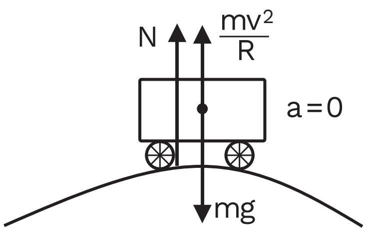
$$\boxed{N = mg - \frac{mv^2}{R}}$$



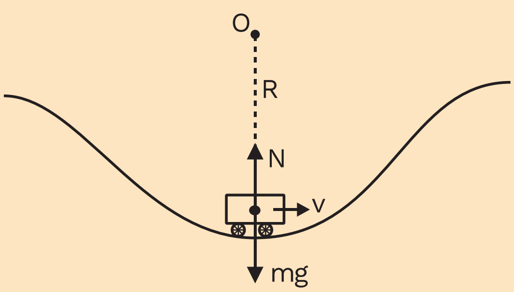
- From frame of reference of cart :

$$N + \frac{mv^2}{R} = mg$$

$$\boxed{N = mg - \frac{mv^2}{R}}$$



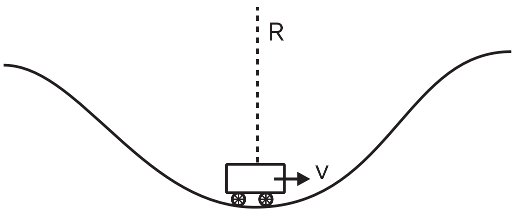
Q. **Find normal reaction on the cart at given position.**



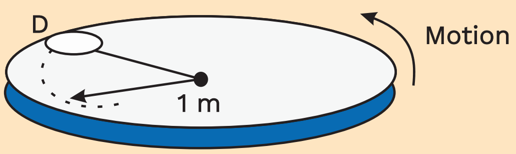
Sol. Net force towards centre = ma_c

$$N - mg = \frac{mv^2}{R}$$

Normal reaction, $N = mg + \frac{mv^2}{R}$



Q. **The 4 kg disk D is attached to the end of cord as shown in figure. The other end of the cord is tied at the centre of a platform. If the platform is rotating rapidly and the disk is placed on it and released from rest as shown, determine the time, in seconds, it takes for the disk to reach a speed great enough to break the cord. The maximum tension the cord can sustain is 100 N and the coefficient of kinetic friction between the disk and the platform is** $\mu_K = 0.1$**. Take** $g = 10\,ms^{-2}$**.**



Sol. Tension acts as centripetal force for the disk.

$$\text{Maximum tension, } T = 100 = \frac{mv^2}{r} = \frac{(4)(v^2)}{(1)}$$

$$\Rightarrow v = 5\,ms^{-1}$$

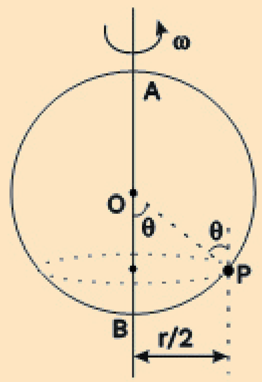
Now, using the equation of motion, v=u+at

$$5 = 0 + \left(\frac{\mu mg}{m}\right)t$$

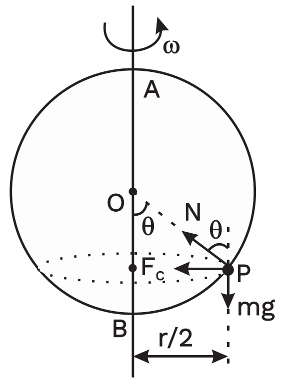
$$5 = \mu g t$$

$$\Rightarrow t = \frac{5}{(0.1)(10)} = 5\,s$$

Q. **A smooth wire of length** $2\pi r$ **is bent into a circle and kept in a vertical plane. A bead can slide smoothly on the wire. When the circle is rotating with angular speed** ω **about the vertical diameter AB, as shown in figure, the bead is at rest with respect to the circular ring at position P as shown. Then the value of** ω^2 **is equal to**



Sol.



Let θ be the angle with the vertical, then

$$\sin\theta = \frac{r/2}{r} = \frac{1}{2}$$

$$\Rightarrow \theta = \sin^{-1}\left(\frac{1}{2}\right) = 30^\circ$$

$$N\cos\theta = mg \qquad \text{.......} (1)$$

$$N\sin\theta = F_c = \frac{m\omega^2 r}{2} \qquad \text{.......} (2)$$

$$\Rightarrow \tan\theta = \frac{\omega^2 r}{2g}$$

$$\Rightarrow \omega^2 = \frac{2g\tan\theta}{r} = \frac{2g}{\sqrt{3}\,r}$$

Turning of vehicles on Circular Roads

When vehicles go through turnings on curved roads, they travel along a nearly circular arc. To make the circular turn possible, there must be some force which will produce the required centripetal acceleration.

If the vehicles take a turn on a horizontal circular path, the resultant force producing centripetal acceleration is also in horizontal direction. The required necessary centripetal force is being provided to the vehicles in following three ways.

(a) By friction only.

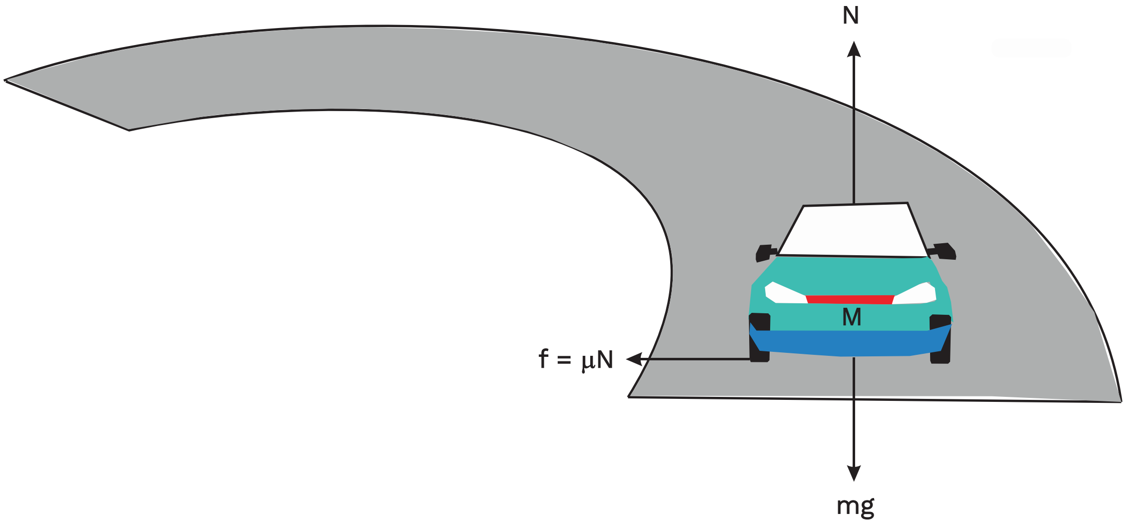
(b) By banking of roads only.

