Centre of Mass

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CENTRE OF MASS

Every physical system has associated with it certain point whose motion characterises motion of whole system. When system moves under some external forces, then this point moves as if entire mass of the system is concentrated at this point and also the external force is applied at this point for translational motion. This point is known as the COM of the system.

KEY POINTS

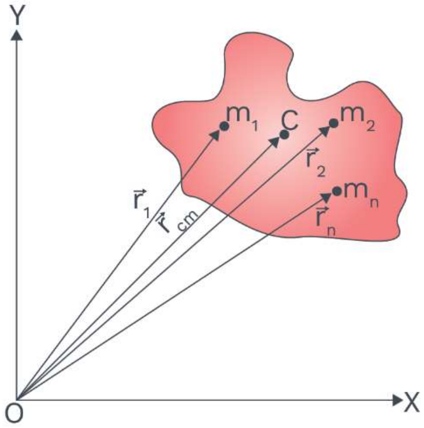
- Centre of mass
- External force
- Translational motion
- Discrete particles

Definitions

The centre of mass of a rigid body is defined as a point, where the entire mass of system is supposed to be concentrated.

Centre of Mass of a System of 'N' Discrete Particles:

Consider a system of 'N' point masses m_1, m_2, m_3, ... m_n whose position vectors from origin 'O' are given by $\vec{r}_1, \vec{r}_2, \vec{r}_3 ......\vec{r}_n$ respectively. Then the position vector of the COM 'C' of the system is given as.



$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + ... + m_n\vec{r}_n}{m_1 + m_2 + ... + m_n}$$

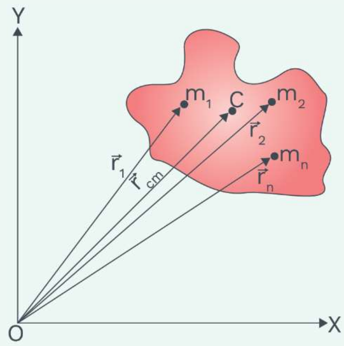
$$\vec{r}_{cm} = \frac{\sum_{i=1}^{n} m_i\vec{r}_i}{\sum_{i=1}^{n} m_i} \Rightarrow \vec{r}_{cm} = \frac{1}{M}\sum_{i=1}^{n} m_i\vec{r}_i$$

where, $m_i\vec{r}_i$ is known as the moment of mass of particle w.r.t. origin.

$M = \left(\sum_{i=1}^{n} m_i\right)$ is total mass of system.

Further,

Concept Reminder



$$\vec{r}_{cm} = \frac{1}{M}\sum_{i=1}^{n} m_i\vec{r}_i$$

$$\vec{r}_i = x_i\,\hat{i} + y_i\,\hat{j} + z_i\,\hat{k}$$

and $\vec{r}_{COM} = x_{COM}\,\hat{i} + y_{COM}\,\hat{j} + z_{COM}\,\hat{k}$

So, the cartesian co-ordinates of the centre of mass will be

$$x_{COM} = \frac{m_1x_1 + m_1x_2 + \ldots + m_nx_n}{m_1 + m_2 + \ldots + m_n} = \frac{\sum_{i=1}^{n} m_ix_i}{\sum_{i=1}^{n} m_i} \quad \text{or}$$

$$x_{COM} = \frac{\sum_{i=1}^{n} m_ix_i}{M}$$

Similarly,

$$y_{cm} = \frac{m_1y_1 + m_2y_2 + \ldots + m_ny_n}{m_1 + m_2 + \ldots + m_n}$$

$$z_{cm} = \frac{m_1z_1 + m_2z_2 + \ldots + m_nz_n}{m_1 + m_2 + \ldots + m_n}$$

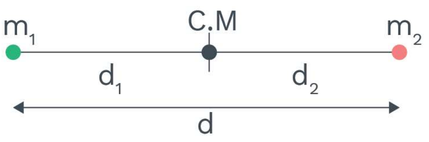
Note: If origin is taken at centre of mass then $\sum_{i=1}^{n} m_i\vec{r}_i = 0$. Hence, the COM is point about which sum of mass moments of the system is zero.

If we change the origin then $\vec{r}_1, \vec{r}_2, \vec{r}_3$..... changes. So, $\vec{r}_{cm}$ also changes but exact location of centre of mass does not change.

Case-I: Position of CM of two particle system:

- In case of two bodies, the ratio of distance of centre of mass from the bodies is in inverse ratio of their masses. If m_1 and m_2 are masses of two bodies separated by a distance 'd' then the sum of moment of weights about centre of mass is zero.

$$m_1gd_1 = m_2gd_2 \quad \text{or} \quad \frac{d_1}{d_2} = \frac{m_2}{m_1}$$



In figure, $d = d_1 + d_2$

On solving, $m_1d_1 = m_2d_2$

Concept Reminder

Centre of mass is point about which sum of mass moments of the system is zero.

$$\sum_{i=1}^{n} m_i\vec{r}_i = 0$$

Rack your Brain

Two particles of mass 5 kg and 10 kg respectively are attached to the two ends of a rigid rod of length of 1 m with negligible mass. The centre of mass of the system from 5 kg particle is nearly at a distance of:

(1) 33 cm (2) 50 cm
(3) 67 cm (4) 80 cm

$$m_1d_1 = m_2 \times (d - d_1)$$

$$\Rightarrow \quad d_1 = \frac{m_2 d}{m_1 + m_2} \text{ and } d_2 = \frac{m_1 d}{m_1 + m_2}$$

Here, d_1, d_2 are the distances of CM from m_1, m_2. Thus, CM locates nearer to heavier body.

Note: If m_1, m_2 are located at x_1, x_2 from origin then

$$\boxed{x_{cm} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}}$$

Ex. If the distance between the centres of the atoms of potassium and bromine in KBr (potassium-bromide) molecule is 0.282×10^{-9} m, find the centre of mass of this two particle system from potassium (mass of bromine = 80 u, and of potassium = 39 u)

Sol. Let position coordinate of potassium,

$$x_K = 0$$

Position co-ordinate of bromine,

$$x_{Br} = 0.282 \times 10^{-9} \text{ m}$$

$\therefore$ Position co-ordinate of centre of mass.

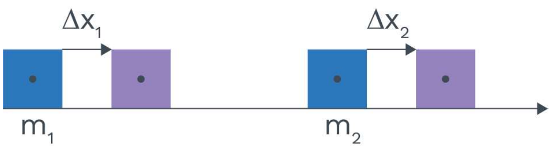
$$x_{cm} = \frac{m_K x_K + m_{Br} x_{Br}}{m_K + m_{Br}}$$

$$\Rightarrow x_{cm} = \frac{39 \times 0 + 80 \times 0.282 \times 10^{-9}}{39 + 80} = 0.189 \times 10^{-9} \text{m}$$

Ex. Two blocks of masses 30 kg and 10 kg are placed on X-axis. The first mass is moved on the axis by a distance of 2 cm right. By what distance should 2nd mass be moved to keep the position of centre of mass unchanged.

Sol. Mass of the first block,

$$m_1 = 10 \text{ kg}$$



Mass of the second block,

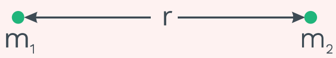
Concept Reminder

If m_1, m_2 are located at x_1, x_2 from origin then-

$$x_{cm} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

Rack your Brain

Two particles having masses m_1 and m_2 are placed at a separation r between them. Locate the centre of mass of the arrangement.



$$m_2 = 30 \text{ kg}$$

$$\Delta x_{cm} = \frac{m_1\Delta x_1 + m_2\Delta x_2}{m_1 + m_2}$$

$$0 = \frac{10 \times 2 + 30\,\Delta x_2}{40}$$

$$\therefore \quad \Delta x_2 = -\frac{2}{3}$$

Therefore, the second block should be moved left through a distance of 2/3cm to keep the position of centre of mass unchanged.

Case-II: Center of mass of a system of n particles in one dimension:

Consider n-particles having masses m_1, m_2, ..., m_n along X-axis. The COM of this system is given by

$$x_{cm} = \frac{m_1x_1 + m_2x_2 + ... + m_nx_n}{m_1 + m_2 + ... + m_n}$$

Ex. When 'n' no. of particles of masses m, 2m, 3m, ... nm are at distance $x_1 = 1$, $x_2 = 2$, $x_3 = 3$ ..... $x_n = n$ units respectively from the origin on X-axis, then find distance of COM of the system from origin.

Sol. $$x_{cm} = \frac{m(1) + 2m(2) + 3m(3) + ... + (nm)n}{m + 2m + 3m + ... + nm}$$

$$x_{cm} = \frac{m(1^2 + 2^2 + 3^2 + ... + n^2)}{m(1 + 2 + 3 + ... + n)}$$

$$= \frac{\left(\frac{n(n+1)(2n+1)}{6}\right)}{\left(\frac{n(n+1)}{2}\right)} = \frac{2n+1}{3}$$

Ex. When 'n' number of particles of masses m, 2m, 3m, ... nm are at distances $x_1 = 1$, $x_2 = 4$, $x_3 = 9$, ... $x_n = n^2$ units respectively from the origin on the X-axis, then find distance of their COM from origin.

Sol. $$x_{cm} = \frac{m(1) + 2m(4) + 3m(9) + ... + nm(n^2)}{m + 2m + 3m + ... + nm}$$

Concept Reminder

Consider n-particles having masses m_1, m_2, ..., m_n along X-axis. The centre of mass of this system is given by

$$x_{cm} = \frac{m_1x_1 + m_2x_2 + ... + m_nx_n}{m_1 + m_2 + ... + m_n}$$

Rack your Brain

Three masses are placed on the x-axis: 300g at origin, 500g at x = 40 cm and 400 g at x = 70 cm. The distance of centre of mass from origin is:

(1) 40 cm (2) 45 cm
(3) 50 cm (4) 30 cm

$$x_{cm} = \frac{m(1+2^3+3^3+.....+n^3)}{m(1+2+3+....+n)}$$

$$= \frac{\left(\frac{n(n+1)}{2}\right)^2}{\frac{n(n+1)}{2}} = \frac{n(n+1)}{2}$$

Case-III: Center of mass of a system of particles in (two dimensional) Plane:

Consider n-particles in x-y plane having masses $m_1, m_2, ..., m_n$ with coordinates $(x_1, y_1), (x_2, y_2), ...,(x_n, y_n)$ respectively.

The distance of centre of mass from origin in a plane is $d = \sqrt{x_{cm}^2 + y_{cm}^2}$

where, $x_{cm} = \frac{\sum_{i=1}^{n} m_i x_i}{M}$ and $y_{cm} = \frac{\sum_{i=1}^{n} m_i y_i}{M}$

Case-IV: Centre of mass of a system of 'n' particles in (Three dimensional) Space:

Consider n-particles in space having masses $m_1, m_2, ... m_n$ with coordinates $(x_1, y_1, z_1), (x_2, y_2, z_2)$ and (x_n, y_n, z_n) respectively, then distance of centre of mass from origin in space is

$$d = \sqrt{x_{cm}^2 + y_{cm}^2 + z_{cm}^2}$$

Where,

$$x_{cm} = \frac{\sum_{i=1}^{n} m_i x_i}{M}, \quad y_{cm} = \frac{\sum_{i=1}^{n} m_i y_i}{M}$$

and $z_{cm} = \frac{\sum_{i=1}^{n} m_i z_i}{M}$

Ex. If COM of three particles of masses of 1 kg, 2 kg, 3 kg is at (2, 2, 2), then where should a fourth particle of mass 4kg be placed so that the combined centre of mass may be at (0, 0, 0).

Sol. Let $(x_1, y_1, z_1), (x_2, y_2, z_2)$ and (x_3, y_3, z_3) be the positions of masses 1kg, 2kg, 3kg and let

Concept Reminder

Consider n-particles in x-y plane having masses $m_1, m_2,....., m_n$ with coordinates $(x_1, y_1), (x_2, y_2), ...,(x_n, y_n)$ respectively.

where,

$$x_{cm} = \frac{\sum_{i=1}^{n} m_i x_i}{M} \text{ and } y_{cm} = \frac{\sum_{i=1}^{n} m_i y_i}{M}$$

Rack your Brain

The centre of mass of system of particles does not depend on:

(1) Position of particles

(2) Relative distance between particles

(3) Masses of particles

(4) Forces acting on particles

the co-ordinates of centre of mass of the three particle system is (x_{cm}, y_{cm}, z_{cm}) respectively.

$$x_{cm} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$$

$$\Rightarrow 2 = \frac{1 \times x_1 + 2 \times x_2 + 3 \times x_3}{1 + 2 + 3}$$

or $x_1 + 2x_2 + 3x_3 = 12$ ....(i)

Suppose the fourth particle of mass 4kg is placed at (x_4, y_4, z_4) so that centre of mass of new system shifts to (0,0,0). For x coordinate of new centre of mass we have

$$0 = \frac{1 \times x_1 + 2 \times x_2 + 3 \times x_3 + 4 \times x_4}{1 + 2 + 3 + 4}$$

$\Rightarrow$ $x_1 + 2x_2 + 3x_3 + 4x_4 = 0$ ...(ii)

From equations (i) and (ii)

$$12 + 4x_4 = 0 \Rightarrow x_4 = -3$$

Similarly,

$$y_4 = -3 \text{ and } z_4 = -3$$

Therefore 4 kg should be placed at (–3, –3, –3).

Case-V: Position vector of Centre of mass:

Let $\vec{r}_1, \vec{r}_2, \vec{r}_3, ..., \vec{r}_n$ be the position vectors of n-particles having masses m_1, m_2, ..., m_n respectively. If $\vec{r}_{CM}$ is position vector of their CM then

$$\vec{r}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + ... + m_n\vec{r}_n}{m_1 + m_2 + ... + m_n} = \frac{\sum_{i=1}^{n} m_i\vec{r}_i}{M}$$

$$\vec{r}_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$$

$$\vec{r}_2 = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$$

$$\vec{r}_n = x_n\hat{i} + y_n\hat{j} + z_n\hat{k}$$

$\therefore$ $\vec{r}_{CM} = x_{CM}\hat{i} + y_{CM}\hat{j} + z_{CM}\hat{k}$

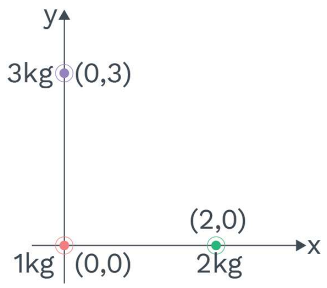
Ex. Find position of COM.

Concept Reminder

Let $\vec{r}_1, \vec{r}_2, \vec{r}_3, ..., \vec{r}_n$ be the position vectors of n-particles having masses m_1, m_2, ..., m_n respectively. If $\vec{r}_{CM}$ is position vector of their CM then

$$\vec{r}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + ... + m_n\vec{r}_n}{m_1 + m_2 + ... + m_n}$$

$$= \frac{\sum_{i=1}^{n} m_i\vec{r}_i}{M}$$



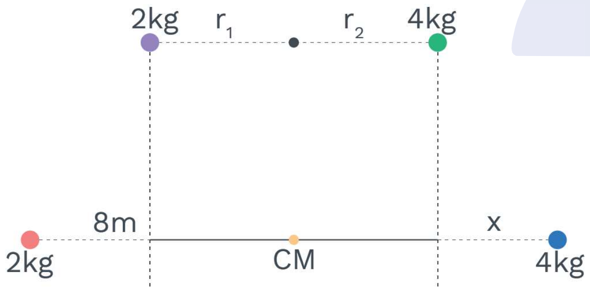
Sol. $x_{cm} = \dfrac{1(0) + 2(2) + 3(0)}{6} = \dfrac{2}{3}$

$y_{cm} = \dfrac{1(0) + 2(0) + 3(3)}{6} = \dfrac{3}{2}$

So, position of COM $\left(\dfrac{2}{3}, \dfrac{3}{2}\right)$

Ex. 2 particles of mass 2 kg and 4kg are present at a fixed distance from each other as shown. If 2 kg mass is displaced towards left by 8m than find out by what distance 4kg mass should be displaced so, COM of system remains at its initial position

Sol. COM of system remains at its initial position



$m_1r_1 = m_2r_2$

$2\ r_1 = 4\ r_2$ ...(i)

$2(r_1 + 8) = 4(r_2 + x)$

$2\ r_1 + 16 = 4\ r_2 + 4x$

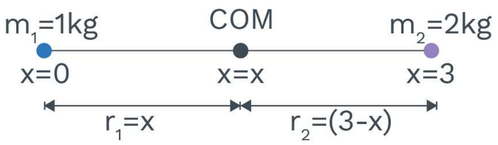
$x = 4$ m

Concept Reminder

The concept of centre of mass of a system enables us, n describing the overall motion of the system by replacing the system by an equivalent single point.

Ex. 2 particle of mass 1 kg and 2 kg are placed at $x = 0$ and $x = 3$ m. Find the position of their COM.

Sol. Since, both particles lie on x-axis, the COM will also lie on the x-axis.
Let COM is located at $x = x$, then
r_1 = distance of COM from particle of mass 1 kg = x



and r_2 = distance of COM from particle of mass 2 kg = (3 – x)

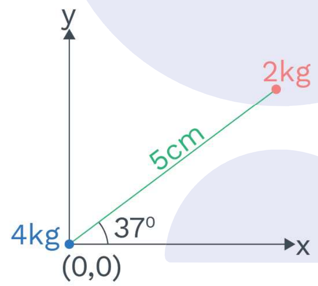
Using $\frac{r_1}{r_2} = \frac{m_2}{m_1}$

or $\quad \frac{x}{3-x} = \frac{2}{1}$

or $\quad x = 2$ m

thus, the COM of two particles is located at x = 2 m.

Ex. Two particle of mass 2 kg & 4 kg are located as shown in figure then find out the position of COM.



Rack your Brain

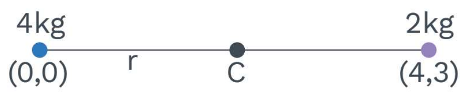
Two bodies of mass 1 kg and 3 kg have position vectors $\hat{i} + 2\hat{j} + \hat{k}$ and $-3\hat{i} - 2\hat{j} + \hat{k}$ respectively. Find the position vector of centre of mass of given system.

Sol. First find out position of 2 kg mass

$x_{2kg} = 5 \cos 37° = 4$ m

$y_{2kg} = 5 \sin 37° = 3$ m

So, these system is like two particle system of mass 4 kg and 2kg are located (0, 0) and (4, 3) respectively, then



$$x_{COM} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} = \frac{0 + 2 \times 4}{4 + 2} = \frac{8}{6} = \frac{4}{3} \text{m}$$

$$y_{COM} = \frac{m_1y_1 + m_2y_2}{m_1 + m_2} = \frac{0 + 3 \times 2}{4 + 2} = 1 \text{ m}$$

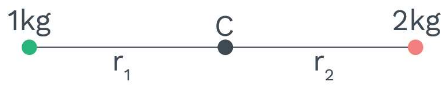
So, position of COM is $\left(\frac{4}{3}, 1\right)$

Ex. Two particles of mass 2 kg and 1 kg lie on same line. If 2kg is moved 10m rightwards then by what distance 1kg should move so that centre of mass will be displaced 2m right wards.

Sol. Initially let us assume that COM is at point C which is r_1 and r_2 distance apart from mass m_1 and m_2 respectively as shown in figure.

Concept Reminder

For two particles of equal masses the centre of mass lies exactly midway between the straight line joining them.



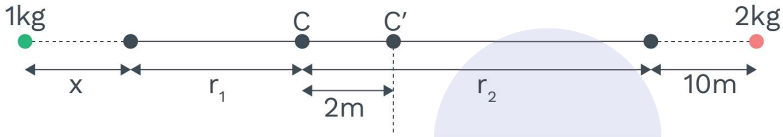
From relation $m_1 r_1 = m_2 r_2$

$\Rightarrow$ $(1)\ r_1 = 2\ r_2$

Now 2kg is moved 10 m rightwards then we assume that 1 kg is displaced x m leftward to move the COM 2 m rightwards

So, from relation

$m_1 r_1' = m_1 r_2'$



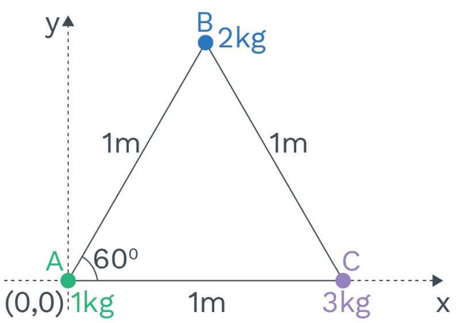
Therefore, $1\ (x + r_1 + 2) = 2(10 + r_2 - 2)$

or $\quad 20 + 2\ r_2 - 4 = x + r_1 + 2 \quad$ ..(ii)

From eq. (i) & (ii) x = 14 m (leftwards)

Ex. 3 particles of 1 kg, 2 kg, and 3 kg are located at corners A, B and C respectively of equilateral triangle ABC of edge 1 m. Find the distance of their COM from 'A'.