(c) Both by the friction force and banking of roads.

In real life, the centripetal force is provided by both friction force and banking of roads.

By Friction only: Vehicle on a level road

When a vehicle goes around a curved road, it shows a tendency to skid sideways i.e., away from the centre of the curved road. Due to this tendency, the static friction f_s acts towards the centre and provides the necessary centripetal force for motion along the curved path.



(a) The forces acting on the vehicle are
(b) Weight (Mg) acting vertically downward
(c) Normal reaction (N)

Static frictional force (f_s)

The static friction is self adjusting and if μ_s is coefficient of static friction, then $f_s \le \mu_s N$. So, For vertical equilibrium, N = Mg

As frictional force provides the necessary centripetal force, so $f_s = \frac{Mv^2}{r}$

As $f_s \le \mu_s N$

$$\Rightarrow \frac{Mv^2}{r} \le \mu_s N$$

$$\Rightarrow \frac{Mv^2}{r} \le \mu_s Mg$$

$$\Rightarrow v \le \sqrt{\mu_s rg}$$

So, maximum speed for no skidding is

$$\boxed{v_{max} = \sqrt{\mu_s rg}}$$

Q. **A circular curve on a level road has a radius of 100 m. What is the maximum speed which a car turning this bend can have without skidding. Given: μ = 0.6.**

Sol.

$$v_{max} = \sqrt{\mu rg} = \sqrt{0.6 \times 100 \times 10}$$

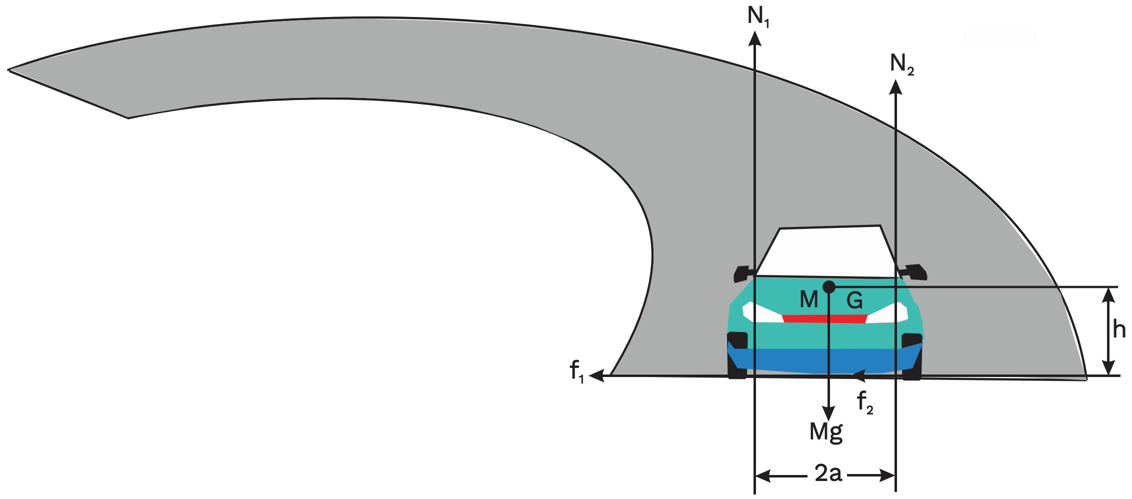
$$\Rightarrow \quad v_{max} = \sqrt{0.6 \times 100 \times 10}$$

$$\Rightarrow \quad v_{max} = \sqrt{600}$$

$$\Rightarrow \quad v_{max} = 10\sqrt{6}\, ms^{-1}$$

Maximum Velocity for Skidding and Overturning

Let A and B be inner and outer wheels of a vehicle moving on a circular track.



The forces acting on the vehicle are-

(a) normal reactions, N_1 and N_2 (vertically upwards)

(b) frictional forces, $f_1 = \mu N_1$ and $f_2 = \mu N_2$

(c) Weight of vehicle, Mg (vertically downwards)

(d) centripetal force F (horizontally towards centre of turn)

For translational equilibrium

$$N_1 + N_2 = Mg \qquad \text{........ (1)}$$

And, total friction force provides the necessary centripetal force

$$F = f_1 + f_2 = \mu N_1 + \mu N_2 \geq \frac{Mv^2}{r}$$

$$\Rightarrow \quad \mu(N_1 + N_2) \geq \frac{Mv^2}{r} \qquad \text{........ (2)}$$

Where r is the radius of the circular path.

Using (1), equation (2) gives.

$$\Rightarrow \quad \mu Mg \geq \frac{Mv^2}{r} \text{ (for no skidding)}$$

$$\Rightarrow \quad v \leq \sqrt{\mu rg}$$

Thus, maximum speed for no skidding is

$$v_{max} = \sqrt{\mu rg} \qquad \text{......... (3)}$$

For No Overturning

If the wheels A and B are at a distance "2a" apart, then taking moments about G, we get

$$N_2 a = N_1 a + Fh, \text{ where } F = \frac{Mv^2}{r}.$$

The car tends to overturn when reaction N_1 on the inner wheel is zero, i.e., when inner wheel leaves contact with the ground. Then, $N_2 . a \geq Fh$

If $N_1 = 0$, then from (1), $N_2 = Mg$

$$\Rightarrow Mga \geq \frac{Mv^2}{r} h$$

$$\Rightarrow v \leq \sqrt{\frac{gra}{h}}$$

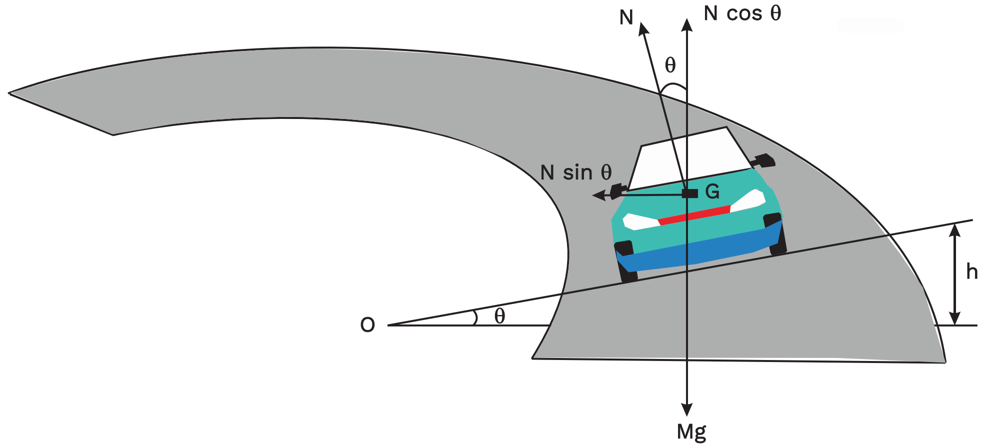
So, the maximum speed for no overturning is

$$v_{max} = \sqrt{\frac{gra}{h}}$$

By Banking of Roads / Tracks

When a vehicle moves round a curve on the road with sufficient speed, then there is a tendency of overturning the vehicle. To avoid this, the road is given a slope rising outwards. The phenomenon is known as banking of roads.