Sol. Assume that 1kg mass is placed at origin as shown in figure.



Co-ordinate of A = (0, 0)
Co-ordinate of B,

$$B = (1 \cos 60°, 1 \sin 60°) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

Co-ordinate of C = (1, 0)
Let us suppose that position of COM is given by

$$\vec{r}_{COM} = x_{COM}\hat{i} + y_{COM}\hat{j}$$

Now, $x_{COM} = \frac{1(0) + 2(1/2) + 3(1)}{1 + 2 + 3} = \frac{2}{3}$

$$y_{COM} = \frac{1(0) + 2\left(\frac{\sqrt{3}}{2}\right) + 3(0)}{1 + 2 + 3} = \frac{\sqrt{3}}{6}$$

Position of centre of mass = $\left(\frac{2}{3}, \frac{\sqrt{3}}{6}\right)$

Distance of COM from point 'A',

$$A = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{\sqrt{3}}{6}\right)^2} = \frac{\sqrt{19}}{6} m$$

Concept Reminder

If two particles having masses m_1 and m_2 are placed at a separation r between them and m_2 be displaced away from m_1 by distance x, then m_1 should be displaced by $\frac{m_2 x}{m_1}$ away from origin to keep position of centre of mass unchanged.

KEY POINTS

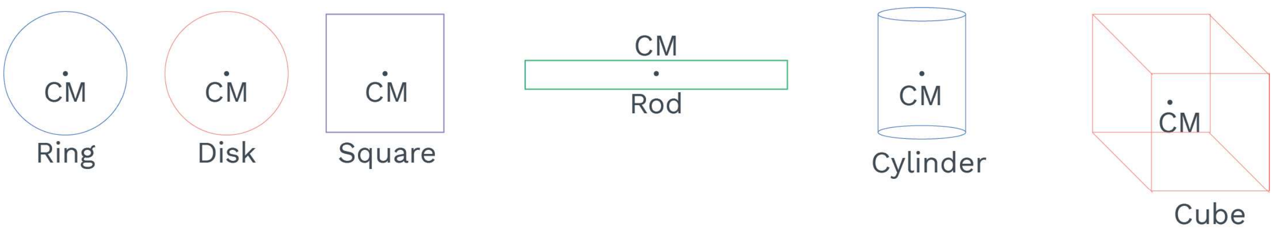
- Centre of gravity
- Continuous mass distribution
- Geometric centre
- Poles

CENTRE OF GRAVITY (COG)

Centre of gravity is a point where gravitational force is supposed to be applied. In normal case centre of gravity and centre of mass coincide but in some case it may differ. For high altitude centre of mass and centre of gravity lie at two different points.

Centre of mass due to continuous mass distribution:

For uniform object:
COM of uniform object lie at their geometric centre.



For continuous mass distribution the COM can be placed by replacing summation sign with integral sign. Proper limits for integral are chosen according to situation

$$x_{COM} = \frac{\int x\,dm}{\int dm},\; y_{COM} = \frac{\int y\,dm}{\int dm},\; z_{COM} = \frac{\int z\,dm}{\int dm} \qquad ...(i)$$

$\int dm$ = M (mass of the body)

Here x, y, z in numerator of eq. (i) is the coordinate of the C.O.M of the dm mass.

$$\vec{r}_{COM} = \frac{1}{M}\int \vec{r}\,dm$$

Concept Reminder

For continuous mass distribution-

$$x_{COM} = \frac{\int x\,dm}{\int dm},\; y_{COM} = \frac{\int y\,dm}{\int dm},$$

$$z_{COM} = \frac{\int z\,dm}{\int dm}$$

or $\vec{r}_{COM} = \frac{1}{M}\int \vec{r}\,dm$

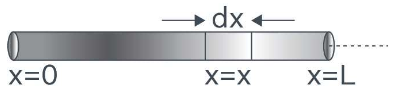
Note: If an object has the symmetric mass distribution about 'x' axis then 'y' coordinate of C.O.M is zero and vice-versa.

(a) Centre of Mass of a Uniform Rod:

Assume rod of mass 'M' and length 'L' is lying along x-axis with its one end at x = 0 and other at x = L. Mass per unit length of the rod $\lambda = \frac{M}{L}$

Hence, dm, (mass of element 'dx' situated at x = x is) = λ dx

The coordinates of the element dx are (x, 0, 0). Therefore, x-coordinate of the COM of the rod will be



$$x_{COM} = \frac{\int_0^L x\,dm}{\int_0^L dm} = \frac{\int_0^L (x)(\lambda\,dx)}{\int_0^L \lambda\,dx} = \frac{1}{L}\int_0^L x\,dx = \frac{L}{2}$$

The y-coordinate of COM is

$$y_{COM} = \frac{\int y\,dm}{\int dm} = 0$$

Similarly, $z_{COM} = 0$

that means, the coordinates of COM of the rod are $\left(\frac{L}{2}, 0, 0\right)$ i.e, it lies at centre of the rod.

Rack your Brain

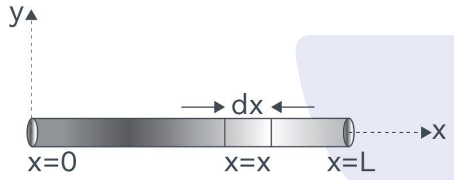
Two bodies of masses 1 kg and 3 kg are lying in x-y plane at (0, 0) and (2, –1) respectively. What are the coordinates of the centre of mass.

Ex. A rod of length 'L' is placed along x-axis between x = 0 and x = L. The linear density (mass/ length)λ of rod changes with distance 'x' from origin as λ = Rx. Here, R is a +ve constant. Find the position of C.O.M of this rod.

Sol. Mass of the element dx placed at x = x is

dm = λ dx = Rx dx

The COM of the element has coordinates (x_{COM}, 0, 0). Therefore, x-coordinates of COM of rod will be



$$x_{COM} = \frac{\int_0^L x\,dm}{\int dm} = \frac{\int_0^L (x)(Rx)\,dx}{\int_0^L (Rx)\,dx}$$

$$= \frac{R\int_0^L x^2\,dx}{\int_0^L x\,dx} = \frac{\left[\frac{x^3}{3}\right]_0^L}{\left[\frac{x^2}{2}\right]_0^L} = \frac{2L}{3}$$

The y-coordinates of COM of the rod is

$$y_{COM} = \frac{\int y\,dm}{\int dm} = 0 \text{ (as } y = 0)$$

Similarly, $z_{COM} = 0$

Hence, the C.O.M of the rod lies at $\left[\frac{2L}{3}, 0, 0\right]$

(b) Centre of mass of a Semi-circular Ring:

Figure shows the object (semi-circular ring). By observation we can comment that the x-coordinate of the C.O.M of the ring is zero as half ring is symmetrical about y-axis on both sides of origin. Only we are needed to find the y-coordinate of C.O.M

Concept Reminder

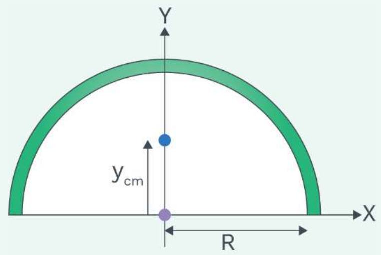
The coordinates of COM of the rod are $\left(\frac{L}{2}, 0, 0\right)$ i.e, it lies at the centre of the rod.

Rack your Brain

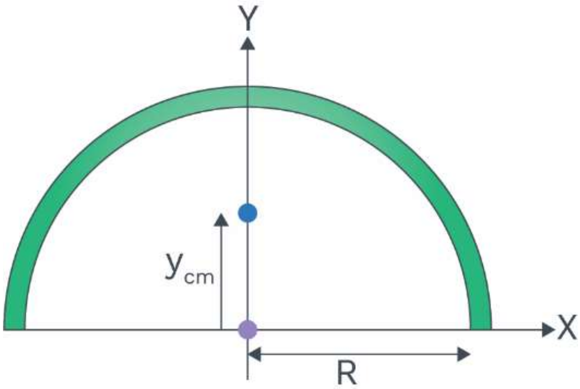
The linear mass density λ. If rod varies with distance x as λ = Ax + B where A and B are constant. If length of rod is L then find position of centre of mass of rod.

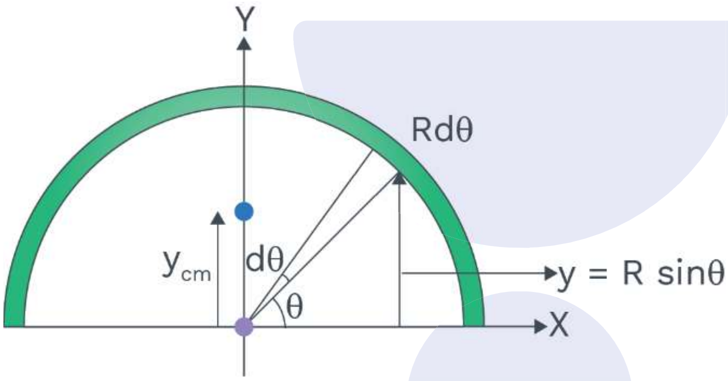
Concept Reminder

Centre of mass of a Semi-circular Ring:



$$y_{cm} = \frac{2R}{\pi}$$





To find y_{cm} we use

$$y_{cm} = \frac{\int (dm)\, y}{\int dm} \qquad ...(i)$$

Here y is the position of COM of dm mass.
Here for dm we think about an elemental arc of ring at an angle θ from x-direction of an angular width $d\theta$. If radius of the ring is 'R', then its y coordinate-will be R $\sin\theta$, here dm is given as
$dm = \lambda R d\theta$
where λ = mass density of semi-circular ring.
So, from equation ...(i), we have

$$y_{cm} = \frac{\int_0^{\pi} \lambda R d\theta (R \sin\theta)}{\int_0^{\pi} \lambda R d\theta} = \frac{R}{\pi}\int_0^{\pi} \sin\theta\, d\theta$$

$$y_{cm} = \frac{2R}{\pi} \qquad ...(ii)$$

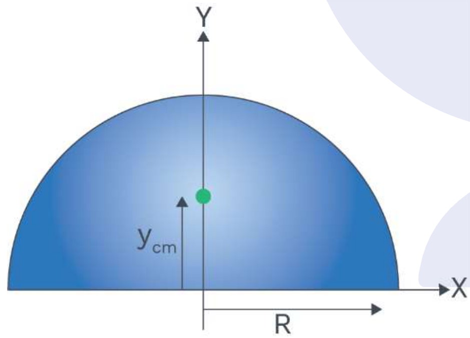
(c) Centre of mass of Semi-circular Disc:

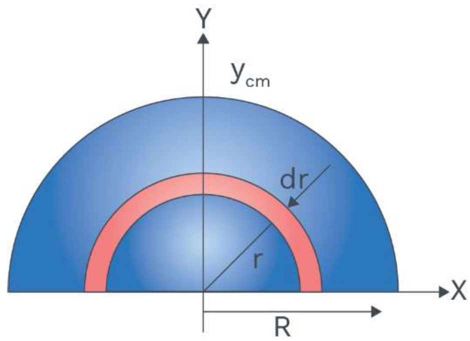
Figure shows the half disc of mass 'M' and radius 'R'. Here, we are only needed to find the y-coordinate of C.O.M of this disc as centre of mass will be situated on its half vertical diameter. Here to calculate y_{cm}, we assume a small elemental ring of mass dm of radius 'r' on disc (disc can be considered to be made up of such thin rings of increasing radii) which will be integrated from '0' to 'R'. Here dm is given as

$dm = \sigma\pi\, r dr$

Where σ is the mass density of the semi circular disc.

$$\sigma = \frac{M}{\pi R^2 / 2} = \frac{2M}{\pi R^2}$$

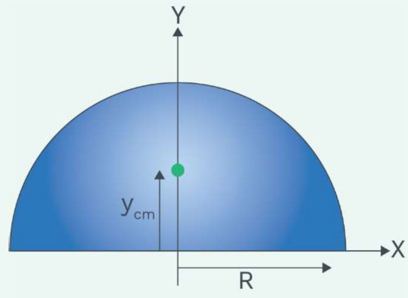




Now the y-coordinate of an element is taken as $\frac{2r}{\pi}$, (as in previous section, we have derived that the centre of mass of the semi-circular ring is placed at $\frac{2R}{\pi}$)

Concept Reminder

Centre of mass of Semi-circular Disc:



$$y_{cm} = \frac{4R}{3\pi}$$

Rack your Brain

A uniform square plate and a disc having same density are kept in contact. Locate the position of centre of mass of the system w.r.t. centre of the square.

$$y_{cm} = \frac{\int_0^R dm \cdot y}{\int_0^R dm}$$

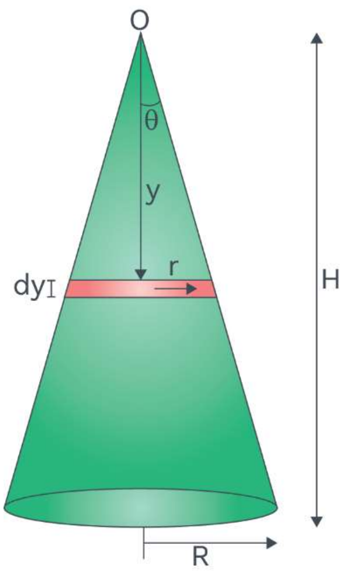
Here 'y' is the position of COM of dm mass.
Here y_{cm} is given as

$$y_{cm} = \frac{\int_0^R dm \frac{2r}{\pi}}{\int_0^R \sigma \pi r\, dr} = \int_0^R \frac{4}{\pi R^2} r^2 dr$$

$$\Rightarrow \quad y_{cm} = \frac{4R}{3\pi}$$

(d) Centre of mass of a Solid Cone:

A solid cone has mass 'M', height 'H' and base radius 'R'. Obviously the C.O.M of this cone will lie somewhere on its axis, at a height less than H/2. To locate the centre of mass we consider an elemental disc of width 'dy' and radius 'r', at a distance 'y' from the apex of the cone. Let mass of this disc be dm, which can be defined as

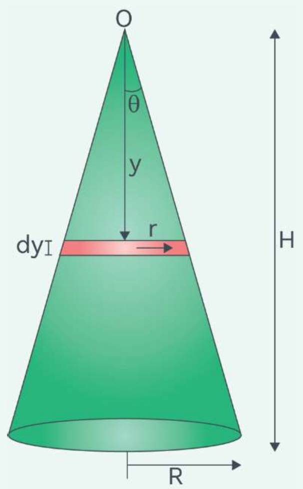


$dm = \rho \times \pi r^2 dy$

Here ρ is mass density of solid cone

Concept Reminder

Centre of mass of a Solid Cone:



$$y_{cm} = \frac{3H}{4}$$

Here y_{COM} can be given as

$$y_{COM} = \frac{1}{M}\int_0^H y\,dm$$

$$= \frac{1}{M}\int_0^H \left(\frac{3M}{\pi R^2 H}\pi\left(\frac{Ry}{H}\right)^2 dy\right)y$$

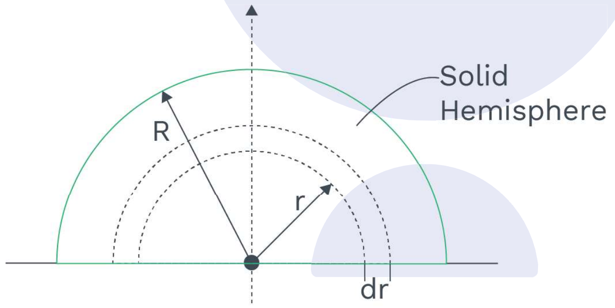
$$= \frac{3}{H^3}\int_0^H y^3 dy = \frac{3H}{4}$$

Concept Reminder

The centre of mass of a uniform hollow hemisphere of radius R is $y_{cm} = \frac{R}{2}$.

(e) C.O.M of a solid Hemisphere:

A hemisphere is of mass density 'ρ' and radius 'R'. To find its COM (only y co-ordinate) we consider an elemental hollow hemisphere of radius 'r' on solid hemisphere (solid hemisphere can be considered to be made up such hollow hemisphere of increasing radii) which will be integrate from 'O' to 'R'.



Here 'y' Co-ordinate of centre of mass of elemental hollow hemisphere is (0, r/2, 0)

$dm = \rho\, 2\pi r^2 dr$

$$y_{COM} = \frac{\int_0^R dm \cdot y}{\int_0^R dm}; \quad y_{COM} = \frac{\int_0^R \rho(2\pi r^2)dr(r/2)}{\int_0^R \rho \cdot 2\pi r^2 \cdot dr}$$

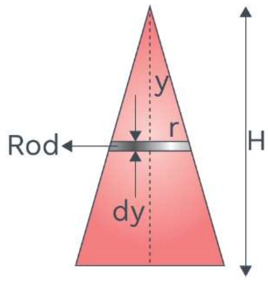
$$y_{COM} = \frac{3R}{8}$$

Concept Reminder

The centre of mass of a uniform solid hemisphere of radius R is $y_{cm} = \frac{3R}{8}$.

(f) Centre of mass of Triangular Plate:

A triangular plate has mass density of σ height 'H' and base of '2R'. Obviously COM of this plate will exist some where on its axis at height less than H/2. To locate COM we consider an elemental rod of width dy and length 2r at a distance 'y' from the apex of the plate. Let mass of this rod be dm which can be given as



$dm = \sigma\,(2r)dy$

From the theorem of triangle

$$\frac{H}{R} = \frac{y}{r}$$

$$\Rightarrow \quad r = \frac{Ry}{H}$$

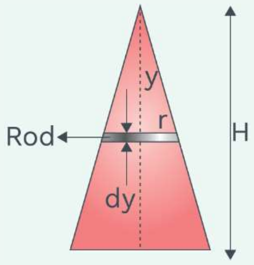
Here y_{COM} can be given as

$$y_{COM} = \frac{\int_0^H y\,dm}{\int_0^H dm}; \quad y_{COM} = \frac{\int_0^H \sigma(2r)dy \cdot y}{\int_0^H \sigma(2r)dy}$$

$$y_{COM} = \frac{\int_0^H \sigma\left(\frac{2Ry}{H}\right) y\,dy}{\int_0^H \sigma\left(\frac{2Ry}{H}\right) dy}; \quad y_{COM} = \frac{2H}{3}$$

Concept Reminder

Centre of mass of Triangular Plate:

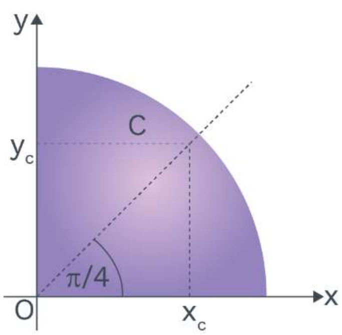


$$y_{cm} = \frac{2H}{3}$$

Circular Arc Sector of a Circle:

Ex. Find the coordinates of COM of a quarter sector of a uniform disk of radius 'r' placed in the 1st quadrant of Cartesian coordinate system with centre at origin.

Sol. From the result obtained for sector of circular plate distance OC of the COM form the center is



Concept Reminder

(i) If the rod is of uniform density i.e, λ = constant

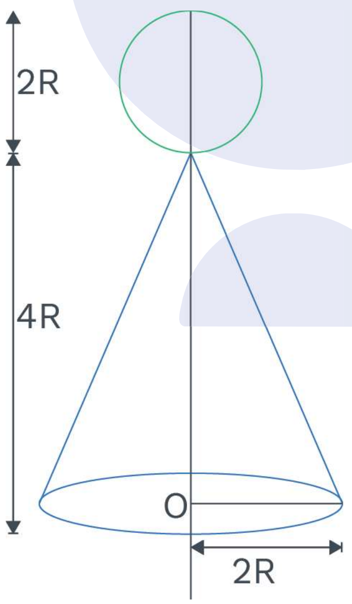
then, $x_{cm} = \frac{L}{2}$.