Consider a vehicle on a road having a slope θ. N is the normal reaction of the ground. This may be resolved into two components: A vertical component $N\cos\theta$ which balances the weight of vehicle and a horizontal component $N\sin\theta$ which provides the necessary centripetal force i.e.,



$$N\sin\theta = \frac{Mv^2}{r}$$

And, $N\cos\theta = Mg$

Dividing equation (1) by (2), we get

$$\tan\theta = \frac{v^2}{rg}$$

This equation gives the angle of banking required.

Let l(= OB) be the width of track and h (= AB) be its height.

Assuming θ (the angle of banking to be small), then

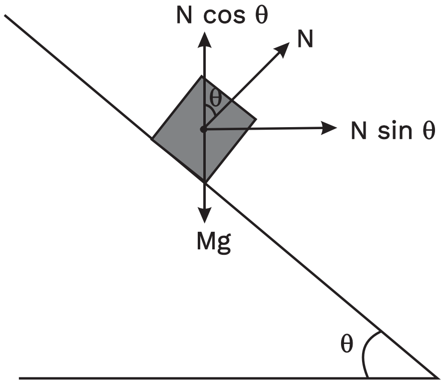
$$\tan\theta = \sin\theta = \frac{h}{l}$$

$$\Rightarrow \frac{h}{l} = \frac{v^2}{rg}$$

$\Rightarrow h = \dfrac{v^2 l}{rg}$, is the height through which outer part of the track has to be raised.

Q. **A circular track of radius 600 m is to be designed for cars at an average speed of 180 kmh^{-1}. What should be the slope of the track?**

Sol. Let the angle of banking be θ. The forces on the car are (shown in figure)
(a) weight of the car (Mg, downwards) and
(b) normal reaction (N).



For proper banking of the road, we do not need the static friction force.
For vertical direction the acceleration is zero. So,

$$N\cos\theta = Mg \qquad \text{.......... (1)}$$

For horizontal direction, the acceleration is $\frac{v^2}{r}$ to towards the centre, so that

$$N\sin\theta = \frac{mv^2}{r} \qquad \text{.......... (2)}$$

From (1) and (2)

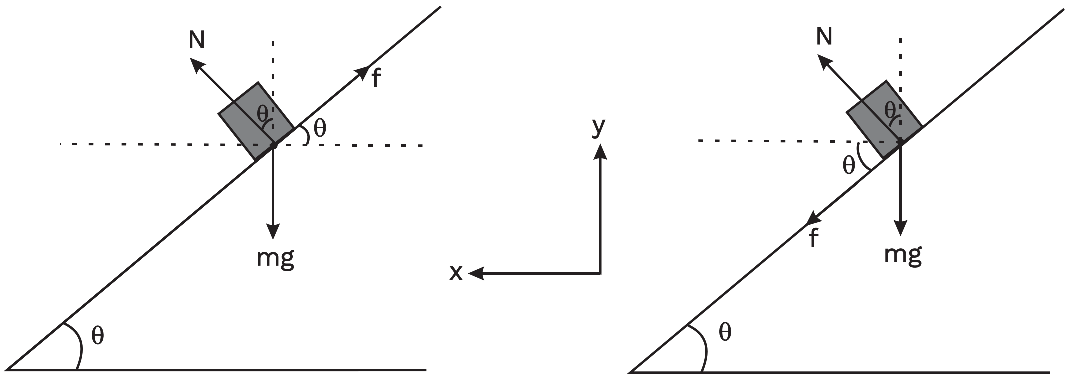
$$\tan\theta = \frac{v^2}{rg}$$

Substituting the values

$$\tan\theta = \frac{180^2\ km^2h^{-2}}{600\,m \times 10\,ms^{-2}} = \frac{\left(180 \times \frac{5}{18}\right)^2 \left(ms^{-1}\right)^2}{600\,m \times 10\,ms^{-2}} = 0.4167$$

Slope= $\tan\theta = 0.4167$

By both Friction and Banking of Road
If the vehicle moves on a circular road which is rough and banked also, then three forces may act on the vehicle. Out of these three forces, the weight (mg) is fixed in magnitude as well as in direction.



The direction of second force, i.e., normal reaction N is also fixed (perpendicular to road).

The direction of the third force i.e., friction f can be either inwards or outwards and its magnitude can be varied up to maximum limit $\left(f_L = \mu N\right)$.

So, the magnitude of normal reaction N, direction of friction and magnitude of friction f are so adjusted so the resultant of the three forces mentioned above is $\frac{mv^2}{r}$ towards the centre.

Since, m and r are also constant so, magnitude of normal reaction N, direction of friction and magnitude of friction mainly depends on the speed of the vehicle v. Thus, situation varies from problem to problem.

(a) friction f will be outwards if the vehicle is at rest (v = 0), because in this case the component of weight $mg\sin\theta$ is balanced by f.

(b) friction f will be inwards if

$$v > \sqrt{rg\tan\theta}$$

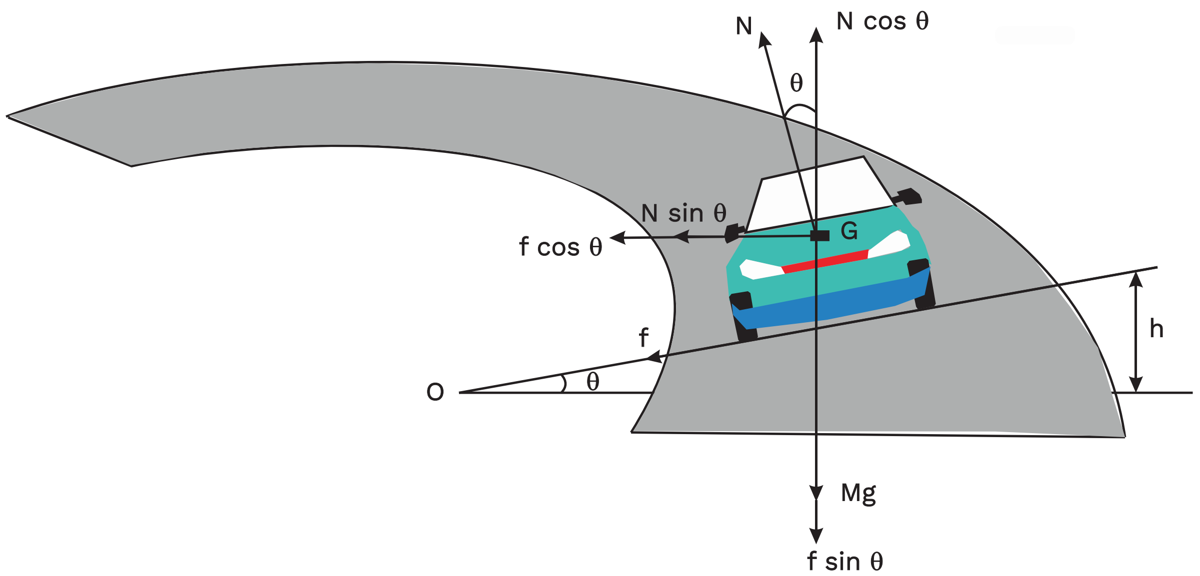
(c) friction f will be outwards if

$$v < \sqrt{rg\tan\theta}\ , \text{ and}$$

(d) friction f will be zero if

$$v = \sqrt{rg\tan\theta}$$

(e) for maximum safe speed (shown in figure below),



$$N\sin\theta + f\cos\theta = \frac{Mv^2}{r} \quad \text{.........} (1)$$

$$N\cos\theta - f\sin\theta = Mg \quad \text{.........} (2)$$

As maximum value of friction is $f_{max} = \mu N$,

So, $N\cos\theta - \mu N\sin\theta = Mg$, and $N\sin\theta + \mu N\cos\theta = \frac{Mv^2}{r}$

Then, $\frac{Mg}{(\cos\theta - \mu\sin\theta)}(\sin\theta + \mu\cos\theta) = \frac{Mv^2}{r}$

$$\Rightarrow \frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta} = \frac{v^2}{rg}$$

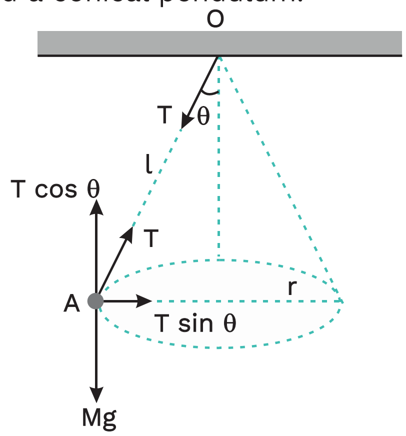
$$\Rightarrow \boxed{v_{max} = \sqrt{\frac{rg(\tan\theta + \mu)}{(1 - \mu\tan\theta)}}}$$

(f) Similarly, for minimum possible safe speed, friction starts acting up the inclined plane, and we can find out that,

$$\boxed{v_{min} = \sqrt{\frac{rg(\tan\theta - \mu)}{(1 + \mu\tan\theta)}}}$$

Conical Pendulum

If consists of a string OA, whose upper end O is fixed, and a bob is tied at the other free end. The bob is given a horizontal push through angular displacement θ and arranged such that the bob describes a horizontal circle with uniform angular velocity ω in such a way that the string always makes an angle θ with the vertical. As the string traces the surface of the cone, the arrangement is called a conical pendulum.



Let T be the tension in the string of length l and r be the radius of circular path. The vertical component of tension T balances the weight of the bob and horizontal component provides the necessary centripetal force.

Thus,

$$T\cos\theta = Mg \qquad \text{........ (1)}$$

And, $$T\sin\theta = Mr\omega^2 \qquad \text{........ (2)}$$

Dividing (2) by (1), we get

$$\tan\theta = \frac{r\omega^2}{g} \quad \text{i.e.,} \quad \omega = \sqrt{\frac{g\tan\theta}{r}} \qquad \text{........ (3)}$$

But $r = l\sin\theta$ and $\omega = \frac{2\pi}{\tau}$, τ being time period of completing one revolution.

$$\Rightarrow \quad \frac{2\pi}{\tau} = \sqrt{\frac{g\tan\theta}{l\sin\theta}}$$

This gives,

$$\tau = 2\pi\sqrt{\frac{l\sin\theta}{g\left(\frac{\sin\theta}{\cos\theta}\right)}}$$

$$\Rightarrow \quad \boxed{\tau = 2\pi\sqrt{\frac{l\cos\theta}{g}}}$$

Q. **A circular turn on a road having a radius 20 m is banked for the vehicle of mass 200 kg going a speed of 10 ms^{-1}. Find the direction and magnitude of frictional force acting on a vehicle if it moves with a speed (a) 5 ms^{-1} (b) 15 ms^{-1}.**
Take g = 10 ms^{-2} and assume that friction is sufficient to prevent slipping.

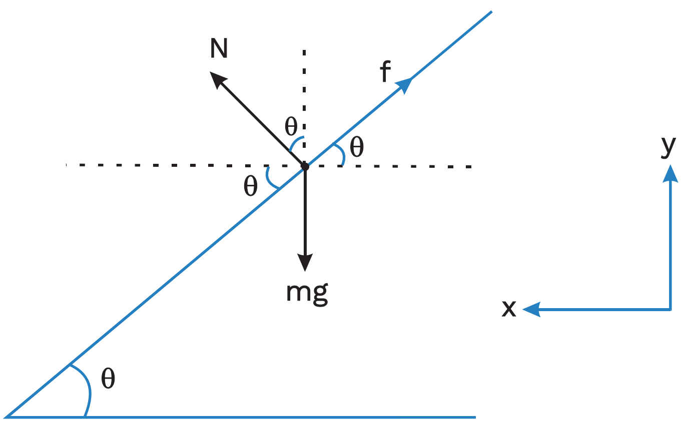
Sol. (a) The turn is banked for speed $v = 10\,ms^{-1}$. If θ is the angle of banking, then

$$\tan\theta = \frac{v^2}{rg} = \frac{(10)^2}{(20)(10)} = \frac{1}{2}$$

Now, as the speed is decreased, force of friction f acts outwards.

Using the equations

$$\Sigma F_x = \frac{mv^2}{r} \text{ and } \Sigma F_y = 0\text{, we get}$$

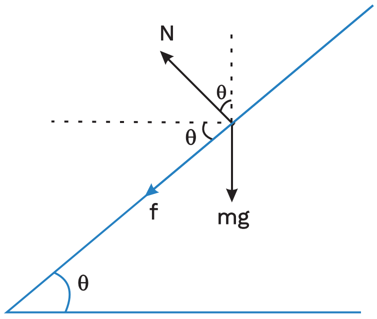


$$N\sin\theta - f\cos\theta = \frac{mv^2}{r} \qquad \text{.......... (1)}$$

$$N\cos\theta + f\sin\theta = mg \qquad \text{.......... (2)}$$

Substituting, $\theta = \tan^{-1}\left(\frac{1}{2}\right), v = 5\,ms^{-1}, m = 200\,kg$ and r = 20 m, in equation (1) and (2), we get $f = 300\sqrt{5}\,N$ (outwards)

(b) In the second case, force friction will now act inwards.