(ii) If the density of rod varies linearly with x then $x_{cm} = \frac{2L}{3}$.

$$OC = \frac{2r\sin\left(\frac{\pi}{4}\right)}{\frac{3\pi}{4}} = \frac{4\sqrt{2}r}{3\pi}$$

Coordinates of COM (x_c, y_c) are $\left(\frac{4r}{3\pi}, \frac{4r}{3\pi}\right)$

Ex. A man has made a toy as shown. If density of material of the sphere is 12 times that of the cone then find the position of the centre of mass. [Centre of mass of a cone of height h is at height of $\frac{h}{4}$ from its base.]



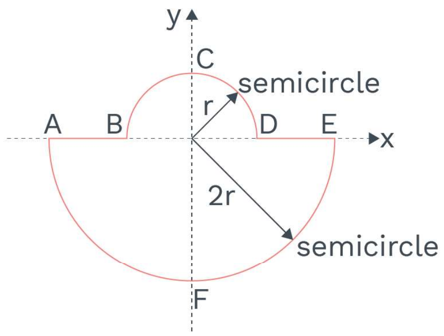
Sol. $m_2 = 12\rho \times \frac{4}{3}\pi R^3 = 3m$

$m_1 = \rho \times \frac{1}{3}\pi(2R)^2 \times (4R) = m$

$$y_{COM} = \frac{m(R) + 3m(5R)}{4m} = 4R$$

So, location of COM is [0, 4R]

Ex. A uniform thin rod is bent in form of closed loop ABCDEFA as shown. The y-coordinate of the COM of the system is.



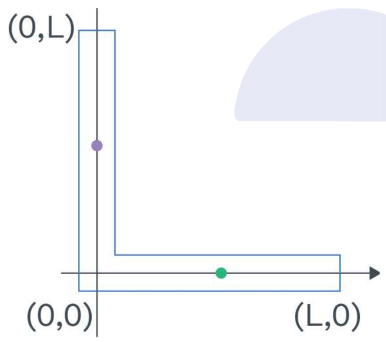
(1) $\frac{2r}{\pi}$ (2) $-\frac{6r}{3\pi+2}$ (3) $-\frac{2r}{\pi}$ (4) Zero

Sol. The centre of mass of semi-circular ring is at a distance from its centre.

(Let λ = mass/length)

$$\therefore \quad y_{COM} = \frac{\lambda\pi r \times \frac{2r}{\pi} - \lambda \times 2\pi r \times \frac{4r}{\pi}}{\lambda\pi r + \lambda r + \lambda r + \lambda \times 2\pi r} = -\frac{6r}{3\pi+2}$$

Ex. Find position of COM.



Sol. Centre of mass of rods are $\left(\frac{L}{2}, 0\right), \left(0, \frac{L}{2}\right)$

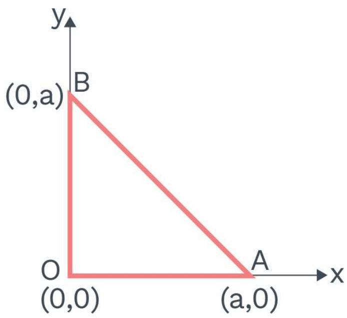
For the system

$$x_{COM} = \frac{m\left(\frac{L}{2}\right) + m(0)}{2m} = \frac{L}{4}$$

$$y_{COM} = \frac{m(0) + m\left(\frac{L}{2}\right)}{2m} = \frac{L}{4}$$

So, position of COM $\left(\frac{L}{4}, \frac{L}{4}\right)$

Ex. Three rods of the same mass are placed as shown in the figure. Find the coordinates of the COM of the system.



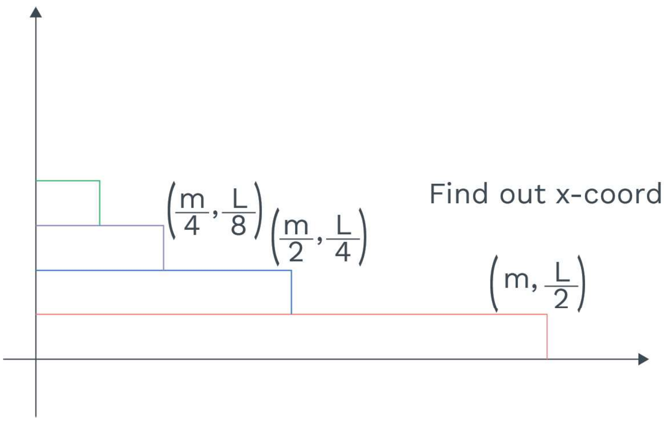
Sol. COM of rod OA is at $\left(\frac{a}{2}, 0\right)$, COM of rod OB is at $\left(0, \frac{a}{2}\right)$ and COM of rod AB is at $\left(\frac{a}{2}, \frac{a}{2}\right)$

For the system,

$$x_{COM} = \frac{m \times \frac{a}{2} + m \times 0 + m \times \frac{a}{2}}{m + m + m} = \frac{a}{3}$$

$$\Rightarrow \quad y_{COM} = \frac{m \times 0 + m \times \frac{a}{2} + m \times \frac{a}{2}}{m + m + m} = \frac{a}{3}$$

Ex. Find out x-coordinate of COM.

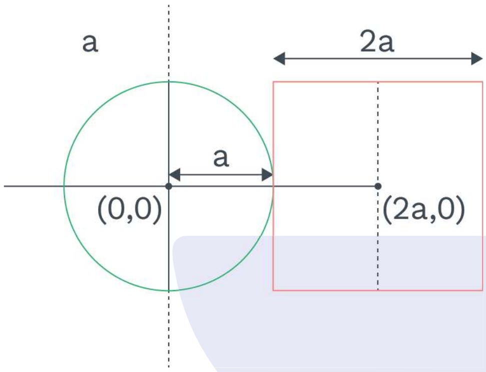


Find out x-coordinate of COM.

Sol. $$x_{COM} = \frac{m\left(\frac{L}{2}\right) + \frac{m}{2}\left(\frac{L}{4}\right) + \frac{m}{4}\left(\frac{L}{8}\right) + ...}{m + \frac{m}{2} + \frac{m}{4} + ...}$$

$$x_{COM} = \frac{\frac{mL}{2}\left[1+\frac{1}{4}+\frac{1}{16}+.....+\infty\right]}{m\left[1+\frac{1}{2}+\frac{1}{4}+.....+\infty\right]} = \frac{\frac{L}{2}\left[\frac{1}{1-1/4}\right]}{\left[\frac{1}{1-1/2}\right]} = \frac{L}{3}$$

Ex. Find out position of COM from given reference point.



(1) If their masses are equal
(2) If their density are equal

Sol. (1) $x_{COM} = \frac{m(0)+m(2a)}{2m} = a$

So, position of COM (a, 0)

(2) On the basis of mass distribution density are of 3 type

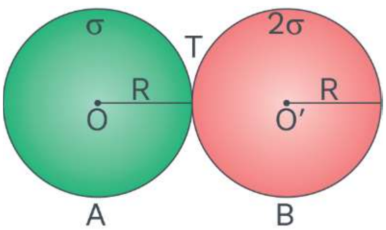
(i) Linear density (λ) = $\frac{m}{\ell} \Rightarrow m = \lambda\ell$

(ii) Area density (σ) = $\frac{m}{A} \Rightarrow m = \sigma A$

(iii) Volume density (ρ) = $\frac{m}{V} \Rightarrow m = \rho V$

$$\Rightarrow \quad x_{COM} = \frac{(\sigma\pi a^2)\times 0 + (\sigma\times 4a^2)\times 2a}{\sigma\times\pi a^2 + \sigma\times 4a^2} = \left(\frac{8a}{\pi+4}\right)$$

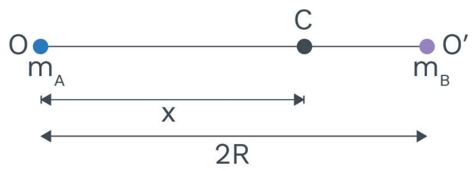
Ex. Two circular disc having radius 'R' and mass density σ and 2σ respectively are placed as shown. Then What will be the position of COM of the system.



Sol. Mass of disc A $m_A = \sigma\pi R^2$

Mass of disc B $m_B = 2\sigma\pi R^2$

Because of the symmetry the COM of disc 'A' lie at point 'O' and COM of disc 'B' lie at point O′. So, we realise the above problem in a following way.

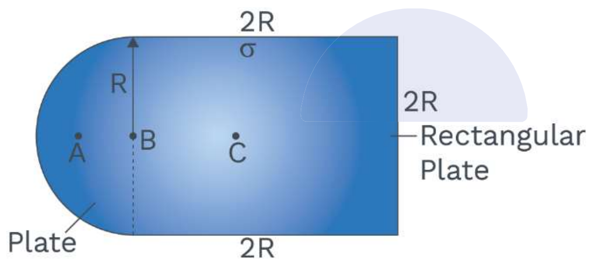


Centre of mass because of the both disc lie at point 'C' (assume), having distance x from m_A

$$\Rightarrow \quad x = \frac{m_B(2R)}{m_A + m_B}; \quad x = \frac{2\sigma\pi R^2(2R)}{\sigma(\pi R^2 + 2\pi R^2)}; \quad x = \frac{4R}{3}$$

So, the centre of mass lie in the disc B having distance $\frac{4R}{3}$ from O.

Ex. Find out the position of centre of mass of the figure shown below.



Sol. We divide the above problem in two parts

(i) First find out position of COM of both semi-circular plate and rectangular plate separately.

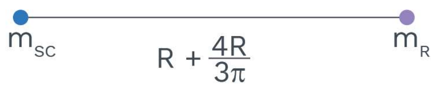
(ii) Then find the position of COM of given structure.

COM of semi-circular disc lie at $\frac{4R}{3\pi}$

$$\Rightarrow \quad AB = \frac{4R}{3\pi}$$

COM of rectangular plate lie at centre of plate at point C

$\Rightarrow \quad BC = R$



$$m_{SC} = \frac{\sigma\pi R^2}{2}; \quad m_R = \sigma 4R^2$$

Let us assume COM is at r_1 distance from m_R

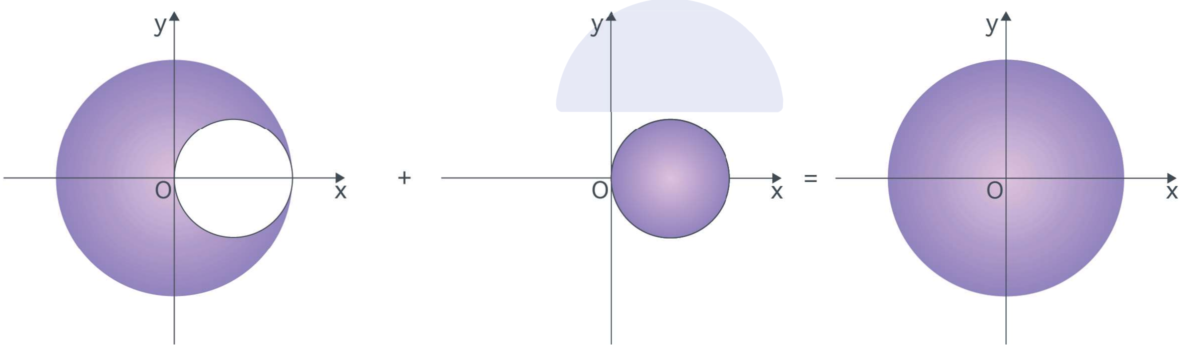
$$\Rightarrow \quad r_1 = \frac{\sigma \cdot \frac{\pi R^2}{2}\left(R + \frac{4R}{3\pi}\right)}{\sigma \cdot \frac{\pi R^2}{2} + \sigma 4R^2}$$

$$\Rightarrow \quad r_1 = \frac{\pi R(3\pi + 4)}{3(\pi + 8)}$$

CAVITY PROBLEMS:

If some mass or area is removed from the rigid body then position of COM of the remaining portion is obtained by assuming that in the remaining part +m & −m mass is there. Further steps are explained by example.

To find the centre of mass of truncated bodies or bodies with cavities we can make use of superposition principle that is, if we restore the removed portion in the same place we obtain the original body. The idea is illustrated in the following figure.



The removed portion is added to the truncated body keeping their location unchanged relative to the coordinate frame.

If a portion of a body is taken out, the remaining portion may be considered as, [Original mass (M)

−mass of the removed part (m)] = {original mass (M)} + { − mass of the removed part (m)}

The formula changes to:

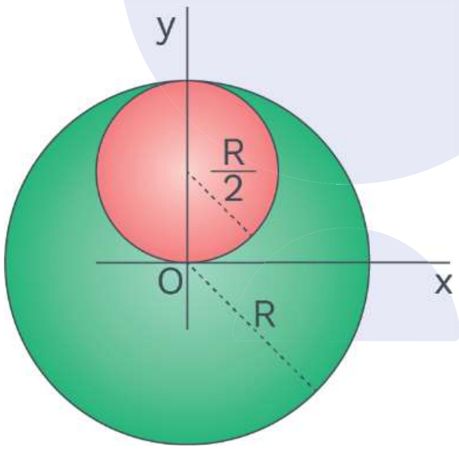
$$x_{COM} = \frac{Mx - mx'}{M - m}$$

$$y_{COM} = \frac{My - my'}{M - m}; \quad z_{COM} = \frac{Mz - mz'}{M - m}$$

Where x', y' and z' represent the coordinates of the centre of mass of the removed part.

Ex. A disc of radius R is cut off from a uniform thin sheet of metal. A circular hole of radius R/2 is now cut out from the disc, with the hole being tangent to the rim of the disc. Find the distance of the centre of mass from the centre of the original disc.

Sol. We treat the hole as a 'negative mass' object that is combined with the original uncut disc. (When the two are overlapped together, the hole region then has zero mass). By symmetry, the CM lies along the +y-axis in figure, so $x_{COM} = 0$. With the origin at the centre of the original circle whose mass is assumed to be m. Mass of original uncut circle $m_1 = m$ and location of COM = (0, 0)



Mass of hole of negative mass,

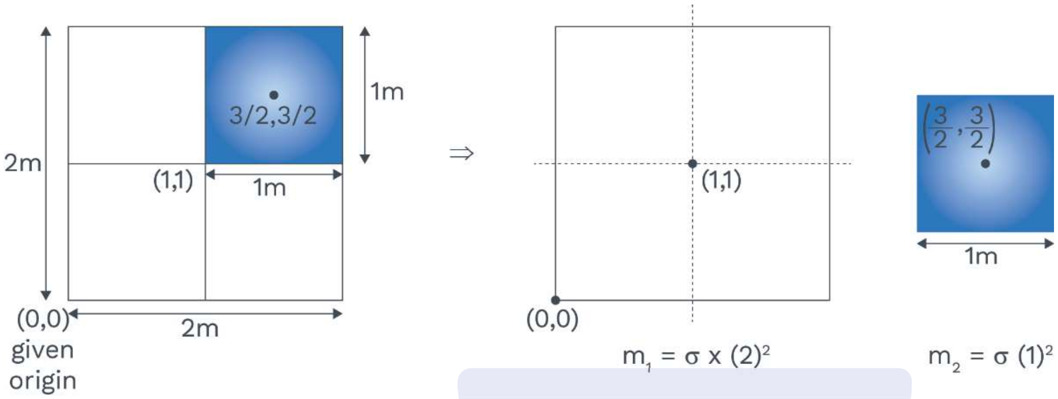
$$m_2 = \frac{m}{4}; \text{ Location of COM} = \left(0, \frac{R}{2}\right)$$

Thus,

$$y_{COM} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2} = \frac{m(0) + \left(-\frac{m}{4}\right)\frac{R}{2}}{m + \left(-\frac{m}{4}\right)} = -\frac{R}{6}$$

So, the centre of mass is at the point $\left(0, -\frac{R}{6}\right)$. Thus, the required distance is R/6.

Ex. From given plate if shaded position is removed than find out COM of remaining.



Sol. $$x_{COM} = \frac{[\sigma \times (2)^2] \times 1 - [\sigma(1)^2] \times \frac{3}{2}}{(\sigma \times 2^2) - (\sigma \times (1)^2)}$$

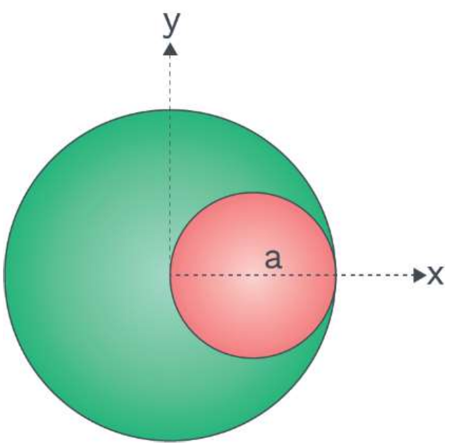
$$x_{COM} = \frac{4 - \frac{3}{2}}{4 - 1} = \frac{5}{6}$$

Similarly, $y_{COM} = \frac{5}{6}$

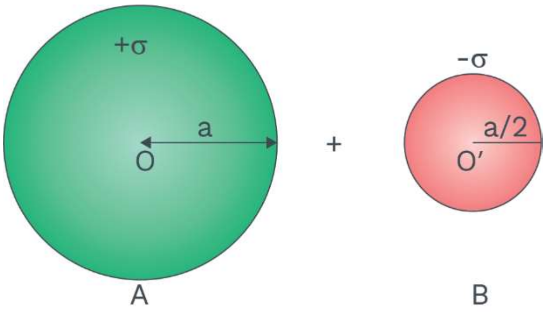
$$x_{COM} = \frac{4 - \frac{3}{2}}{4 - 1} = \frac{5}{6}$$

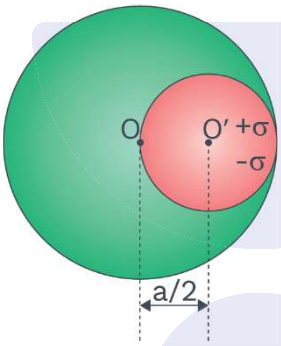
So, position of COM $\left[\frac{5}{6}, \frac{5}{6}\right]$

Ex. Find the position of COM of the uniform lamina shown in figure. If the mass density of lamina is σ.

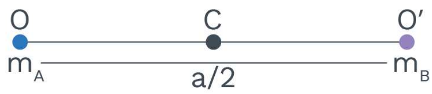


Sol. We assume that in remaining portion a disc of radius a/2 having mass density +σ is there then we also include one disc of a/2 radius having – σ mass density. So now the problem changes in following form





So, the COM of both disc ‘A & B’ lie in their respective centre such as O & O’. Now



$$\Rightarrow \quad \text{COM of the lamina} = \frac{m_A \frac{a}{2}}{m_A + m_B}$$

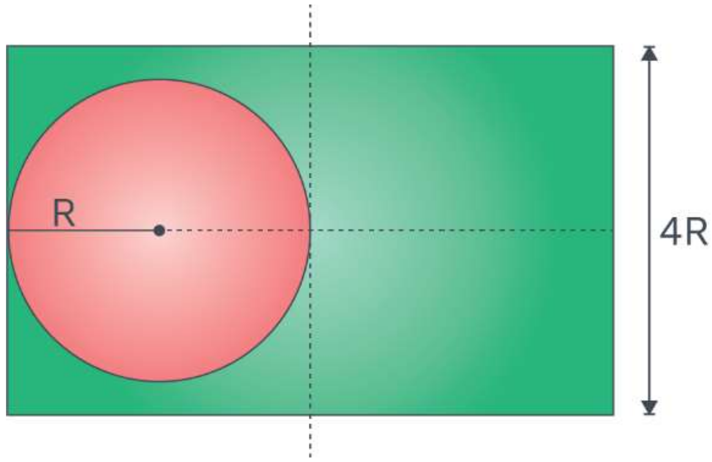
$$m_A = \sigma(\pi a^2)$$

$$m_B = -\sigma(\pi)\left(\frac{a}{2}\right)^2 = -\sigma\pi\frac{a^2}{4}$$

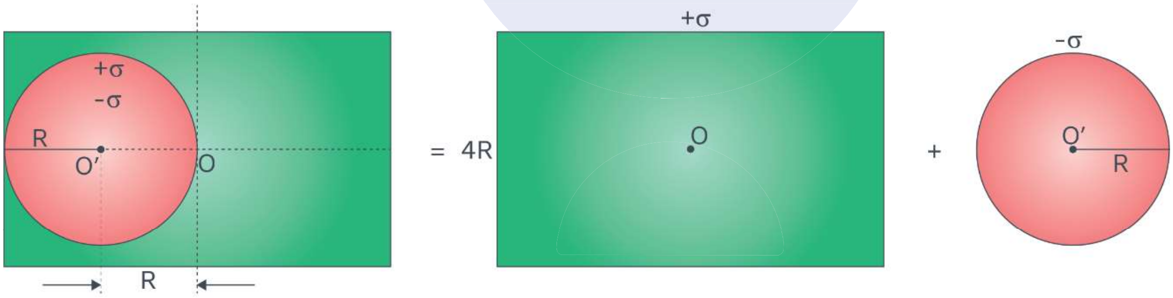
$$\Rightarrow \quad \text{COM} = \frac{\sigma\pi^2 a^2 . \frac{a}{2}}{\sigma\pi a^2 - \sigma\pi\frac{a^2}{4}} = \frac{\frac{a^3}{2}}{\frac{3a^2}{4}} = \frac{a^3}{2} \times \frac{4}{3a^2} = \frac{2a}{3}$$

i.e., C.O.M lie on leftward side from point O.