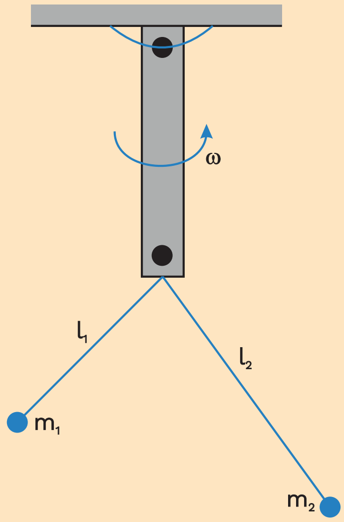


Using $\sum F_x = \frac{mv^2}{r}$ and $\sum F_y = 0$, we get

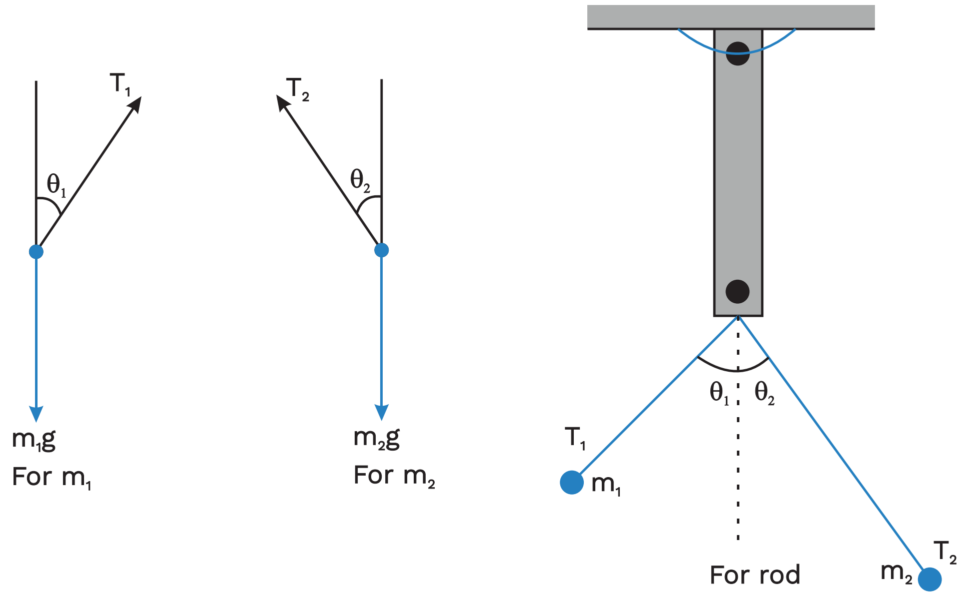
$$N\sin\theta + f\cos\theta = \frac{mv^2}{r} \qquad \text{........ (1)}$$
$$N\cos\theta - f\sin\theta = mg$$

Substituting $\theta = \tan^{-1}\left(\frac{1}{2}\right)$, $v = 15\,ms^{-1}$, $m = 200\,kg$ and $r = 20\,m$ in equation (3) and (4), we get $f = 500\sqrt{5}\,N$ (inwards)

Q. **Two balls of masses m_1 and m_2 are suspended by two threads of length l_1 and l_2 at the end of a freely hanging rod, Determine the angular velocity ω with which the rod must be rotated about the vertical axis so that it remains vertical.**



Sol. Free body diagrams of two masses and the rod are as shown in figure.



Equations of motion are,

$$T_1 \sin\theta_1 = m_1\omega^2 l_1 \sin\theta_1$$

$$T_1 \cos\theta_1 = m_1 g$$

$$T_2 \sin\theta_2 = m_2\omega^2 l_2 \sin\theta_2$$

$$T_2 \sin\theta_2 = m_2 g$$

For the rod to remain in vertical position,

$$T_1 \sin\theta_1 = T_2 \sin\theta_2$$

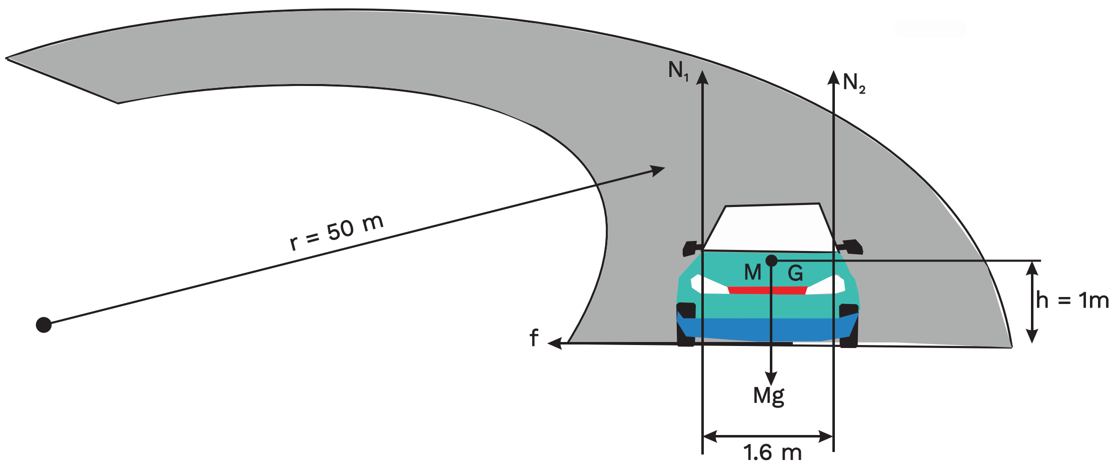
Solving the above equations, we get

$$\omega^4 = \left(\frac{m_1^2 g^2 - m_2^2 g^2}{m_1^2 l_1^2 - m_2^2 l_2^2}\right)$$

Q. **A vehicle whose wheel track is 1.6 m wide, and centre of gravity is 1 m above the road, centred between the wheels, takes a curve whose radius is 50 m, on a level road. Taking $g = 10\,ms^{-2}$, find the speed at which the inner wheel would leave the road.**

Sol. The situation is shown in figure. Let N_1 and N_1 be the reactions at inner wheels and outer wheels respectively. "f" is the frictional force of the tracks. G represents the centre of gravity and Mg is the weight of the vehicle acting downwards at centre of gravity. For vertical equilibrium,

$$N_1 + N_2 = Mg \quad \text{........ (1)}$$



For circular motion,

$$f = \frac{Mv^2}{r} \quad \text{.......... (2)}$$

For rotational equilibrium, net moment of all the forces about G should be zero. Hence,

$$f(1) + N_1\left(\frac{1.6}{2}\right) = N_2\left(\frac{1.6}{2}\right)$$

$$\Rightarrow \quad 0.8N_1 + f = 0.8N_2 \quad \text{.......... (3)}$$

When the inner wheel leaves the road, then $N_1 = 0$. Therefore from (3) we get

$$f = 0.8N_2 \quad \text{........... (4)}$$

And, from equation (1) we get

$$N_2 = mg$$

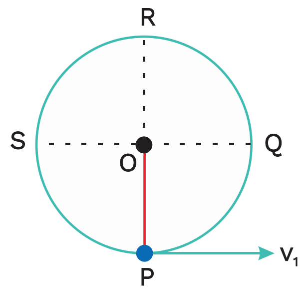
Solving equation (2), (4) and (5), we get

$$v = \sqrt{(0.8)\times(50)\times(10)} = 20\,ms^{-1}$$

Vertical Circular Motion

Motion in a Vertical Circle

Let a particle P be suspended in a vertical plane, by a massless, inextensible string from a fixed point O. In figure the string is vertical with P vertically below the point of suspension O.

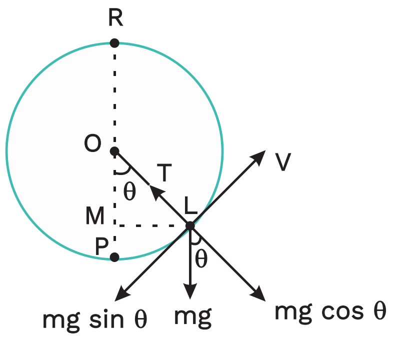


The particle is in equilibrium. Let the particle be given an initial velocity $\bar{v}_1$ in horizontal direction as, shown in figure. The particle moves along a vertical circle (radius = length of the string). The point of suspension is the centre of the circle.

This motion has to be non-uniform circular motion. Velocity of the particle changes both in magnitude and direction.

Speed of particle decreases continuously as it moves up the circle (i.e., from P $\rightarrow$ Q $\rightarrow$ R) due to the work done against the force of gravity.

Speed of particle increases continuously as it moves down the circle (i.e. from R $\rightarrow$ S $\rightarrow$ P) due to work done by the force of gravity on the particle.



To get the basic characteristics of vertical circular motion, consider an instantaneous position of the particle at point L.

This is shown in figure, where the string makes an angle θ with the vertical line OP. At this position, the forces acting on the particle are

(i) Weight (= mg) acting vertically downwards

(ii) Tension (= T) acting along LO (in the string)

Direction of the instantaneous velocity v is along the tangent to the circle at L. The corresponding instantaneous centripetal force $= \frac{mv^2}{r}$ [where r (= length of string l) is radius of the circular path] acts along LO.

Taking components of mg,

mg cos θ acts opposite to LO and mg sin θ acts opposite to $\vec{v}$ (along tangent to the circular path, i.e. perpendicular to LO.)

The net force towards centre of the circle (along LO) = T — mg cosθ this is necessary

centripetal force $\left(=\frac{mv^2}{r}\right)$.

So, $\frac{mv^2}{r} = T - mg\cos\theta$

$\therefore T = \frac{mv^2}{r} + mg\cos\theta$ ...... (1)

Taking horizontal direction at the lowest point P, as the position of zero gravitational potential energy.

As per law of conservation of energy,

Total energy at P = Total energy at L

$\therefore \frac{1}{2}mv_1^2 + 0 = \frac{1}{2}mv^2 + mgh$ ...... (2)

Where MP = h, is the vertical height of the particle above P.

OM = OL cosθ = r cosθ

$\therefore MP = h = OP - OM = r - r\cos\theta = r\left(1 - \cos\theta\right)$

Or, $h = r(1-\cos\theta)$ ...... (3)

From equations (2) and (3)

$$\frac{1}{2}mv_1^2 = \frac{1}{2}mv^2 + mgr(1-\cos\theta)$$

or $v_1^2 = v^2 + 2gr(1-\cos\theta)$ ...... (4)

Put the value of v^2 from equation (4) in equation (1)

$$T = \frac{m}{r}[v_1^2 - 2gr(1-\cos\theta)] + mg\cos\theta$$

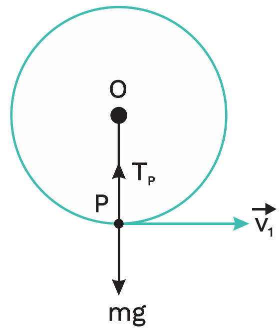
$$T = \frac{mv_1^2}{r} - 2mg(1-\cos\theta) + mg\cos\theta$$

$$T = \frac{mv_1^2}{r} - 2mg + 3mg\cos\theta \quad \text{........ (5)}$$

This equation given tension T as a function of θ. We can use it to see the details when the particle is at (i) lowest point (P) (ii) mid way (horizontal) (Q) and, (iii) highest position (R)

At P, $\theta = 0$

The tension in the string = T_P

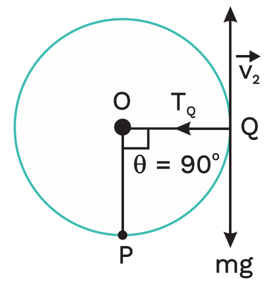


$$T_P = \frac{mv_1^2}{r} - 2mg + 3mg\cos 0^\circ \quad \text{(from equation 5)}$$

$$T_P = \frac{mv_1^2}{r} + mg \qquad \text{........ (6)}$$

At Q, $\theta = 90^\circ$ string is in horizontal position.

Let $\vec{v}_2$ be the instantaneous velocity at Q and let be the instantaneous tension is the string.
So,



$$T_Q = \frac{mv_1^2}{r} - 2mg + 3mg\cos 90^\circ$$

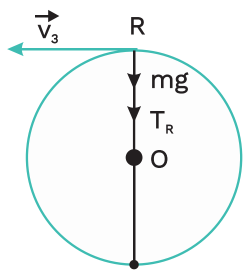
$$T_Q = \frac{mv_1^2}{r} - 2mg \qquad \text{........ (7)}$$

The change in the tension, as the particle moves from P to Q

$$= T_P - T_Q = \left(\frac{mv_1^2}{r} + mg\right) - \left(\frac{mv_1^2}{r} - 2mg\right)$$

$$T_P - T_Q = 3mg$$

At highest point R, $\theta = 180^\circ$



Let $\vec{v}_3$ be the instantaneous velocity at R and let T_R the instantaneous tension in the string.
Then,

$$T_R = \frac{mv_1^2}{r} - 2mg + 3mg\cos 180^\circ$$

$$T_R = \frac{mv_1^2}{r} - 5mg \qquad \text{........ (8)}$$

The change in tension in the string as the particle moves from P to R

$$= T_P - T_R$$

$$= \left(\frac{mv_1^2}{r} + mg\right) - \left(\frac{mv_1^2}{r} - 5mg\right)$$

$$T_P - T_R = 6mg$$

$\Rightarrow$ The tension in the string is maximum at lowest point P.
And, the tension in the string is minimum at highest point R.
This is so because at the highest point, a part of centripetal force, needed to keep the particle moving in circular path is provided by weight (mg) of the particle.
From equation (8), we can see that T_R can be (i) positive (ii) negative or (ii) zero, depending on the value of v_1.