Ex. Find the position of COM of the uniform lamina as shown in figure.



Sol. We assume that a disc of radius ‘R’ having mass density $-\sigma$ is in the removed section. Now the problem change in following form



When a disc of mass density $+\sigma$ and radius ‘R ‘is include than a complete rectangular plate is make having COM at point ‘O’. When consider only disc having mass density $-\sigma$ and radius ‘R’ then C.O.M of this disc lie at point O

Then the position of COM

$$= \frac{\sigma(4R)^2 \cdot R}{-\sigma\pi R^2 + \sigma(4R)^2} = \frac{\sigma 16 R^3}{\sigma R^2(16-\pi)} = \frac{16R}{16-\pi}$$

i.e., centre of mass lie in the rightwards side from the cavity.

MOTION OF COM AND CONSERVATION OF MOMENTUM:

The position of COM is given by

$$\vec{r}_{COM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + ...}{m_1 + m_2 + m_3 + ...} \quad ...(i)$$

Here m_1, m_2, m_3 ..... are the mass in the system and $\vec{r}_1, \vec{r}_2, \vec{r}_3$ ..... is the corresponding position vector of m_1, m_2, m_3 respectively.

(a) Velocity of COM of system:

To find the velocity of COM we differentiate equation (i) with respect to time

$$\frac{d\vec{r}_{COM}}{dt} = \frac{m_1 \frac{d\vec{r}_1}{dt} + m_2 \frac{d\vec{r}_2}{dt} + m_3 \frac{d\vec{r}_3}{dt} + ...}{m_1 + m_2 + m_3 + ...}$$

$$\vec{v}_{COM} = \frac{m_1 \frac{d\vec{r}_1}{dt} + m_2 \frac{d\vec{r}_2}{dt} + m_3 \frac{d\vec{r}_3}{dt} + ...}{m_1 + m_2 + m_3 + ...}$$

$$\vec{v}_{COM} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + ...}{m_1 + m_2 + m_3 + ...} \quad ...(ii)$$

(b) Acceleration of COM of the system:

To find acceleration of C.O.M we differentiate equation (ii)

$$\frac{d\vec{v}_{COM}}{dt} = \frac{m_1 \frac{d\vec{v}_1}{dt} + m_2 \frac{d\vec{v}_2}{dt} + m_3 \frac{d\vec{v}_3}{dt} + ...}{m_1 + m_2 + m_3 + ...}$$

$$\vec{a}_{COM} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + ...}{m_1 + m_2 + m_3 + ...} \quad ...(iii)$$

We can write

$$M\vec{v}_{COM} = m_1\vec{v}_1 + m_2\vec{v}_2 + ...$$

$$\vec{p}_{COM} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + ... [\because \vec{p} = m\vec{v}]$$

$$M\vec{v}_{COM} = \vec{p}_{COM} \quad [\because \Sigma\vec{p}_i = \vec{p}_{COM}]$$

Linear momentum of the system of particles is equal to the product of mass of the system with velocity of its centre of mass. From Newton's second law

$$\vec{F}_{ext} = \frac{d(M\vec{v}_{COM})}{dt}$$

$$\Rightarrow \quad M\vec{a}_{COM} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + ...$$

$$\Rightarrow \quad \vec{F}_{net} = M\vec{a}_{COM}$$

Concept Reminder

The position of centre of mass is given by-

$$\vec{r}_{COM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + ...}{m_1 + m_2 + m_3 + ...}$$

Velocity of COM of system:

$$\vec{v}_{COM} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + ...}{m_1 + m_2 + m_3 + ...}$$

Acceleration of centre of mass of the system:

$$\vec{a}_{COM} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + ...}{m_1 + m_2 + m_3 + ...}$$

If $\vec{F}_{ext} = \vec{0}$, then $\vec{v}_{COM}$ = constant.

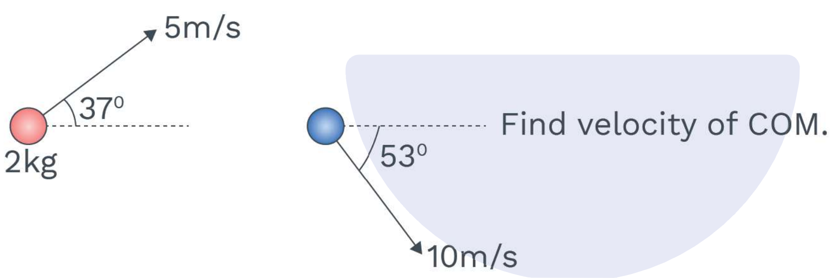
If no external force acts on the system velocity of its COM remains constant, i.e., velocity of COM is unaffected by internal forces.

$$\boxed{\vec{F}_{net} = M\vec{a}_{COM}}$$

Rack your Brain

Two particles of mass 1 kg and 2 kg are moving along same line with speed 2 m/s and 4 m/s respectively. Calculate speed of centre of mass of system if both particles are moving in same direction.

Ex. Find velocity of COM.



Sol. $v_{cm} = \dfrac{2\times(4\hat{i}+3\hat{j})+1(6\hat{i}-8\hat{j})}{3}$

$$= \frac{8\hat{i}+6\hat{j}+6\hat{i}-8\hat{j}}{3} = \frac{14\hat{i}-2\hat{j}}{3}$$

$$= \left(\frac{14\hat{i}}{3}-\frac{2\hat{j}}{3}\right) m/s$$

Concept Reminder

$$\vec{F}_{ext} = M\vec{a}_{cm}$$

If net external force on the system is zero, the linear momentum of the system, is conserved and the centre of mass will move with constant velocity.

Motion of COM in a moving system of particles:

(a) COM at Rest:

If F_{ext} = 0 and v_{COM} = 0, then COM remains at rest. Individual components of system may move and have non-zero momentum due to the mutual forces (internal), but net momentum of system remains zero.

(i) All the particles of system are at rest.

(ii) Particles are moving in a way such that their net momentum is zero.

Example:

(iii) A bomb at rest suddenly explodes into the various smaller fragments, all moving in the different directions then, since explosive forces are internal & there is no external force on system for explosion therefore, the COM of bomb will remain at the original position and fragment fly such that their net momentum will remain zero.

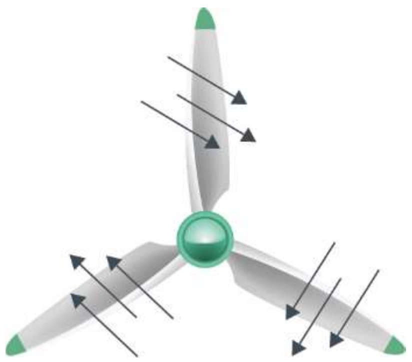
(iv) Two men standing on a frictionless platform, push each other, then also their net momentum will remain zero because push forces are internal for two men system.

(v) A boat floating in a lake, also has the net momentum zero if people on it changes their position, because friction force required to move the people is internal of boat system.

(vi) Objects which are initially at rest, if moving under the mutual forces (electrostatic or gravitation) also have the net momentum zero.

(vii) A light spring of spring constant 'k' kept compressed between two blocks of masses m_1 and m_2 on the smooth horizontal surface. When released, blocks acquire velocities in opposite directions, such that net momentum is zero.

(viii) In a fan, all the particles are moving but COM is at rest



(b) COM moving with uniform velocity:

If $F_{ext} = 0$, then v_{COM} remains constant

KEY POINTS

- Motion of centre of mass
- Velocity of centre of mass
- Acceleration of centre of mass
- Conservation of linear momentum

Concept Reminder

When a radioactive nucleus initially at rest decays, the daughter nuclei fly off in different directions with different velocities obeying the principle of linear momentum conservation.

therefore, the net momentum of system also remains conserved. Individual components of system may have variable velocity and the momentum because of the mutual forces (internal), but the net momentum of system remains constant and COM continues to move with initial velocity.

(i) All the particles of system are moving with same velocity.
e.g., A car moving with the uniform speed on a straight road, has its COM moving with a constant.

(ii) Internal explosions/breaking does not change motion of COM and net momentum remains conserved. A bomb going in a straight line suddenly explodes into the various smaller fragments, all moving in different directions then, since explosive forces are internal & there is no external force on system for explosion therefore, the COM of bomb will continue original motion and the fragment fly such that their net momentum remains conserved.

(iii) Man jumping from the cart or buggy also exert internal forces therefore the net momentum of the system and hence, Motion of the COM remains conserved.

(iv) Two moving blocks connected by a light spring on the smooth horizontal surface. If the acting forces is only due to spring then COM will remain in its motion and the momentum will remain conserved.

(v) Particles colliding in absence of the external impulsive forces also have their momentum conserved.

Rack your Brain

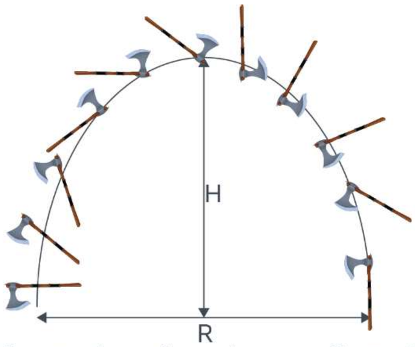
Two bodies A and B are attracted towards each other due to gravitation. Given A is much higher than B then centre of mass will-
(1) Shifts towards A
(2) Shifts towards B
(3) Remain at rest
(4) Cannot say anything

(c) COM moving with acceleration:

If an external force is there then COM continues its original motion as if external force is acting on it, irrespective of the internal forces.

Example:

Projectile motion: An axe has thrown in air at an angle θ with horizontal will perform a complicated motion of rotation as well as the parabolic motion under effect of gravitation.



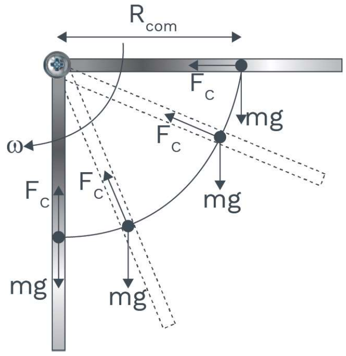
The motion of axe is complicated but the COM is moving in a parabolic motion.

$$H_{COM} = \frac{u^2 \sin^2 \theta}{2g}$$

$$R_{COM} = \frac{u^2 \sin^2 2\theta}{g}; \quad T = \frac{2u\sin\theta}{g}$$

Example:

Circular Motion: A rod hinged at end, rotates, than its COM performs circular motion. The centripetal force (F) required in the circular motion is assumed to be acting on the COM.



$$F_c = m\omega^2 R_{COM}$$

Ex. Two forces $\vec{F}_1 = (4\hat{i} - 2\hat{j} + 3\hat{k})N$,

$\vec{F}_2 = (7\hat{i} - 8\hat{j} + 5\hat{k})N$

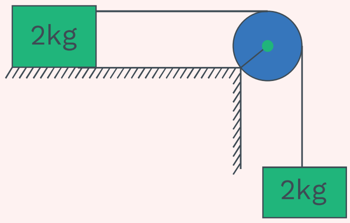
are acting on two particles of mass 10 kg and 5 kg respectively than find out $\vec{a}_{COM}$

Concept Reminder

To determine the motion of centre of mass no knowledge of internal forces of the system of particles is required. For this purpose we need to know only the external forces.

Rack your Brain

Find acceleration of centre of mass of system shown.

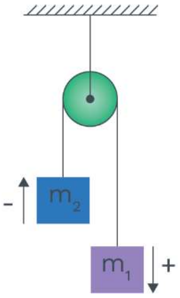


Sol. $\vec{a}_{COM} = \dfrac{\vec{F}_{net}}{m_{total}}$

$$\vec{a}_{COM} = \frac{(4\hat{i} - 2\hat{j} + 3\hat{k}) + (7\hat{i} + -8\hat{j} + 5\hat{k})}{15}$$

$$= \frac{(11\hat{i} - 10\hat{j} + 8\hat{k})}{15} m/s^2$$

Ex. For a given system find out $\vec{a}_{COM}$.



Sol. $\vec{a}_{COM} = \dfrac{m_1\vec{a}_1 + m_2\vec{a}_2}{m_1 + m_2} = \dfrac{(m_1 - m_2)\vec{a}}{(m_1 + m_2)}$

$$\because \quad a = \frac{(m_1 - m_2)g}{m_1 + m_2}$$

$$a_{COM} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) g \left(\frac{m_1 - m_2}{m_1 + m_2}\right) = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2 g$$

$$a_{COM} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2 \times g$$

Ex. Find the acceleration of COM of the blocks of masses m_1 and m_2 ($m_1 > m_2$) in Atwood's machine.

Sol. We know from Newton's laws of motion magnitude of acceleration of each block.

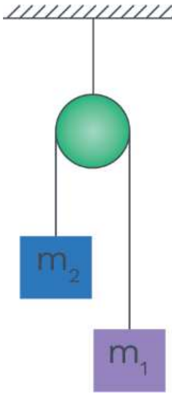
Concept Reminder

When an object of finite size is through with some initial velocity at an angle with horizontal, it follows a parabolic path. The centre of mass of such object also follow the parabolic path, even if the object were to disintegrate in mid-air.

Rack your Brain

The motion of the centre of mass of system of two particles is not affected by internal forces-

(1) Irrespective of their directions
(2) Only when they act along the line joining the particles
(3) Only when the forces are perpendicular to each other
(4) When the angle between the lines of action of forces lies between 0° and 90°



$$\vec{a} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)g$$

So, $a_{COM} = \frac{m_1 a + m_2(-a)}{m_1 + m_2}$

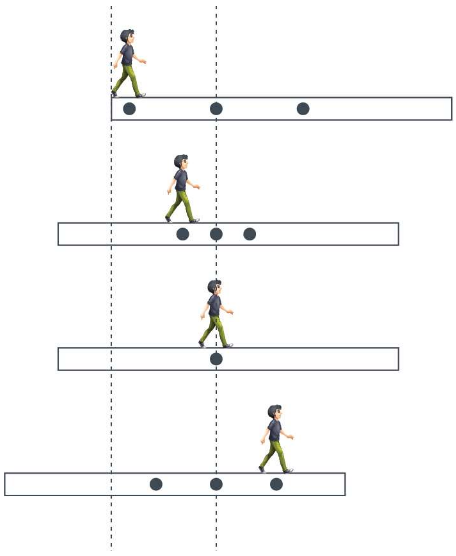
Acceleration of centre of mass

$$a_{COM} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2 g$$

Note: The magnitude of displacement of centre of mass in time 't' is

$$S_{COM} = \frac{1}{2} a_{COM} t^2$$

Man and Plank:



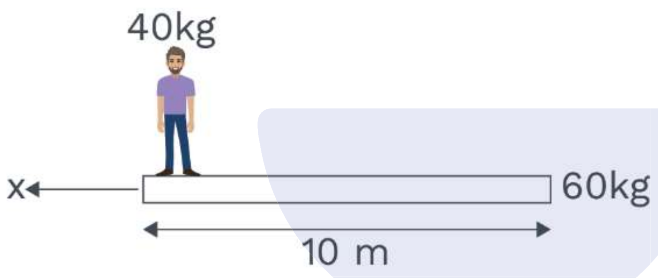
$$x_{COM} = \frac{m_1x_1 + m_2x_2 + ...}{M} \quad ...(i)$$

$$x_{COM} = \frac{m_1x'_1 + m_2x'_2 + ...}{M} \quad ...(ii)$$

From equation (i) and (ii)

$$\boxed{m_1(\Delta x_1) + m_2(\Delta x_2) + ... + m_n(\Delta x_n) = 0}$$

Ex. A person of mass 40 kg is standing on a plank of length 10 m. If person moves on the other side of plank find distance moved by the plank.



Sol. Let plank moves distance 'x'.

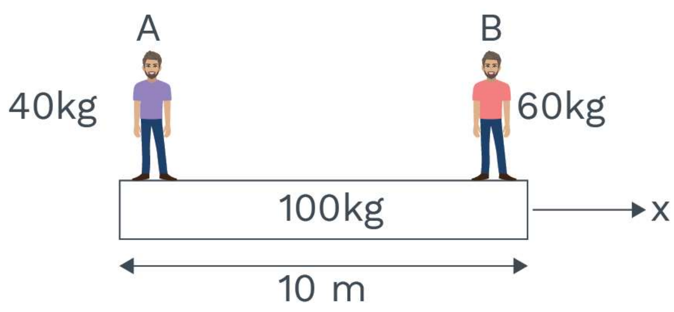
$40(10 - x) + 60(-x) = 0$

$40(10 - x) = 60 \times (x)$

$40 - 4x = 6x$

$10x = 40 \quad \Rightarrow \quad x = 4 \text{ m}$

Ex. Two person A & B are standing on a plank as shown in figure. If A & B exchange, then position find displacement of the plank.



Sol. Since there is no external force is acting on system, so COM of system does not change its position. Let plank moves 'x' distance.

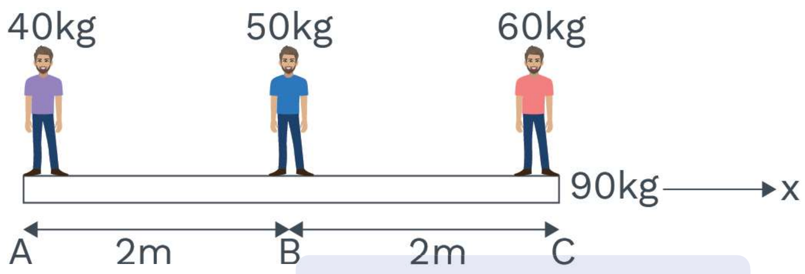
$40(10 + x) + 100(x) + 60(x - 10) = 0$

$400 + 40x + 100x + 60x - 600 = 0$

$40 - 60 + 10x + 4x + 6x - 20 + 20x = 0$

x = 1 m

Ex. Three man A, B, & C of masses 40 kg, 50 kg, 60 kg are standing on a plank of mass 90 kg. Which is kept on a smooth horizontal plane. If A & C exchange, their position then mass B will shift.



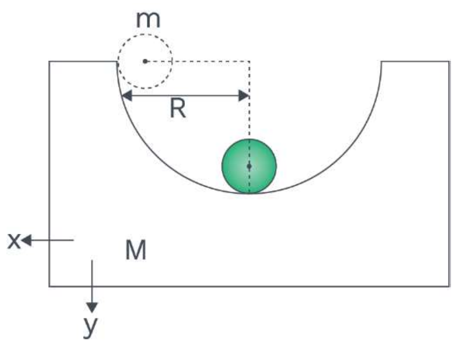
Sol. Let plank shift distance 'x'.

$40(4 + x) + 50 (x) + 60 (-4 + x) + 90x = 0$

$240x = 80$

$\Rightarrow \quad x = +\frac{1}{3} m$

Ex. A sphere of mass m is rolling down on a wedge of mass M. When sphere reaches at lowest point then find distance moved by wedge in horizontal direction.



Sol. $m(R - x) - Mx = 0$

$\Rightarrow mR - mx - Mx = 0$

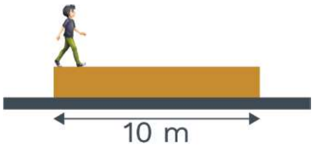
$\Rightarrow mR - x (M + m) = 0$

$\therefore \quad x = \frac{mR}{M + m}$

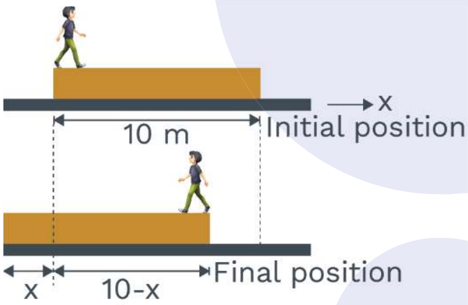
Ex. A wooden plank of 20 kg is at rest on a smooth horizontal floor. A boy of mass 60 kg starts moving from one end of plank to other end.

Centre of Mass

The length of plank is 10 m. Find displacement of the plank over floor when the boy reaches the other end of the plank.