When T_R becomes negative, string slackens and the particle will fall down before completing its circular path.
$\Rightarrow$ Minimum value of T_R should be zero for completing the vertical circle.

So, $(T_R)_{min} = \dfrac{m(v_1)^2_{min}}{r} - 5mg = 0$

$\therefore \boxed{(v_1)_{min} = \sqrt{5gr}}$ ...... (9)

Using equation (4), the minimum speed, which the particle must have at the highest point R, so that it completes the vertical circle, is given by

$$(v_1)^2_{min} = (v_3)^2_{min} + 2gr(1 - \cos 180^\circ) \quad (\text{As } \theta = 180^\circ \text{ at R})$$

$$5gr = (v_3)^2_{min} + 4gr$$

or $\boxed{(v_3)_{min} = \sqrt{gr}}$ ... (10)

When the particle completes its motion along the vertical circle, it is called 'looping the loop'. For this, minimum speed at the lowest point must be $\sqrt{5gr}$.
Let us calculate tension in the string, when the particle is just able to do 'looping the loop' corresponding to

$$v_1 = (v_1)_{min} = \sqrt{5gr}$$

We can see that for $v_1 = \sqrt{5gr}$,

$$T_P = \frac{m}{r}(5gr) + mg = 6mg \quad \text{and,}$$

$$T_R = \frac{m}{r}(5gr) - 5mg = 0$$

Note:

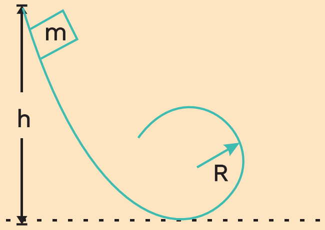
If $v_0 \geq \sqrt{5gl}$, the bob will complete full circular path.

If $v_0 = \sqrt{5gl}$, the tension at the top is zero but the velocity is $\sqrt{gl} > 0$.

If $\sqrt{2gl} < v_0 < \sqrt{5gl}$, the bob leaves the circular trajectory when the tension in the string is zero, but the speed of the bob is not zero. When the bob leaves the circular trajectory, $90^\circ < \theta < 180^\circ$, where θ is the angular displacement from the lowest position of the bob.

If $0 < v_0 \le \sqrt{2gl}$, the bob will oscillate about the lowest position having maximum angular displacement $\theta(\le 90^\circ)$. For $\theta_0 = 90^\circ$, the speed as well as tension will be zero.

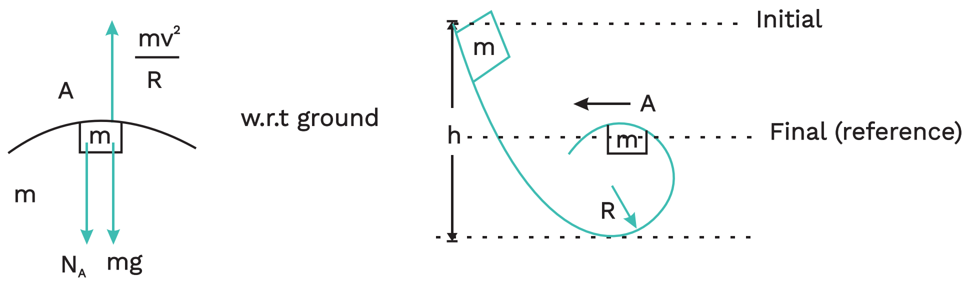
Q. **A particle of mass m slides down a smooth curved surface, which ends into a vertical loop of radius R. What should be the minimum height h such that if the particle released from it, does not fall at the uppermost point of the loop?**



Sol. If the particle does not fall at the highest point, then normal reaction at highest point A

$N_A \ge 0$

F.B.D. at topmost point A



$N + mg = \frac{mv^2}{r} \Rightarrow v = v_{min}$, when $N_A = 0$

$v_{min} = \sqrt{gR}$ .... (1)

Applying conservation of energy between initial and final points.

$K_i + U_i = K_f + U_f$ [Assuming final point as a reference]

$\Rightarrow 0 + mg(h - 2R) = \frac{1}{2}mv^2$

$\Rightarrow 2g(h-2R) = gR$

$\Rightarrow 2h - 4R = R$

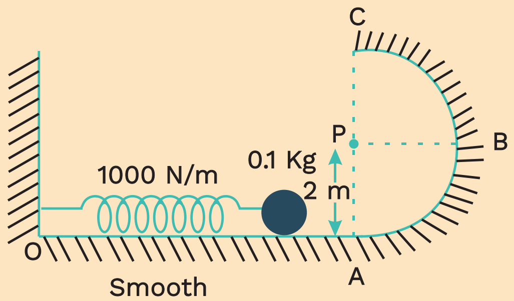
$$\boxed{h = \frac{5R}{2}}$$

Q. **A small stone of mass 0.4 kg tied to a massless inextensible string is made to loop the loop. Radius of the path is 4 m. Find its speed at the highest point. How would this speed change if mass of the stone is decreased by 10% ($g = 10\ ms^{-2}$).**

Sol. Speed at highest point $= \sqrt{gr} = \sqrt{10 \times 4} = 6.32\,ms^{-1}$

It does not depend on the mass of the stone. So, this speed remains the same on changing mass of the stone.

Q. **The figure shows a small ball of mass 0.1 kg placed on a smooth plane surface OA which acquires a semi-circular shape ABC of radius 2m. The ball just touches a light spring of stiffness 1000 N/m. The ball is pushed to the left to compress the spring by a distance x and released. This ball then starts moving towards the circular track ABC. ($g = 10\ m/s^2$)**



(a) Find the minimum work done by external agent to push the ball to the left through 50 cm.

(b) If the ball is pushed to the left by 5 cm and released, calculate

(i) Normal force on the ball just after crossing A.

(ii) Maximum angle covered with respect to PA on the circular track before it comes to rest.

(c) What is the minimum distance x_{min} by which the ball should be pushed to the left and released so that is can reach up to C?

(d) If the ball is pushed to left by 0.7 x_{min}, calculate

(i) Reaction force between ball and track at point B

(ii) Maximum height attained by the ball above horizontal surface OA.