Sol. Here, the system is man + plank. Net force on this system in the horizontal direction is zero and initially the COM of the system is at rest. Therefore, the COM does not move in horizontal direction.
Let 'x' be the displacement of Plank. Taking the origin, i.e., x = 0 at position shown below.



As we said earlier also, COM will not move in horizontal direction (x-axis). Therefore, for COM to remain stationary, $x_i = x_f$

$20x = 60 \times (10 - x)$

or $x = \frac{30}{4}\text{m}$ or $x = 7.5\text{m}$

Ex. A boy of mass m_1 is standing on a platform of mass m_2 kept on a smooth horizontal surface. The boy starts moving on the platform with a velocity v_r relative to the platform. Find the recoil velocity of platform.

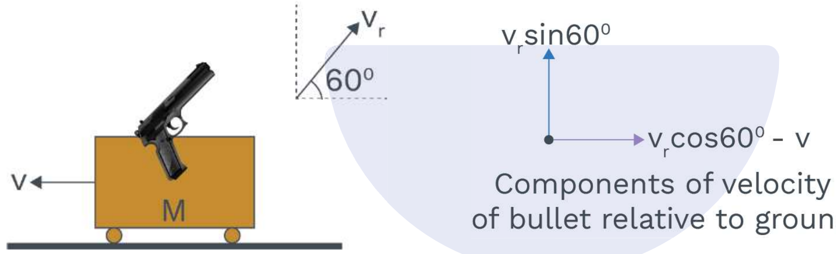
Sol. Absolute velocity of boy = $v_r - v$ where v = recoil velocity of platform. Taking the platform and the boy a system, net external force on system in horizontal direction is zero. The linear momentum of system remains constant. Initially both the boy and the platform were at rest.

Hence, $0 = m_1(v_r - v) - m_2 v$

$$\therefore \quad v = \frac{m_1 v_r}{m_1 + m_2}$$

Ex. A gun of mass 'M' fires a bullet of mass 'm' with speed v_r relative to barrel of gun which is inclined at an angle of 60° with the horizontal. The gun is placed over the smooth horizontal surface. Find recoil speed of gun.

Sol. Let the recoil speed of gun is 'v'. Taking gun + bullet as system. Net external force on



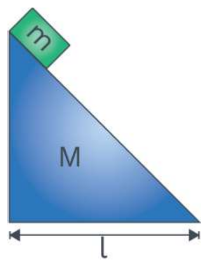
Components of velocity of bullet relative to ground

System in the horizontal direction is zero. Initially the system was at rest. Therefore, applying principle of the conservation of linear momentum in the horizontal direction, we get

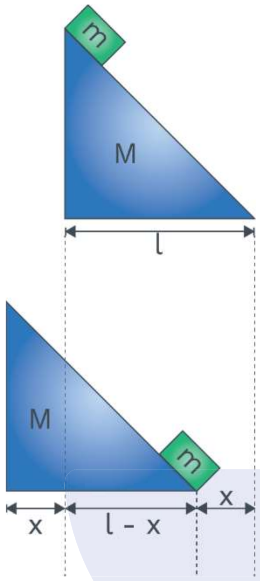
$Mv - m(v_r \cos 60° - v) = 0$

$$\therefore \quad v = \frac{mv_r \cos 60°}{M + m} \quad \text{or} \quad v = \frac{mv_r}{2(M + m)}$$

Ex. A particle of mass 'm' is placed at rest on top of a smooth wedge of mass 'M', which in turn is placed at rest on the smooth horizontal surface as shown in figure. Then distance moved by the wedge as particle reaches the foot of wedge is.



Sol. There will be no external force in the horizontal direction on wedge block system, so t he x-coordinate of the COM of the wedge block system is at rest.



Let us assume that wedge move ‘x’ when block reaches ground. We can use following relation when x-coordinate of COM is at rest

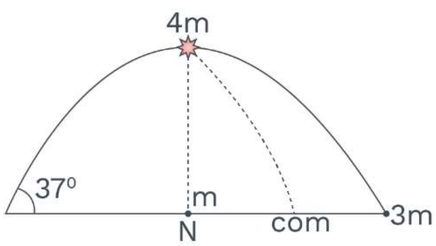
$m_1x_1 = m_2x_2$

$Mx = m(\ell - x)$

$$x = \frac{m\ell}{m + M}$$

Ex. A projectile is fired at a speed 100 m/s at an angle of 37° above horizontal. At highest point, projectile breaks into two parts of mass ratio 1 : 3, lighter piece coming to rest. Find distance from the launching point to the point where the heavier piece lands.

Sol. Internal force do not effect motion of the COM, the COM hits the ground at the position where original projectile would have landed. The range of original projectile is,



$$x_{COM} = \frac{2u^2 \sin\theta \cos\theta}{g} = \frac{2 \times 10^4 \times \frac{3}{5} \times \frac{4}{5}}{10} m$$

$= 960$ m

The COM will hit the ground at this position. As the smaller block comes to the rest after breaking, it falls down vertically and hits ground at half of range, i.e., at x = 480 m. If heavier block hits ground at x_2, then

$$x_{COM} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

$$\Rightarrow \quad 960 = \frac{(m)(480) + (3m)(x_2)}{(m + 3m)}$$

$x_2 = 1120$ m

Ex. A particle of mass '2 m' is projected at an angle of 45° with the horizontal with a velocity of $20\sqrt{2}$ m/s. After 1 s explosion takes place and particle is broken into 2 equal pieces. As a result of explosion one part comes to the rest. Find maximum height attained by the other part. (Take g = 10 m/s²)

Sol. Applying the conservation of linear momentum at time of collision, or at t = 1 s,

$$m\vec{v} + m(0) = 2m(20\hat{i} + 10\hat{j})$$

$$\therefore \quad \vec{v} = 40\hat{j} + 20\hat{j}$$

At 1 sec, the masses will be at height,

$$h_1 = u_y t + \frac{1}{2}v_y t^2 = (20)(1) + \frac{1}{2}(-10)(1)^2 = 15\,m$$

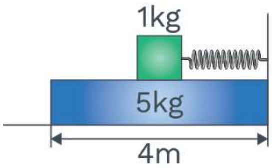
After the explosion other mass will further rise to a height

$$h_2 = \frac{u_y^2}{2g} = \frac{(20)^2}{2 \times 10} = 20\,m$$

$u_y = 20$ m/s just after collision

$\therefore$ Total height $h = h_1 + h_2 = 35$ m

Ex. A plank of mass 5 kg placed on the frictionless horizontal plane. Further a block of 1 kg which placed over plank. A massless spring of natural length 2 m is fixed to plank by its one end. The other end of the spring is compressed by block by half of spring's natural length. They system is now released from rest. What is velocity of the plank when block leaves plank? (The stiffness constant of the spring is 100 N/m)



Sol. Let velocity of the block and the plank, when block leaves the spring be 'u' and 'v' respectively. By the conservation of energy

$\frac{1}{2}kx^2 = \frac{1}{2}mu^2 + \frac{1}{2}Mv^2$ [M = mass of the plank, m = mass of the block]

$\Rightarrow \quad 100 = u^2 + 5v^2$ ...(i)

By conservation of momentum

$mu + Mv = 0$

$\Rightarrow \quad u = -5v$ ...(ii)

Solving equations (i) and (ii)

$30v^2 = 100 \Rightarrow v = \sqrt{\frac{10}{3}}$ m/s

From this moment until block falls, both the plank and block keep their velocity uniform.

Thus,

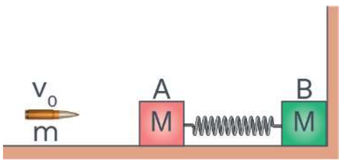
When the block falls, velocity of plank = $\sqrt{\frac{10}{3}}$ m/s.

Ex. Two identical blocks of mass M = 9 kg are placed on the rough horizontal surface of frictional coefficient μ = 0.1. The 2 blocks are joined by a light spring and block 'B' is in contact with a vertical fixed wall as shown. A bullet of mass m = 1 kg and v_0 = 10 m/s hits block 'A' and gets embedded in it. Find maximum compression of spring. (Take: Spring constant = 240 N/m, g = 10 m/s²)

Sol. For collision

$1 \times 10 = 10 \times v$

$\Rightarrow \quad v = 1$ m/s



If 'x' be the maximum compression

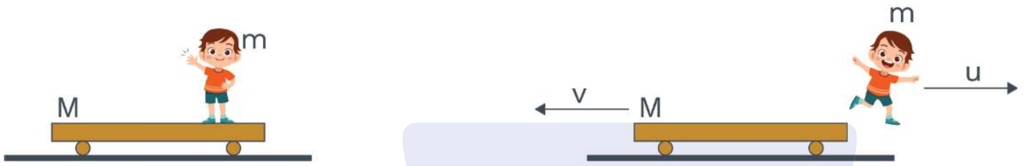
$\frac{1}{2} \times 10 \times 1^2 = \mu(m+M)gx + \frac{1}{2}kx^2$

$5 = 10x + 120x^2$

$$\Rightarrow \quad x = \frac{1}{6} m$$

Ex. A flat car of mass 'M' is at rest on a frictionless floor with a child of mass 'm' standing at its edge. If child jumps off from car towards right with an initial velocity 'u', w.r.t the car, find the velocity of car after its jump.

Sol. Let car attains a velocity 'v', and net velocity of child wr.t earth will be u – v, as 'u' is its velocity w.r.t car.



Initially, system was at rest, thus according to the momentum conservation, momentum after the jump must be zero, as

m(u – v) = Mv

$$v = \frac{mu}{m + M}$$

Ex. A flat car of mass 'M' with a child of mass m is moving with a velocity v_1 on a friction less surface. The child jumps in direction of motion of car with a velocity 'u' w.r.t car. Find the final velocities of child and that of car after jump.

Sol. This case is similar to previous example, except now car is moving before jump. Here also there is no external force is acting on the system in horizontal direction, hence the momentum remains conserved in this direction.

After the jump car attains a velocity v_2 in same direction, which is less than v_1, due to backward push of child for jumping. After jump the child attains a velocity u + v_2 in the direction of motion of the car, w.r.t ground.

According to the momentum conservation

$(M + m)\, v_1 = Mv_2 + m\,(u + v_2)$

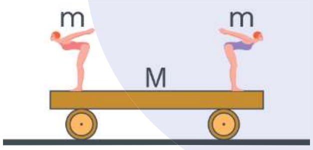
Velocity of the car after jump is

$$v_2 = \frac{(M+m)\,v_1 mu}{M+m}$$

Velocity of the child after jump is

$$u + v_2 = \frac{(M+m)\,v_1 + (M+2m)\,u}{M+m}$$

Ex. Two persons 'A' and 'B', each of mass 'm' are standing at two ends of rail-road car of mass M. The person 'A' jumps to the left with a horizontal speed 'u' with respect to the car. Thereafter, the person 'B' jumps to the right, again with the same horizontal speed 'u' with respect to the car. Find velocity of the car after both the persons have jumped off.



Sol. Let car attain the velocity 'v' in right ward and velocity of man A with respect to ground is v' then

$v' = v - u$

From momentum conservation

$0 = mv' + (M + m)v$

$\Rightarrow \quad m(v - u) + (M + m)v = 0$

$$\Rightarrow \quad v = \frac{mu}{(M+2m)}$$

After wards mass 'B' jumps to right with the same horizontal speed 'u' with respect to car, than car attain v" velocity from linear momentum conservation.

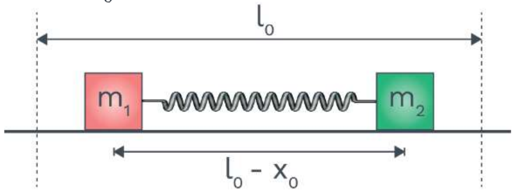
$(M + m)v = m(u + v") + Mv"$

$$(M+m)\left[\frac{mu}{M+2m}\right] = mu + (m+M)v''$$

$$\text{Now, } v'' = \frac{m^2 u}{(M+2m)(M+m)}$$

SPRING BLOCK SYSTEM:

A light spring of spring constant 'k' and natural length l_0 attached in a compressed condition between two blocks of the mass m_1 and m_2 on a smooth horizontal surface as shown in figure. The spring is initially compressed by distance x_0.



When the system is released, block acquire velocities in opposite direction. Let us assume that velocities of block m_1 and m_2 is v_1 and v_2 respectively at natural length of spring and since no external force acts on this system in horizontal direction. Hence linear momentum remains constant. Then from momentum conservation.

$$0 = m_2v_2 - m_1v_1$$

$$m_2v_2 = m_1v_1 \qquad \text{...(i)}$$

From mechanical energy conservation

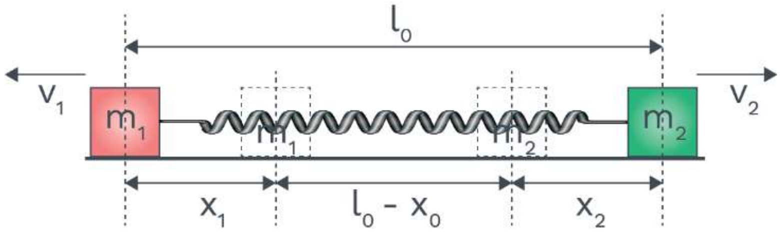
$$K_i + U_i = K_f + U_f$$

$$\Rightarrow \quad 0 + \frac{1}{2}kx_0^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + 0$$

$$\Rightarrow \quad \frac{1}{2}kx_0^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \qquad \text{...(ii)}$$

In the initial condition there is no external force on system and both block is at stationary condition. Therefore centre of mass of system is at rest. So, we can write.

$$m_1x_1 = m_2x_2 \qquad \text{...(iii)}$$



from above figure we can conclude

$$l_0 - x_0 + x_1 + x_2 = l_0$$

$$\Rightarrow \quad x_0 = x_1 + x_2 \qquad \text{(iv)}$$

Concept Reminder

If a gun of mass M (without bullet) fires a bullet of mass m with muzzle velocity υ then recoil speed of gun is $v = \frac{m\upsilon}{m+M}$.

Concept Reminder

If plank of mass M carrying a mass of mass m is moving horizontally with velocity v_0 on frictionless surface and man starts walking towards right with velocity μ w.r.t. the plank then velocity of man w.r.t. ground $= v_0 + \frac{M\upsilon}{m+M}$.

Due to inertia both the block move further from the position of the natural length of the spring. Maximum extension occur when both blocks come to rest. Let us suppose that x_1' & x_2' are extension in the spring from initial position due to block m_1 & m_2 from natural length. So, at maximum extension $v_1 = v_2 = 0$

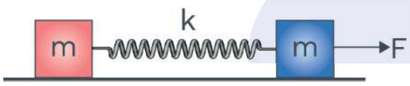
∴ COM is at rest Therefore we can write

$m_1x_1' = m_2x_2'$

$x_1' + x_2' + l_0 - x_0 = l_0 + x_0$

$x_1' + x_2' = 2x_0$

Ex. Two blocks of equal mass 'm' are connected by an unstretched spring and system is kept at rest on frictionless horizontal surface. A constant force 'F' is applied on one of blocks pulling it away from other as shown in figure.



(a) Find displacement of the COM at time t.

(b) If the extension of spring is x_0 at time t, find the displacement of two blocks at this instant.

Sol. (a) The acceleration of the COM is

$$a_{COM} = \frac{F}{2m}$$

The displacement of the COM at time t will be

$$x = \frac{1}{2}a_{COM}t^2 = \frac{Ft^2}{4m}$$

(b) Suppose the displacement of the 1st block is x_1 and that of the second is x_2. Then,

$$x = \frac{mx_1 + mx_2}{2m}$$

or $$\frac{Ft^2}{4m} = \frac{x_1 + x_2}{2}$$

or $x_1 + x_2 = \frac{Ft^2}{2m}$ ...(i)

Further, extension of spring is $x_1 - x_2$.

Therefore, $x_1 - x_2 = x_0$ ...(ii)

From equation (i) and (ii),

$$x_1 = \frac{1}{2}\left(\frac{Ft^2}{2m} + x_0\right) \text{ and } x_2 = \frac{1}{2}\left(\frac{Ft^2}{2m} - x_0\right)$$

IMPULSE:

Impulse of a force $\vec{F}$-acting on a body for the time interval $t = t_1$ to $t = t_2$ is defined as

$$\vec{I} = \int_{t_1}^{t_2} \vec{F}\,dt$$

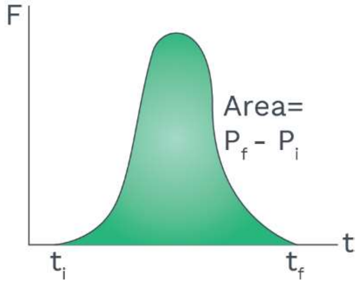
$$\vec{I} = \int \vec{F}dt = \int m\frac{d\vec{v}}{dt}dt = \int m d\vec{v}$$

$\vec{I} = m(\vec{v}_2 - \vec{v}_1) = \Delta\vec{P}$ = change in momentum due to-force $\vec{F}$

Also,

$$\vec{I}_{Re} = \int_{t_1}^{t_2} \vec{F}_{Res}dt = \Delta\vec{R}$$

Note: Impulse applied to an object in a given time the interval can also be calculated from area under force time (F-t) graph in same time interval.



1. Instantaneous Impulse:

There are many cases when force acts for such a short time that effect is instantaneous, for example, a bat striking a ball. In such cases, although magnitude of the force and the time for which it acts may each be unknown but value of their product (i.e., impulse) can be known by the measuring the initial and final momentum. Thus, we can write.

$$\vec{I} = \int \vec{F}dt = \Delta\vec{P} = \vec{P}_f - \vec{P}_i$$

Rack your Brain

In an explosion, a body lying at rest breaks up into two pieces of unequal masses. In this-

(1) Both parts will have numerically equal momentum

(2) Heavier part will have large momentum

(3) Heavier part will have large momentum

(4) Both have same kinetic energy

Important Points:

(1) It is a vector quantity.

(2) Dimensions = [MLT^{-1}]

(3) SI unit = kg m/s

(4) Direction is along the change in momentum.

(5) Magnitude is equal to the area under the F-t. graph.

(6) $\vec{I} = \int \vec{F}dt = \vec{F}_{av}\int dt = \vec{F}_{av}\Delta t$

(7) It is not the property of a particle, but it is a measure of degree to which an external force changes momentum of the particle.

Ex. The hero of stunt film fires 50 g bullets by a machine gun, each at a velocity of 1.0 km/s. If he fires 20 bullets in the 4 seconds, mind the average force he exert against machine gun during this period.

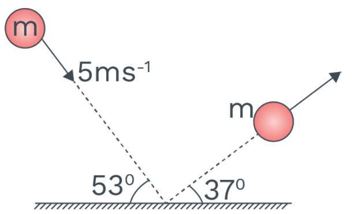
Sol. Momentum of each bullet

= (0.050 kg) (1000 m/s) = 50 kg-m/s

The gun has been exerting this much amount of momentum by the each bullet fired. Thus, the rate of the change of momentum of the gun

$$= \frac{(50\,kg - m/s) \times 20}{4\,s} = 250\,N$$

Ex. A ball of mass m = 1kg strikes the smooth horizontal floor as shown. Find out the impulse exerted on the floor is.



Sol. As ball strike on the surface on impulsive normal force is exerted on ball as shown in figure.

Definitions

IMPULSE:

Impulse of a force $\vec{F}$-acting on a body for the time interval t = t_1 to t = t_2 is defined as

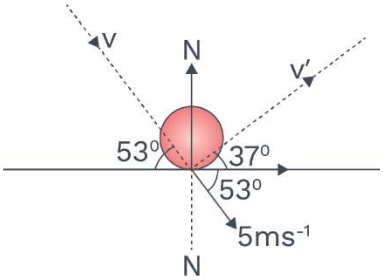
$$\vec{I} = \int_{t_1}^{t_2} \vec{F}\,dt$$

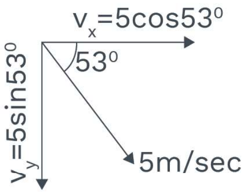
Concept Reminder

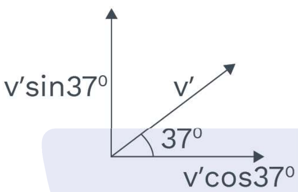
- $\vec{I} = \int \vec{F}dt = \Delta\vec{P} = \vec{P}_f - \vec{P}_i$
- Magnitude of impulse is equal to area under F-t graph.

KEY POINTS

- Impulse
- Instantaneous impulse
- Impulsive force







This normal force can change only component v_y. So in 'x' direction momentum is conserved.

F_{net} in x-direction = 0

$\Rightarrow$ $v' \cos 37° = 5 \cos 53°$

$$v' = \frac{5 \times 3 \times 5}{5 \times 4} = \frac{15}{4} \text{ m / sec}$$

So, $v' = v' \sin 37° = \frac{15}{4} \times \frac{3}{5} = \frac{9}{4}$ m / sec

Impulse = change in the linear momentum in 'y' direction

$$I = \int N \cdot dt = m(v_y - (-v'_y))$$

$$= 1\left(4 + \frac{9}{4}\right) = 6.25 \text{ N-sec}$$

Rack your Brain

A man of mass m jumps from a cart of mass M horizontally. Cart is placed on a horizontal smooth floor. Find speed of cart in ground frame if man jumps off with speed v w.r.t. ground.

2. Impulsive force:

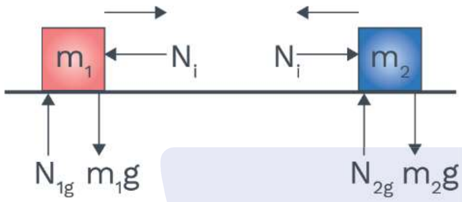
A force, of relatively high magnitude and acting for the relatively shorter time, is known as impulsive force. An impulsive force can change momentum of a body in a finite magnitude in a very short time interval. Impulsive force is a relative term. There is no clear boundary between the impulsive and Non-Impulsive force.

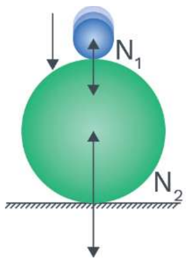
Note: Usually the colliding forces are impulsive in nature.
Since, the application time is very small, hence, very little motion of particle takes place. Important points:

1. Gravitational force and the spring force are always non-Impulsive.
2. Normal, tension and the friction are case dependent.
3. The impulsive force can only be balanced by the another impulsive force.

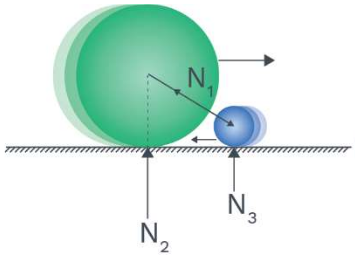
(a) Impulsive Normal:

In case of the collision, normal forces at surface of collision are always impulsive.

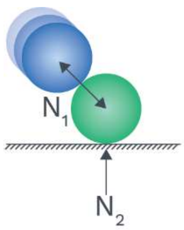




Both normals are Impulsive



N_1, N_3 = Implusive: N_2 =



Both normals are Imp

Definitions

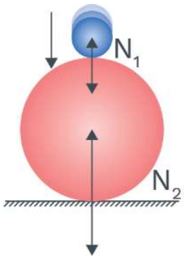
A force, of relatively higher magnitude and acting for relatively shorter time, is called impulsive force.

Concept Reminder

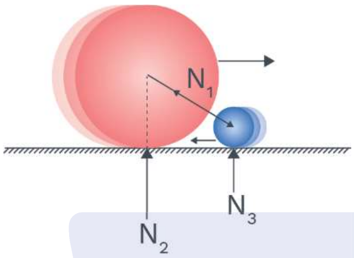
- Gravitational force and spring force are always non-Impulsive.
- In case of collision, normal forces at the surface of collision are always impulsive.

(b) Impulsive Friction:

If normal between two objects is impulsive, then the friction between the two will also be impulsive.



Friction at both surfaces is impulsive



Friction because of N_2 is non-impulsive and due to N_3 and N_1 are impulsive.

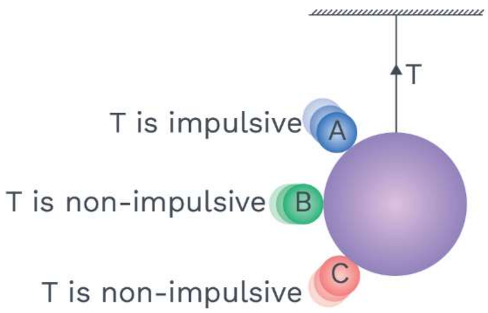
(c) Impulsive Tensions:
When a string jerks, then equal and opposite tension act suddenly at each end. Therefore, equal and opposite impulses act on bodies attached with the string in the direction of string. There are two cases to be considered.

One end of the string is fixed:
The impulse which acts at fixed end of the string cannot change momentum of the fixed object. The object attached to free end however will undergo a change in momentum in the direction of string. The momentum remains unchanged in the direction perpendicular to string where no impulsive forces act.

Both ends of string attached to movable objects:
In this case the equal and opposite impulses act on the objects, producing equal and opposite changes in the momentum. The total momentum of system therefore remains constant, although momentum of each individual object is changed in the direction of string. Perpendicular to the string however, no impulse acts and momentum of each particle in this direction is unchanged.

KEY POINTS

- Impulsive normal
- Impulsive friction
- Impulsive tension



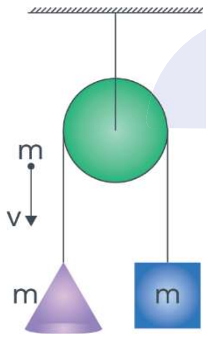
All normal are impulsive but tension
T is impulsive only for the ball A

Concept Reminder

When an apple falls from the branch of a tree then apple is not isolated system. Here apple and earth form the system. Force between apple and earth is internal. Therefore, momentum of system remains constant.

For this example: In case of the rod, tension is always impulsive and in case of the spring, tension is always non-impulsive.

Ex. A block of mass 'm' and a pan of same mass are connected by a string going over a smooth light pulley. Initially system is at rest when a particle of mass 'm' falls on the pan and sticks to it. If particle strikes the pan with a speed 'v', find the speed with which system moves just after collision.



Sol. Let required speed is 'V'.

Further, assume J_1 = impulse between particle and pan

and J_2 = impulse imparted to block and pan by string

By using, Impulse = change in momentum

For the particle

$J_1 = mv - mV$ ...(i)

For pan

$J_1 - J_2 = mV$ ...(ii)

For block

$J_2 = mV$ ...(iii)

Solving, these 3 equation,

we get $V = \frac{v}{3}$

Alternative solution:

Applying the conservation of linear momentum along string;

$$mv = 3mV$$

we get, $V = \frac{v}{3}$

Ex. A heavy nucleus of mass no. A initially at rest emits a-particle with velocity v. Find recoiling speed of daughter nucleus.

Sol. Apply COM

(A × 0) = 4v + (A – 4)v'

$$-\frac{4v}{A-4} = v'$$

Ex. A bomb of mass 4 kg is initially at rest is exploded two fragment 1 kg and 3 kg respectively if 3 kg is moving with velocity 10 m/s. (i) Find velocity of other fragment. (ii) Also find work done by internal forces.

Sol. Apply COLM

$$3(10) = -1v'$$

$$300 = -1v' \Rightarrow v' = -30 \text{ m/s}$$

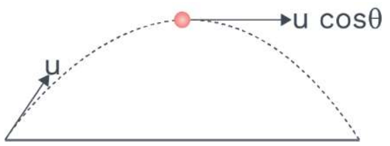
$$w_{int} = \Delta k = \frac{1}{2} \times 3 \times 10^2 + \frac{1}{2} \times 1 \times (-30)^2$$

= 150 + 450

Δk = 600 J

Ex. A bomb of mass m is projected with speed u at an angle of θ, at the top most point it is exploded into two fragments $\frac{m}{2}$ and $\frac{2m}{3}$ respectively. If $\frac{m}{3}$ just fall down find velocity of another fragments just after explosion.

Sol. Apply COLM in horizontal direction



$$mu\cos\theta = \frac{m}{3}(0) + \frac{2m}{3}(v)$$

$$v = \frac{3u\cos\theta}{2}$$

Ex. A bomb of mass 'm' is projected with speed 'u' at an angle θ, at the topmost point it is exploded into two equal fragments. If one fragment retrates its path find velocity of other fragment.

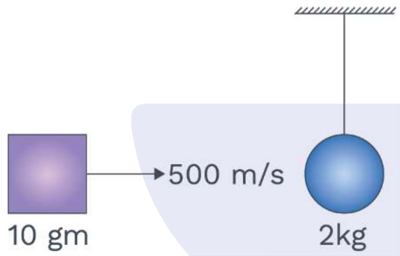
Sol. Apply COLM at topmost point.

$$m(u\cos\theta) = \frac{m}{2}(-u\cos\theta) + \frac{m}{2}v$$

$$\frac{3u}{2}\cos\theta = \frac{v}{2}$$

$$v = 3u\cos\theta$$

Ex. A bullet of mass 10 g is moving with velocity 500 ms⁻¹, strikes a block of mass 2 kg and get embedded into as shown in figure it. Find height upto which block rises.



Sol. Apply COLM

$$\frac{10\times 500}{1000} + 0 = 0 + 2\times v$$

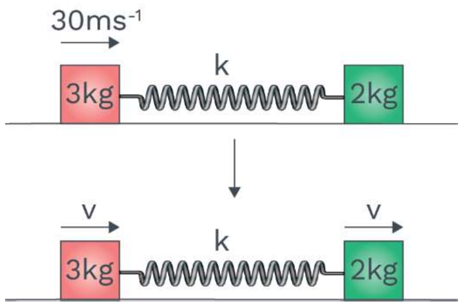
$$v = 2.5\ ms^{-1}$$

Conservation of mechanical energy

$$\frac{1}{2}\times 2\times (2.5)^2 = 2\times 10\times h$$

$$h = \frac{6.25}{20} = .3125\,m$$

Ex. Find maximum compression in the spring.



Sol. **Note:** At the state of max. compression or max. elongation, velocity of connecting blocks must be same.

Apply COLM

$3 \times 30 = 3 \times v + 2v$

$v = 18$ m/s

Apply COME

$$\frac{1}{2} \times 3 \times (30)^2 = \frac{1}{2} \times 3 \times 18^2 + \frac{1}{2} \times 2 \times 18^2 + \frac{1}{2} kx^2$$

$2700 = 18^2 \times 5 + kx^2$

$1080 = kx^2$

$\Rightarrow \quad x = \sqrt{\frac{1080}{K}}$

COLLISION

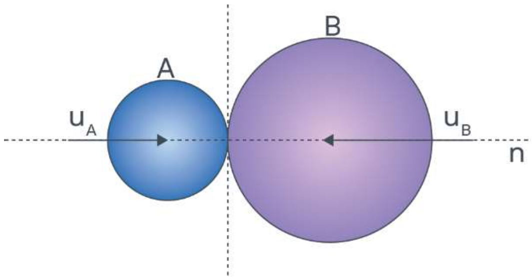
Impact or collision is the interaction between two bodies during very small duration in which they exert relatively large forces on each other. The duration of interaction is short enough to permit us only to consider the states of motion just before and after the event and not during the impact.

Let us take example, when a rubber ball strikes a floor, it remains in contact with the floor for very short time in which it changes its velocity. This is an example of collision where physical contact takes place between the colliding bodies.

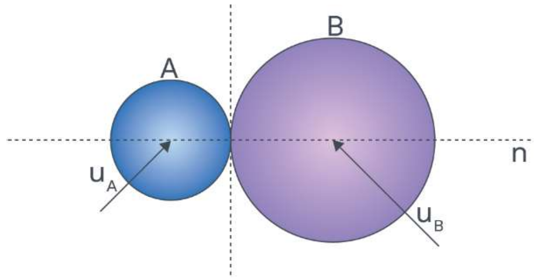
As a result of the collision, the momentum and kinetic energy (K.E) of the interacting bodies change. Forces involved in a collision are action-reaction forces, i.e., the internal forces of the system.

The total momentum will remain conserved in any type of the collision.

Head-on (Direct) and Oblique collision (impact): If velocity vectors of colliding bodies are directed along the line of impact, the impact is known as a direct or head-on impact; and if the velocity vectors of both or of the any one of the bodies are not along LOI, the impact is called an oblique impact.



Direct Impact



Oblique Impact

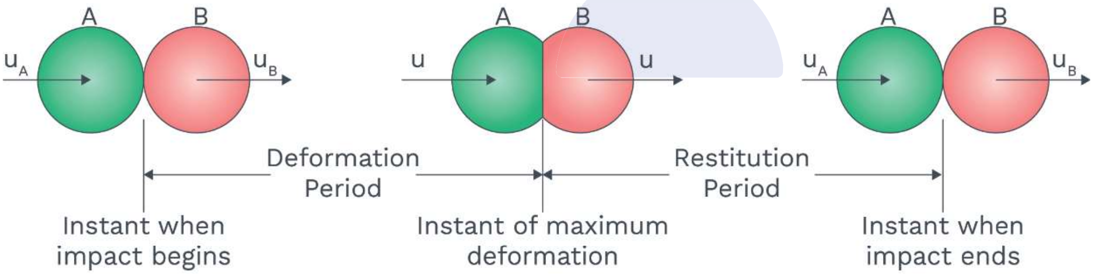
Definitions

Impact or collision is the interaction between two bodies during very small duration in which they exert relatively large forces on each other.

Head-on (Direct) Impact: To understand what happens in a head-on impact let us consider two balls A and B of masses m_A and m_B moving with velocities u_A and u_B in the same direction as shown. Velocity u_A is larger than u_B so the ball A hits the ball B. During the impact, both the bodies push each other and first they get deformed till the deformation reaches a maximum value and then they try to regain their original shapes due to elastic behaviour of the materials forming the balls.

KEY POINTS

- Collision
- Head-on collision
- Oblique collision



The time interval during which deformation takes place is called the deformation period and the time interval in which the bodies try to regain their original shapes is called the restitution period. Due to push applied by the balls on each other during period of deformation speed of ball A decreases and that of ball B increases and at the end of the deformation period, when the deformation is maximum both the balls move with the same velocity say it is u.

Thereafter, the balls will either move together with this velocity or follow the period of restitution. During the period of restitution due to push applied by the balls on each other, speed

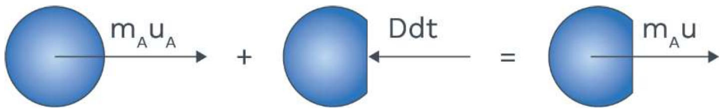
Concept Reminder

In collision duration of interaction is short enough to permit us only to consider state of motion just before and after the event and not during the impact.

of the ball A decreases further and that of ball B increases further till they separate from each other. Let us denote the velocities of the balls A and B after the impact by v_A and v_B respectively.

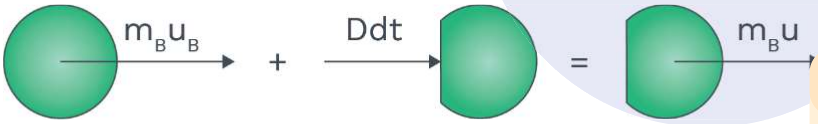
Equation of Impulse and Momentum during impact

Impulse momentum principle describes the motion of ball A during deformation period.



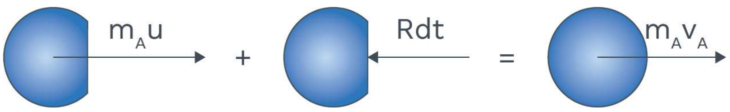
$$m_Au_A - \int Ddt = m_Au \qquad ...(i)$$

Impulse momentum principle describes the motion of ball B during deformation period.



$$m_Au_B + \int Ddt = m_Bu \qquad ...(ii)$$

Impulse momentum principle describes the motion of ball A during restitution period.



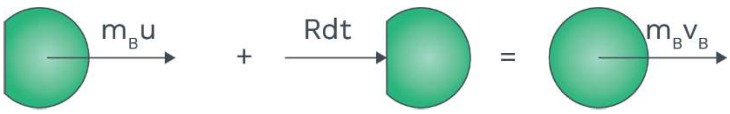
Definitions

Deformation period: The time interval during which deformation takes place is called the deformation period.

Restitution period: the time interval in which the bodies try to regain their original shapes is called the restitution period.

$$m_Au - \int Rdt = m_Av_A \qquad ...(iii)$$

Impulse momentum principle describes the motion of ball B during restitution period.



$$m_Bu + \int Rdt = m_Bv_B \qquad (iv)$$

Conservation of Momentum during impact:

From equations, (i) and (ii) we have

$$m_Au_A + m_Bu_B = (m_A + m_B)u \qquad ...(v)$$

From equations, (iii) and (iv) we have

$$(m_A + m_B)u = m_A v_A + m_B v_B \quad ...(vi)$$

From equations, (v) and (vi) we obtain the following equation.

$$m_A v_A + m_B v_B = m_A u_A + m_B u_B \quad ...(vii)$$

The above equation elucidates the principle of conservation of momentum.

Coefficient of Restitution:

Usually the force D applied by the bodies A and B on each other during the period of deformation differs from the force R applied by the bodies on each other during the period of restitution. Therefore, it is not necessary that the magnitude of impulse $\int Ddt$ due to deformation equals to that of impulse $\int Rdt$ due to restitution.

The ratio of magnitudes of impulse of restitution to that of deformation is called the coefficient of restitution and is denoted by e.

$$e = \frac{\text{Impulse of recovery}}{\text{Impulse of deformation}} = \frac{\int Rdt}{\int Ddt}$$

$$e = \frac{\text{velocity of separation along line of impact}}{\text{velocity of approach along line of impact}} = \frac{|\vec{v}_B - \vec{v}_A|}{|\vec{u}_A - \vec{u}_B|}$$

From equations (i), (ii), (iii) and (iv), we have

Coefficient of restitution depends on various factors as elastic properties of materials forming the bodies, velocities of the contact points before impact, state of rotation of the bodies and temperature of the bodies. In general, its value ranges from zero to one but in collisions where additional kinetic energy is generated, its value may exceed one.

Depending on the values of coefficient of restitution, two particular cases are of special interest.

Rack your Brain

A moving block having mass m, collides with another block of mass 4m. The lighter block comes to rest after collision. when the initial velocity of the lighter block is v, then find value of coefficient of restitution.

Perfectly Plastic or Inelastic Impact: For these impacts e = 0, and bodies undergoing impact stick to each other after the impact.

Perfectly Elastic Impact: For these impacts e = 1.

Strategy to solve problems of head-on impact:

Write the momentum conservation equation

$$m_A v_A + m_B v_B = m_A u_A + m_B u_B \quad \text{...(i)}$$

Write the equation involving coefficient of restitution

$$v_B - v_A = e(u_A - u_B) \quad \text{...(ii)}$$

Types of collisions according to the conservation law of kinetic energy:

(a) Elastic collision:

$$KE_{\text{before collision}} = KE_{\text{after collision}}$$

(b) Inelastic collision: Kinetic energy isn't conserved.

Some energy is lost in the collision.

Therefore, $KE_{\text{before collision}} > KE_{\text{after collision}}$

(c) Perfect inelastic collision: Both the bodies stick together after collision.

Note: Momentum remains conserved in all types of collisions.

KEY POINTS

- Coefficient of restitution

Definitions

- The ratio of magnitudes of impulse of restitution to that of deformation is called the coefficient of restitution and is denoted by e.

Concept Reminder

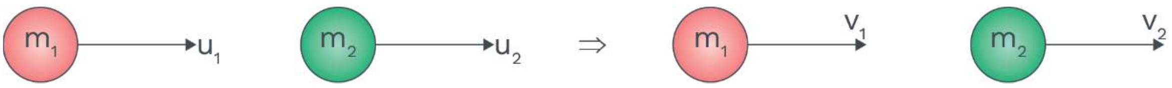
$$e = \frac{\text{velocity of separation along line of impact}}{\text{velocity of approach along line of impact}}$$

$$= \frac{|\vec{v}_B - \vec{v}_A|}{|\vec{u}_A - \vec{u}_B|}$$

TYPES OF COLLISIONS

- On the basis of direction
  - One Dimensional collision or Head on collision
    - A collision, in which the particles move along the same straight line before and after the collision, is defined as one dimensional collision.
  - Two Dimensional collision or Oblique collision
    - A collision, in which the particles move in the same plane at different angles before and after collision, is defined as oblique collision.
- On the basis of kinetic energy
  - Elastic collision
    - A collision is said to be elastic, if the total kinetic energy before and after the collision is remains same
  - In-elastic collision
    - A collision is said to be inelastic, if the total kinetic energy does not remain constant
  - Perfectly inelastic collision
    - The collision, in which particles get sticked together after the collision, is called perfectly inelastic collision. In this type of collision, loss of energy is maximum

Elastic Collision:



$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \qquad \text{...(i)}$$

$$e = 1 = \frac{\text{velocity of sep.(After collision)}}{\text{velocity of app.(Before collision)}}$$

$$1 = \frac{v_2 - v_1}{v_1 - v_2} \qquad \text{...(ii)}$$

In solving eq. (1) and (2)

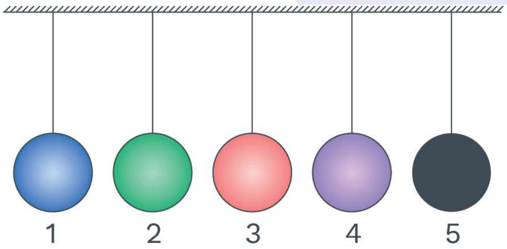
$$\boxed{\vec{v}_1 = \frac{m_1 - m_2}{m_1 + m_2}\vec{u}_1 + \frac{2m_2}{m_1 + m_2}\vec{u}_2}$$

and $$\boxed{\vec{v}_2 = \frac{m_2 - m_1}{m_1 + m_2}\vec{u}_2 + \frac{2m_1}{m_1 + m_2}\vec{u}_1}$$

Concept Reminder

Types of collisions are-
(a) Elastic collision
(b) Inelastic collision
(c) Perfectly inelastic collision

- Two particles of equal masses perform head on elastic collision then, after collision, their velocities get exchanged. Newton's cradle.