Sol. (a) When the work done by external force to push the ball against the spring is minimum, there should be no kinetic energy of the ball. The work done is only responsible for its potential energy.

$$\therefore W = \frac{1}{2}kx^2$$

$$\Rightarrow W = \frac{1}{2} \times 1000 \times \left(\frac{5}{100}\right)^2 = 1.25\,J$$

(b) As the ball leaves the spring, it will be moving towards right with a speed v, such that the potential energy of the spring changes to kinetic energy of the ball.

$$\Rightarrow \frac{1}{2}mv^2 = \frac{1}{2}kx^2$$

$$\Rightarrow v = x\sqrt{\frac{k}{m}}$$

As, k = 1000 N/m, m = 0.1 kg, x = 5 cm=0.05 m, so

$$v = 0.05 \times \sqrt{\frac{1000}{0.1}} = 5\,m/s$$

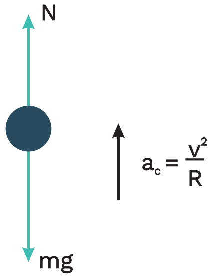
(i) As it crosses A, its path becomes circular, and it experiences a centripetal acceleration.

$$a_c = \frac{v^2}{R}$$

By Newton's 2nd law,

$$N - mg = 2\ a_c$$

$$N = mg + \frac{mv^2}{R} = (0.1 \times 10) + \left(0.1 \times \frac{25}{2}\right) = 2.25\,N$$



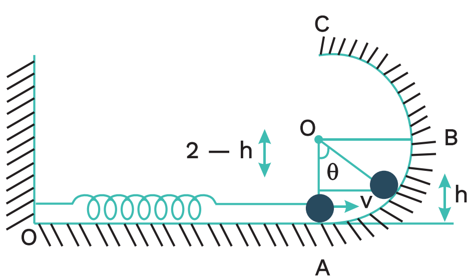
(ii) As the ball rises on the track, its gravitational potential energy increases, and kinetic energy decreases. As it comes to rest, kinetic energy becomes zero. If it happens at a height h, then,
By conservation of energy,

$$mgh + 0 = 0 + \frac{1}{2}mv^2$$

$$h = \frac{v^2}{2g} = \frac{5^2}{2 \times 10} = \frac{25}{20} = 1.25\,m$$

$$\text{Now, } \cos\theta = \frac{2-h}{2} = \frac{2-1.25}{2} = \frac{0.75}{2} = \frac{3}{8}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{3}{8}\right)$$



(c) Let the spring is compressed by x_{min} and the speed of the ball after leaving contact with spring is v, then by energy conservation, $\frac{1}{2}mv^2 = \frac{1}{2}kx_{min}^2$

$$v = x_{min}\sqrt{\frac{k}{m}}$$

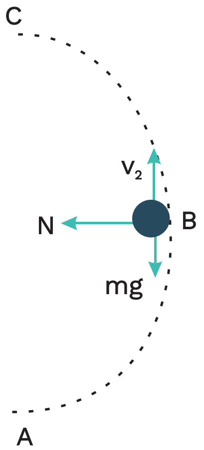
Now, to complete circular motion, minimum speed at A should be equal to $\sqrt{5gR}$.

$$\Rightarrow x_{min}\sqrt{\frac{k}{m}} = \sqrt{5gR}$$

$$\Rightarrow x_{min} = 0.1m$$

(d) When the spring is compressed by $0.7\ x_{min} = 0.07\,m,$ the speed acquired by the ball is $v = x\sqrt{\frac{k}{m}} = 0.07 \times \sqrt{\frac{1000}{0.1}} = 7\ \ m/s$

(i) As the ball reaches B, its speed becomes v_B , then by conservation of mechanical energy,



$$\frac{1}{2}mv^2 = \frac{1}{2}mv_B^2 + mgR$$

$$v_B = \sqrt{v^2 - 2gR} = \sqrt{49 - 40}$$

$$v_B = 3\ \ m/s$$

The FBD of ball at B is shown.

The radial acceleration at this instant is $a_C = \frac{v^2}{R} = 4.5\ m/s^2$

By Newton's 2^{nd} law,

$$N = \frac{mv_B^2}{R} = 0.1 \times \frac{9}{2} = 0.45\,N$$

(ii) As the speed at lowest point is less than that required to complete the circular
track and reach C, the ball will leave the track before it reaches C.

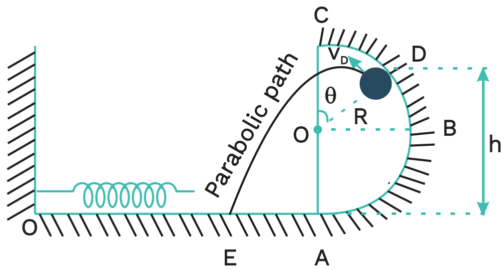
If it occurs at D such that OD makes an angle θ with OC, let v_D be the speed at this position.

The height at D is, h = R(1 + $\cos\theta$)

By conservation of mechanical energy.

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_D^2 + mgR(1+\cos\theta) \quad \text{..... (1)}$$

At D, FBD of the ball is shown.

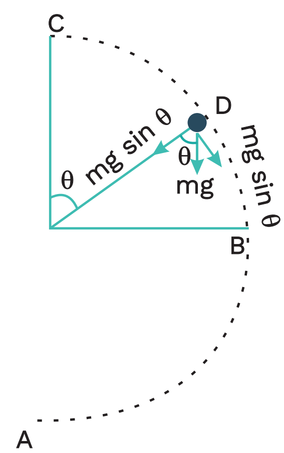


As the ball leaves the track at D, normal reaction is zero.

By Newton's 2^{nd} law in radial direction,

$$mg\cos\theta = \frac{mv_D^2}{R}$$

$$v_0^2 = gR\cos\theta \quad \text{......... (2)}$$



From (1) and (2),

$$\frac{1}{2}mv^2 = \frac{1}{2}mgR\cos\theta + mgR(1+\cos\theta)$$

$$v^2 - 2gR = 3gR\cos\theta$$

$$\Rightarrow \cos\theta = \frac{v^2 - 2gR}{3gR} = \frac{7^2 - (2\times 10\times 2)}{(3\times 10\times 2)} = \frac{3}{20}$$

$$\therefore h = R\left(1+\frac{3}{20}\right) = \frac{23}{20}\times 2 = 2.3\,\text{m}$$

From point D, the ball moves in a parabolic path under the action of gravity alone. From this point onwards, it rises further to a height H given by,

$$H = \frac{v_D^2 \sin^2\theta}{2g}$$

$$\Rightarrow \quad H = \frac{gR\cos\theta\sin^2\theta}{2g} \quad \text{[from (1)]}$$

$$\Rightarrow H = \frac{2}{2}\times\frac{3}{20}\times\left(1-\frac{9}{400}\right) = \frac{3}{20}\times\frac{391}{400} = \frac{1173}{8000}\text{m}$$

Maximum height above OA is $2.3 + \frac{1173}{8000} = 2.45\,\text{m}$