- In elastic collision loss in kinetic energy will be zero but transformation of energy will take place.

$$\boxed{\%\text{ transfer} = \left[1 - \left(\frac{v_1}{u_1}\right)^2\right] \times 100}$$

Inelastic Collision:

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \quad ...(i)$$

$$e = \frac{v_2 - v_1}{u_1 - u_2} \quad ...(ii)$$

$$\boxed{\vec{v}_1 = \left(\frac{m_1 - em_2}{m_1 + m_2}\right)\vec{u}_1 + \left(\frac{(1+e)m_2}{m_1 + m_2}\right)\vec{u}_2}$$

and $\vec{v}_2 = \left(\frac{m_2 - em_1}{m_1 + m_2}\right)\vec{u}_2 + \frac{(1+e)m_1}{m_1 + m_2}\vec{u}_1$

Concept Reminder

General formulas for velocit
after collision are-

$$\vec{v}_1 = \left(\frac{m_1 - em_2}{m_1 + m_2}\right)\vec{u}_1 + \left(\frac{(1+e)m_2}{m_1 + m_2}\right)\vec{u}_2$$

$$\vec{v}_2 = \left(\frac{m_2 - em_1}{m_1 + m_2}\right)\vec{u}_2 + \frac{(1+e)m_1}{m_1 + m_2}\vec{u}_1$$

Special Case:

(1) e = 0

$\Rightarrow$ $v_1 = v_2$

$\Rightarrow$ For the perfectly inelastic collision, both bodies, move with the same velocity after the collision.

(2) e = 1

and $m_1 = m_2 = m$

we get $v_1 = u_2$ and $v_2 = u_1$

i.e., when the two particles of same mass collide elastically and the collision is head on, they exchange their velocities., e.g.

Before collision After collision

Before collision After collision

(3) $m_1 >> m_2$

$m_1 + m_2 \approx m_1$ and $\frac{m_2}{m_1} \approx 0$

$\Rightarrow$ $v_1 = u_1$ (no change)

and $v_2 = u_1 + e(u_1 - u_2)$

Now if e = 1,

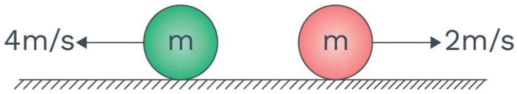
$v_2 = 2u_1 - u_2$

Ex. Two similar balls are approaching towards each other on a straight line

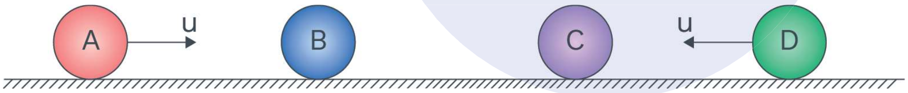
Centre of Mass

with the velocity 2 m/s and 4 m/s respectively. Find final velocities, after elastic collision between them.

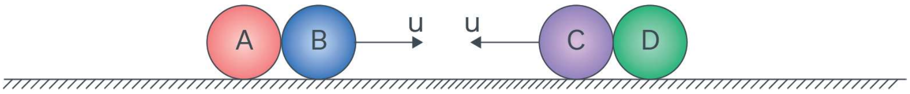
Sol. The two velocities will be exchanged and final motion is reverse of initial motion for both.



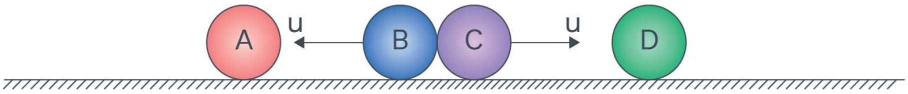
Ex. Four similar balls A, B, C and D are placed in a line on the frictionless horizontal surface. A and D are moved with same speed 'u' towards the middle as shown. Assuming elastic collisions, find final velocities.



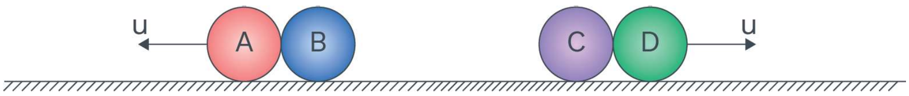
Sol. 'A' and 'D' collides elastically with 'B' and 'C' respectively and come to rest but 'B' and 'C' starts moving with velocity u towards each other as shown.



'B' and 'C' collides elastically and exchange their velocities to move in opposite directions.



Now, 'B' and 'C' collides elastically with 'A' and 'D' respectively and come to rest but 'A' and 'D' starts moving with velocity 'u' away from each other as shown.



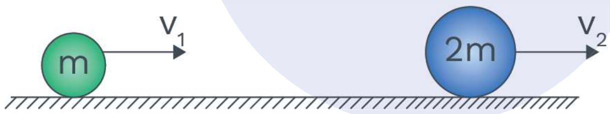
∴ Final velocities

$v_A = u(\leftarrow); v_B = 0;$

$v_C = u$ and $v_D = u\ (\rightarrow)$

Ex. Two particles of mass 'm' and '2m' moving in the opposite directions on a frictionless surface collide elastically with velocity 'v' and '2v' respectively. Calculate their velocities after collision, also find the fraction of kinetic energy lost by the colliding particles.

Sol. Let final velocities of m and 2m be v_1 and v_2 respectively as shown in the figure.



By conservation of momentum:

$$m(2v) + 2m(-v) = m(v_1) + 2m(v_2)$$

or $\quad 0 = mv_1 + 2mv_2$

or $\quad v_1 + 2v_2 = 0 \qquad$ ...(i)

and since the collision is elastic:

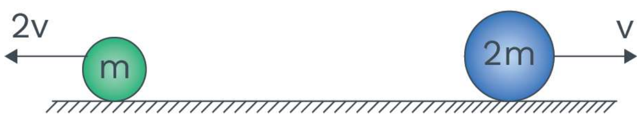
$$v_2 - v_1 = 2v - (-v)$$

or $\quad v_2 - v_1 = 3v \qquad$ ...(ii)

Solving the above two equations, we get,

$v_2 = v$ and $v_1 - 2v$

i.e., the mass 2m returns with the velocity v while the mass m returns with velocity '2v' in the direction shown in figure:



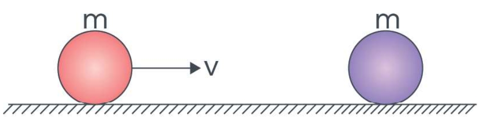
The collision was elastic therefore, no loss in K.E

K.E loss = $K.E_i - K.E_f$

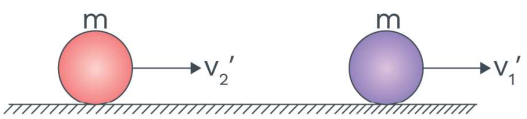
or $\left(\frac{1}{2}m(2v)^2 + \frac{1}{2}(2v)(-v)^2\right) - \left(\frac{1}{2}m(-2v)^2 + \frac{1}{2}(2m)v^2\right) = 0$

Ex. On a frictionless surface, a ball of mass 'm' moving at a speed 'v' makes head on collision with an identical ball at rest. Kinetic energy (K.E) of the balls after the collision is $\left(\frac{3}{4}\right)^{th}$ of the original. Find the coefficient of restitution.

Sol. As we have seen in above discussion, that under the given conditions:



Before collision



After collision

By using the conservation of linear momentum and equation of 'e', we get,

$v'_1 = \left(\frac{1+e}{2}\right)v$ and $v'_2 = \left(\frac{1-e}{2}\right)v$

Given that, $K_f = \frac{3}{4}K_i$

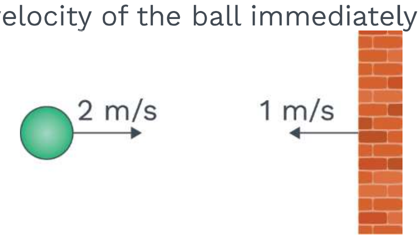
or $\frac{1}{2}mv'^2_1 + \frac{1}{2}mv'^2_2 = \frac{3}{4}\left(\frac{1}{2}mv^2\right)$

Substituting the value, we get

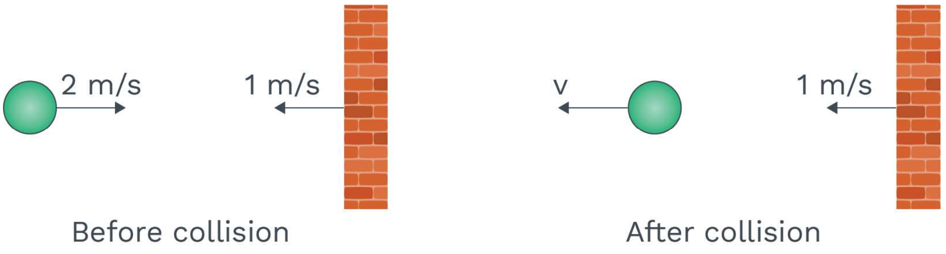
$$\left(\frac{1+e}{2}\right)^2 + \left(\frac{1-e}{2}\right)^2 = \frac{3}{4}$$

or $e = \frac{1}{\sqrt{2}}$

Ex. A ball is moving with the velocity 2 m/s towards a heavy wall moving towards ball with speed of 1 m/s as shown. Assuming the collision to be elastic, find velocity of the ball immediately after collision.



Sol. The speed of wall will not change after collision. So, let 'v' be the velocity of the ball after the collision in the direction shown in figure. Since the collision is elastic (e = 1),

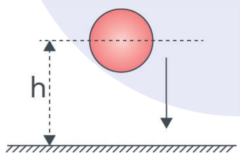


Separation speed = approach speed

or $v - 1 = 2 + 1$ or $v = 4$ m/s

Ex. If a body falls vertically on a surface from height 'h', what will be height regained after collision if coefficient of restitution is 'e'?

Sol. If a body falls from height 'h', from the equations of motion we know that it will hit the ground with a velocity say $u = \sqrt{2gh}$ which is also velocity of approach here.

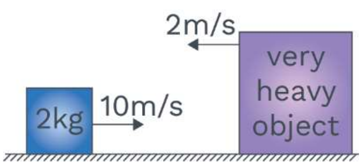


Now if after the collision it regains a height h_1 then again by the equations of motion $v = \sqrt{2gh_1}$ which is also velocity of separation.

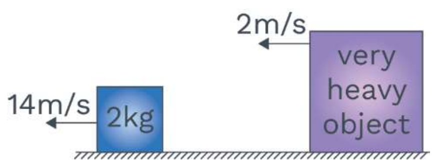
So, by the definition of 'e',

$$e = \sqrt{\frac{2gh_1}{2gh}} \text{ or } h_1 = e^2 h$$

Ex. A block of 2 kg is pushed towards a very heavy object moving with 2 m/s closer to block (as shown). Assuming elastic collision and frictionless surface, calculate the final velocities of blocks.



Sol. Let v_1 and v_2 be the final velocities of 2kg block and heavy object respectively then,



$$v_1 = u_1 + 1(u_1 - u_2) = 2u_1 - u_2$$
$$= -14 \text{ m/s}$$
$$v_2 = -2 \text{ m/s}$$

Line of Motion

The line passing through centre of the body along the direction of resultant velocity.

Line of Impact

The line passing through common normal to surfaces in contact during the impact is known as line of impact. The force during the collision acts along this line on both bodies.

Direction of LOI can be determined by:

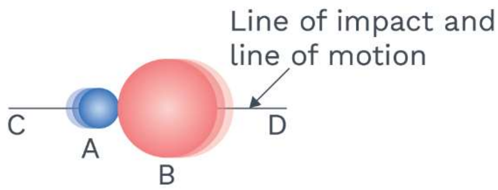
(a) Geometry of colliding objects like spheres, discs, wedge etc.

(b) Direction of change of momentum.

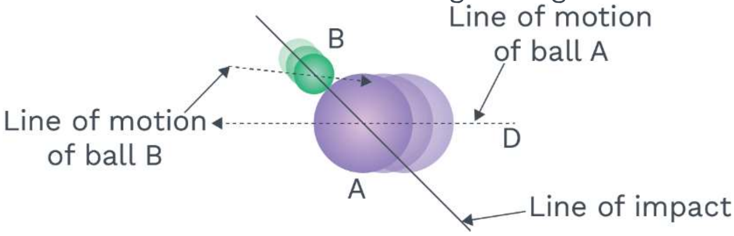
If one particle is stationary before the collision then the LOI will be along its motion after collision.

Examples of line of impact:

(i) Two balls 'A' and 'B' are approaching each other such that their centres are moving along line CD.



(ii) Two balls 'A' and 'B' are approaching each other such that their centre are moving along the dotted lines as shown



(iii) Ball is falling on a stationary wedge.

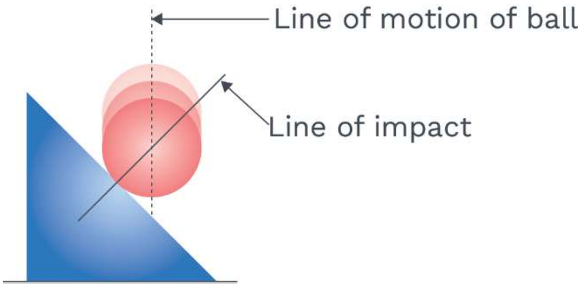
Rack your Brain

A ball is thrown vertically downwards from a height of 20 m with an initial velocity v_0. It collides with the ground, loses 50 percent of its energy in collision and rebounds to same height. Find value of v_0.

Definitions

Line of Motion: The line passing through the centre of the body along the direction of resultant velocity.

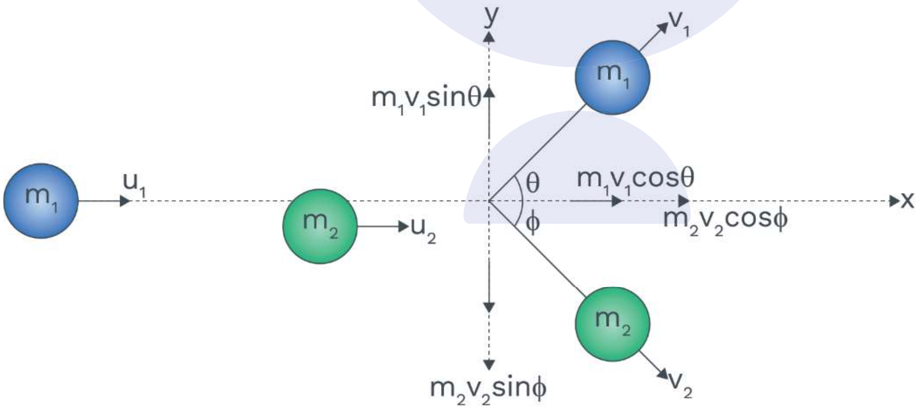
Line of Impact: The line passing through the common normal to the surfaces in contact during impact is called line of impact.



Note: In previous discussed examples line of motion is same as line of impact. But in problems in which line of impact and line of motion is different then e will be.

$$e = \frac{\text{velocity of separation along line of impact}}{\text{velocity of approach along line of impact}}$$

Oblique Collision:



By COLM along x-axis

$$m_1u_1 + m_2u_2 = m_1v_1 \cos\theta + m_2v_2 \cos\phi$$

By COLM along y-axis

$$0 + 0 = m_1v_1 \sin\theta - m_2v_2 \sin\phi$$

If collision is elastic then,

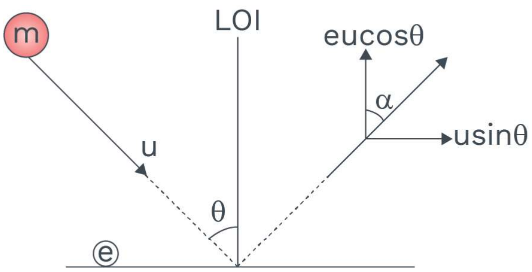
By conservation of kinetic energy,

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

Ex. A ball of mass m is projected at angle θ with vertical as shown in figure find:

(1) Velocity first after collision.

(2) Angle at which particle ball will go.



Sol. $v = \sqrt{(eu\cos\theta)^2 + (u\sin\theta)^2}$

$\tan\alpha = \dfrac{u\sin\theta}{eu\cos\theta}$

$\tan\alpha = \dfrac{\tan\theta}{e}$

Ex. For projectile motion:

Find (i) Time of flight

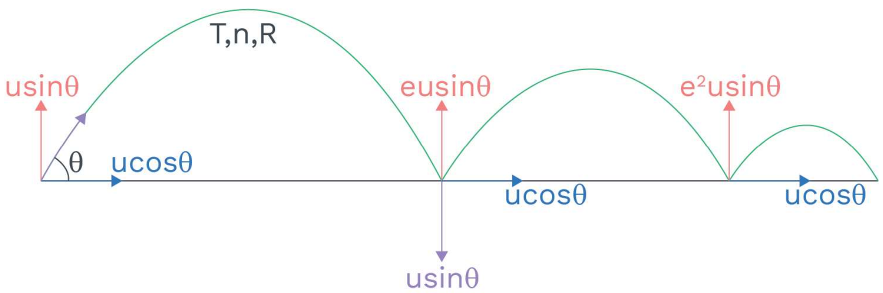
(ii) Maximum height

(iii) Range after collision (coeff. of res.)

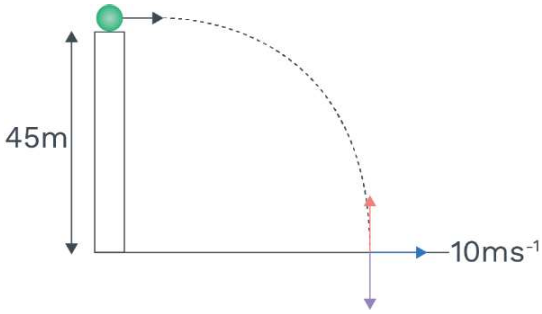
Sol. $T = \dfrac{2u\sin\theta}{g}$

$H = \dfrac{u^2\sin\theta}{g} = \dfrac{(u\sin\theta)^2}{2g}$

$R = \dfrac{u^2\sin 2\theta}{g} = u_x \cdot T$



Ex. Find velocity of the particle just after collision. If collision is perfectly inelastic also find kinetic energy.



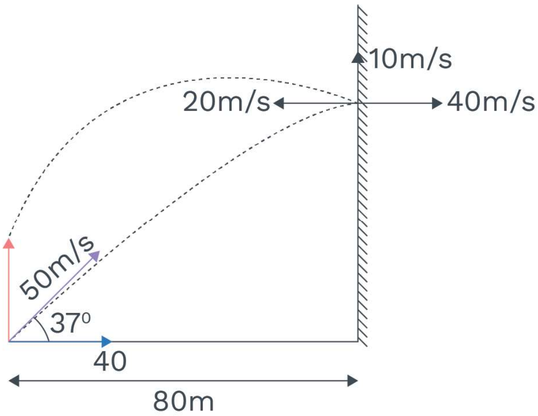
A particle is projected horizontally will velocity 10 ms^{-1} as shown in the figure. If collision with ground is perfectly inelastic. Then find KE after collision.

Sol. $KE = \frac{1}{2} \times 1 \times 100 = 50\,J = E_{T_j}$

$E_{T_i} = 1 \times 450 + \frac{1}{2} \times 1 \times 100$

$\Rightarrow \quad E_{T_i} = 450\,J = 90$

Ex. A particle is projected with speed 50 m/s at an angle of 37° with horizontal. If it strikes to a wall which is at a distance of 80m from point of projection. If $e = \frac{1}{2}$ find distance at which particle will strike the ground.



Sol. $T = \frac{24 \sin \theta}{g} = 6\,sec$

Horizontal velocity before collision = 40 m/s
Horizontal velocity after collision = 20 m/s
Time to reach at wall = 2 sec
Remaining time = 4 sec

Concept Reminder

In a collision, the colliding bodies may or may not come in real physical touch.

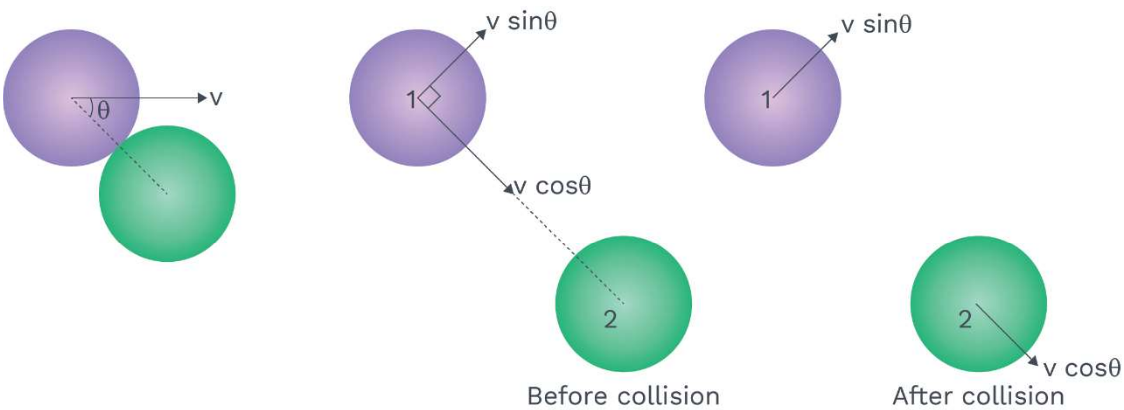
So distance travelled by parallel after collision
= 20 × 4 = 80 m/s

Collision in two dimension (oblique):

1. A pair of the equal and opposite impulses acts along the common normal direction. Hence, linear momentum of individual particles change along common normal direction. If mass of the colliding particles remain constant during collision, then we can say that linear velocity of the individual particles change during collision in this direction.
2. No component of the impulse act along common tangent direction. Hence, the linear momentum or linear velocity of individual particles (if mass is constant) remain unchanged along this direction.
3. Net impulse on both particles is zero during collision. Hence, net momentum of both particles remain conserved before and after collision in any direction.
4. Definition of the coefficient of restitution can be applied along the common normal direction, i.e., along the common normal direction we can apply Relative speed of separation = e (relative speed of approach)

Ex. A ball of mass 'm' makes an elastic collision with another identical ball at the rest. Show that if collision is oblique, bodies go at the right angles to each other after collision.

Sol. In the head on elastic collision between the two particles, they exchange their velocities. In this case, component of ball '1' along the common normal direction, v cos θ becomes zero after the collision, while.



that of 2 becomes v cos θ. While components along the common tangent direction of both particles remain unchanged. Thus, the components along the common tangent and the common normal direction of both the balls in tabular form are given a head:

BALL	COMPONENT ALONG COMMON TANGENT DIRECTION		COMPONENT ALONG COMMON NORMAL DIRECTION	
	Before collision	After collision	Before collision	After collision
1	v sin θ	v sin θ	v cos θ	0
2	0	0	0	v cos θ

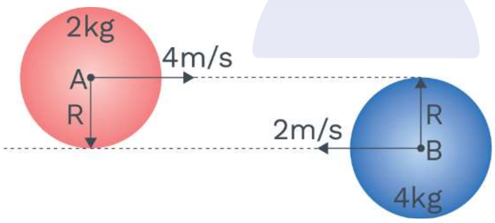
From above table and figure, we see that both balls move at right angles after the collision with velocities v sin θ and v cos θ.

Note: When the two identical bodies have an oblique elastic collision, with one body at rest before collision, then two bodies will go in ⊥ directions.

Concept Reminder

If two bodies of masses M_1 and M_2 collide head-on elastically then maximum transfer of kinetic energy occurs if $M_1 = M_2$ and either v_1 or v_2 is zero.

Ex. 2 spheres are moving towards each-other. Both have equal radius but their masses are 4 kg and 2 kg. If velocities are 4m/s and 2m/s respectively and coefficient of restitution is e = 1/3, find-

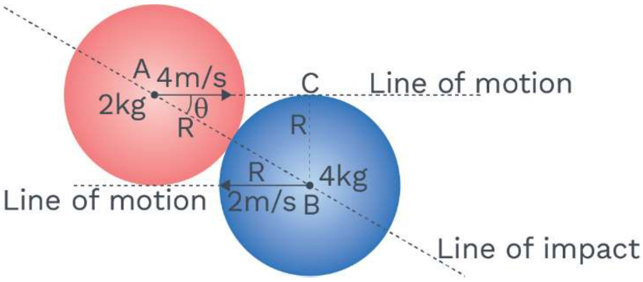


(a) The common velocity along line of impact.
(b) Final velocities along LOI
(c) Impulse of the deformation.
(d) Impulse of the reformation
(e) Maximum potential energy of deformation
(f) Loss in K.E due to collision.

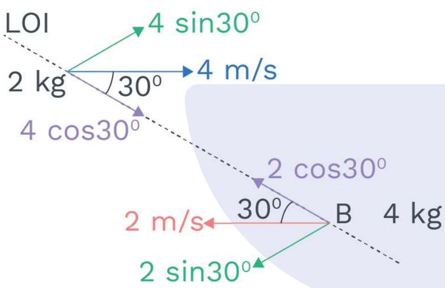
Sol. In ΔABC,

$$\sin\theta = \frac{BC}{AB} = \frac{R}{2R} = \frac{1}{2}$$

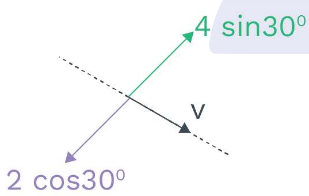
or $\theta = 30°$



(a) By conservation of momentum along line of impact.



Just Before Collision Along LOI



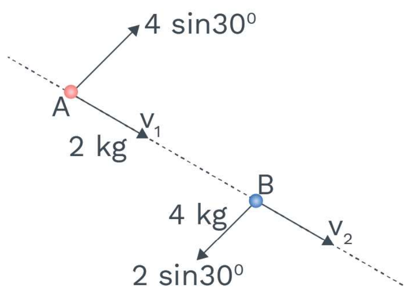
Maximum Deformed State

$$2(4\cos 30°) - 4(2\cos 30°) = (4+2)v$$

or $v = 0$ (common velocity along the LOI)

(b) Let v_1 and v_2 be final velocity of 'A' and 'B' respectively then, by conservation of momentum along line of impact,

$$2(4\cos 30°) - 4(2\cos 30°) = 2(v_1) + 4(v_2)$$



Just After Collision Along LOI

or $\quad 0 = v_1 + 2v_2$ ...(i)

By coefficient of restitution,

$$e = \frac{\text{velocity of separation along LOI}}{\text{velocity of approach along LOI}}$$

or $\quad \frac{1}{3} = \frac{v_2 - v_1}{4\cos 30° + 2\cos 30°}$

or $\quad v_2 - v_1 = \sqrt{3}$ ...(ii)

From the above two equations,

$$v_1 = \frac{-2}{\sqrt{3}} \text{ m/s and } v_2 = \frac{1}{\sqrt{3}} \text{ m/s}$$

(c) $\quad J_0 = m_1(v - u_1)$

$\quad = 2\ (0 - 4 \cos 30°) = -4\sqrt{3}$ N-s

(d) $\quad J_R = eJ_0 = \frac{1}{3}(-4\sqrt{3}) = -\frac{4}{\sqrt{3}}$ N-s

(e) Max. potential energy of deformation is equal to the loss in kinetic energy during deformation upto maximum deformed state,

$$U = \frac{1}{2}m_1(u_1\cos\theta)^2 + \frac{1}{2}m_2(u_2\cos\theta)^2 - \frac{1}{2}(m_1 + m_2)v^2$$

$$= \frac{1}{2}2(\cos 30°)^2 + \frac{1}{2}4(-2\cos 30°) - \frac{1}{2}(2+4)(0)$$

or $\quad U = 18$ joule

(f) Loss in kinetic energy

$$\Delta KE = \frac{1}{2}m_1(u_1\cos\theta)^2 + \frac{1}{2}m_2(u_2\cos\theta)^2 - \left(\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2\right)$$

$$= \frac{1}{2}2(4\cos 30°) + \frac{1}{2}4(-2\cos 30°)$$

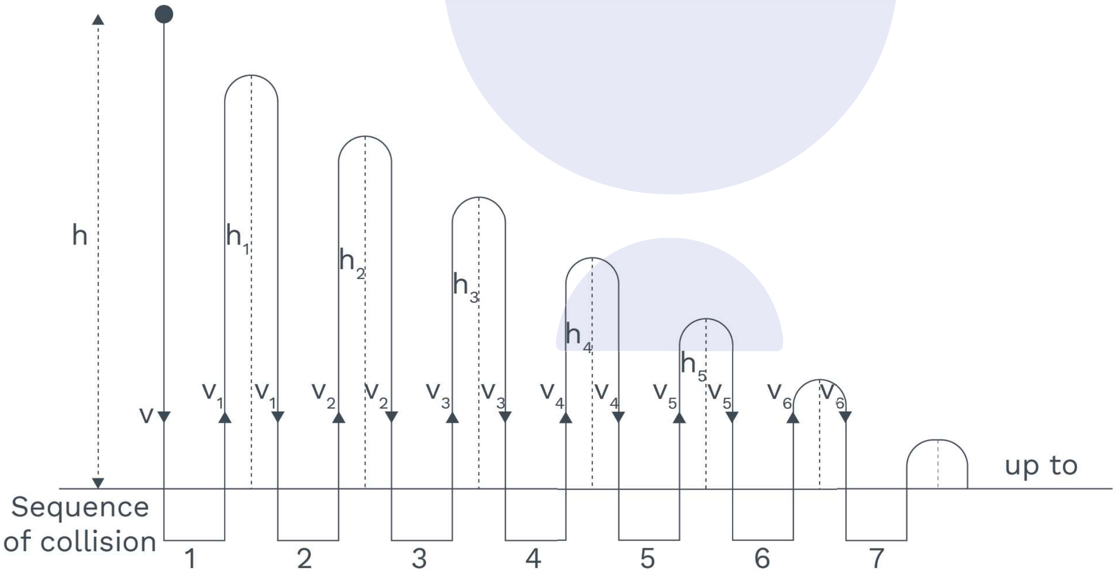
$$-\left(\frac{1}{2}2\left(\frac{2}{\sqrt{3}}\right)^2 + \frac{1}{2}4\left(\frac{1}{\sqrt{3}}\right)^2\right)$$

$\Delta KE = 16$ Joule

Bouncing of Ball:

Let the ball fall from a height (h) and let it touch the ground with a velocity v taking time (t) to reach the ground. Let v_1, v_2, v_3, ... be the velocities immediately after first, second, third......collisions with the ground.

- **Velocity immediately After the 'n'th Rebound:**



By Newton's formula,

$$(\vec{v}_2 - \vec{v}_1) = e(\vec{u}_1 - \vec{u}_2)$$

$$v = \sqrt{2gh}$$

Here $\vec{v}_2 = 0$, $\vec{u}_2 = 0$ (surface at rest)

$v_1 = ev$ (opposite direction)

$v = u_1$

$v_1 = ev$ ...(i)

$v_2 = ev_1$ ...(ii)

$v_2 = e(ev) \Rightarrow v_2 = e^2v$

Similarly,

$v_3 = e^3v, \;\; v_4 = e^4v......$ $\boxed{v_n = e^n v}$

$\boxed{v_n = e^n\sqrt{2gh}}$

- **Height Attained by the Ball After the n^{th} Rebound:**

$v_1 = ev \Rightarrow \sqrt{2gh_1} = e\sqrt{2gh}$

$\Rightarrow \quad h_1 = e^2h$

$v_2 = e^2v \Rightarrow \sqrt{2gh_2} = e^2\sqrt{2gh}$

$\Rightarrow \quad h_2 = e^4h$

Similarly, $\boxed{h_n = e^{2n}h}$

- **Time Taken in n^{th} Rebound:**

$h_1 = e^2h \Rightarrow \frac{1}{2}gt_1^2 = e^2\frac{1}{2}gt^2$

$\Rightarrow \quad t_1^2 = e^2t^2, \; t_1 = et$

$t_1 = e\sqrt{\frac{2h}{g}}, \quad h_2 = e^4h,$

$\frac{1}{2}gt_2^2 = e^4\left(\frac{1}{2}gt^2\right)$

$\Rightarrow \quad t_2^2 = e^4t^2, \; t_2 = e^2t, \; t_2 = e^2\sqrt{\frac{2h}{g}}$

Similarly,

$\boxed{t_n = e^n\sqrt{\frac{2h}{g}}}$ $\boxed{t_n = e^n t}$

Total time taken in bouncing. (i.e., total time elapsed before the ball stops)

$T = t + 2t_1 + 2t_2 + ...$

$= t + 2et + 2e^2t + 2e^3t + ...$

$= t + 2t(e + e^2 + e^3 + ...)$

$= t + 2t\left(\frac{e}{1-e}\right) = t\left(\frac{1+e}{1-e}\right) = \sqrt{\frac{2h}{g}}\left(\frac{1+e}{1-e}\right)$

$\boxed{T = \sqrt{\frac{2h}{g}}\left(\frac{1+e}{1-e}\right)}$

Concept Reminder

- $\boxed{v_n = e^n\sqrt{2gh}}$
- $\boxed{h_n = e^{2n}h}$

- **Distance Covered by The Ball Before it Stops:**

$$s = h + 2h_1 + 2h_2 + ... + \infty$$

$$= h + 2e^2h + 2e^4h + 2e^6h + ...$$

$$= h + 2e^2h\left(\frac{1}{1-e^2}\right) = h\left[1 + \frac{2e^2}{1-e^2}\right]$$

$$\boxed{s = h\left(\frac{1+e^2}{1-e^2}\right)}$$

Average Speed:

$$v_{av} = \frac{\text{Total distance}}{\text{Total time}} = \frac{h\left(\frac{1+e^2}{1-e^2}\right)}{\sqrt{\frac{2h}{g}}\left(\frac{1+e}{1-e}\right)}$$

$$\boxed{v_{av} = \sqrt{\frac{gh}{2}}\left[\frac{1+e^2}{(1+e)^2}\right]}$$

Average Velocity:

$$v_{av} = \frac{\text{Total displacement}}{\text{Total time}} = \frac{h}{\sqrt{\frac{2h}{g}}\left(\frac{1+e}{1-e}\right)}$$

$$\boxed{v_{av} = \sqrt{\frac{gh}{2}}\left(\frac{1-e}{1+e}\right)}$$

e.g.:

A particle of mass 'm' is projected upward with speed 'u' find:

(1) Total five taken by the particle in all possible collisions

(2) Total distance travelled in all possible collisions.

Total time,

$$t = \frac{2u}{g} + \frac{2eu}{g} + \frac{2e^2u}{g} + ...\infty$$

Rack your Brain

A rubber ball is dropped from a height of 5m on ground. On bouncing it rises to 1.8 m. The ball loses its velocity on bouncing by what factor?

Concept Reminder

Total time elapsed before the ball stops:

$$T = \sqrt{\frac{2h}{g}}\left(\frac{1+e}{1-e}\right)$$

Concept Reminder

Distance Covered by The Ball Before it Stops:

$$s = h\left(\frac{1+e^2}{1-e^2}\right)$$

$$= \frac{2u}{g}(1 + e + e^2 + ... + \infty)$$

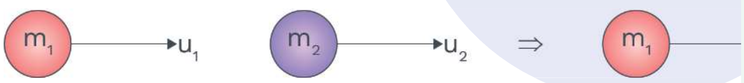
$$\boxed{t = \frac{2u}{g}\left(\frac{1}{1-e}\right)}$$

Total distance,

$$S = \frac{u^2}{g} + \frac{(eh)^2}{g} + \frac{(e^2u)^2}{g} + ... + \infty$$

$$= \frac{u^2}{g}(1 + e^2 + e^4 + ... + \infty) = \frac{u^2}{g}\left[\frac{1}{1-e^2}\right]$$

Loss in kinetic energy:



Loss in kinetic energy

$$= \left[\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2\right] - \left[\frac{1}{2}m_1v_1^2 + m_2v_2^2\right]$$

$$= \frac{1}{2}\frac{m_1m_2}{m_1 + m_2}[1 - e^2][\vec{u}_1 - \vec{u}_2]^2$$

Concept Reminder

Average speed:

$$v_{av} = \sqrt{\frac{gh}{2}}\left[\frac{1+e^2}{(1+e)^2}\right]$$

Concept Reminder

Average velocity in bouncing:

$$v_{av} = \sqrt{\frac{gh}{2}}\left(\frac{1-e}{1+e}\right)$$

Concept Reminder

Loss in kinetic energy:

$$\frac{1}{2}\frac{m_1m_2}{m_1 + m_2}[1 - e^2][\vec{u}_1 - \vec{u}_2]^2$$

EXAMPLES

Q1 **A bus with a gun mounted on it is kept on horizontal friction less surface. Total mass of bus, gun and shell is 50 kg. Mass of each shell is 1 kg. If shell is fired horizontally with relative velocity 100 m/sec with respect to gun. What is the recoil speed of bus after second shot ?**

Sol. For first shot

$(100 - v_1) \times 1 = 49v_1$

For second shot

$(100 - v_2) = 48v_2$

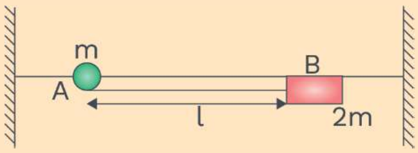
$v_1 = \frac{100}{49+1}$ and $v_2 = \frac{100}{48+1}$

$v_{net} = v_1 + v_2$

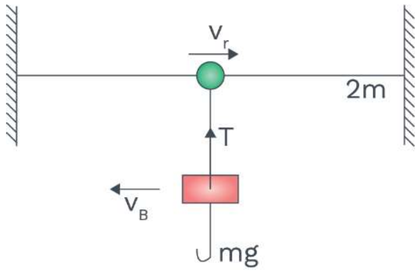
$= \frac{100}{50} + \frac{100}{49}$

$v_{net} = 100\left(\frac{1}{50} + \frac{1}{49}\right) m / s$

Q2 **A small ring A of mass 'm' is attached at an end of a light string other end of which is tied to the block 'B' of mass 2m. The ring is free to move on the fixed smooth horizontal rod as shown in figure. Find:**
(i) Velocity of ring A when string becomes vertical.
(ii) Tension in the string when it becomes vertical.



Sol.



By mechanical energy conservation

$$\frac{1}{2}(2m)v_b^2 + \frac{1}{2}mv_r^2 = 2mgl$$

$$2v_b^2 + v_r^2 = 4gl \qquad ...(1)$$

Using momentum conservation

$$mv_r = 2mv_b, v_r = 2v_b \qquad ...(2)$$

$$v_b^2 + 4v_b^2 = 4gl, \; 6v_b^2 = 4gl, \; v_b = \sqrt{\frac{2}{3}gl}$$

(a) $v_{rg} = 2v_b = 2\sqrt{\frac{2}{3}gl}$

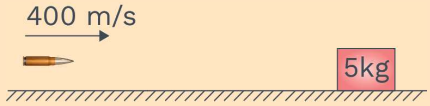
When string be comes vertical velocity of block wrt to string.

$$v_{br} = v_b - (-v_r) = 3v_b = 3\sqrt{\frac{2}{3}gl}$$

(b) $T - 2mg = \frac{2m(v_{br})^2}{\ell}$

$$T = 2mg + \frac{9 \times 2gl \times (2)m}{3\ell} = 14 \text{ mg.}$$

Q3 **A bullet fired horizontally with a speed of 400 m/sec. It strikes a wooden block of 5 kg initially at rest placed on the horizontal floor as shown. It emerges with speed of 200 m/sec and the block slides a distance 20 cm before coming to rest. calculate the coefficient of friction between block and the surface. Mass of bullet is 20 g. (take g = 10 m/s²).**



Sol. $m = 20 \times 10^{-3} kg;$ M = 5 kg

u = 400 d = 0.2 m

v = 200 μ = ?

$$\Delta P_{Bullet} = \Delta P_{Block}$$

$$m(u - v) = 20 \times 10^{-3}(400 - 200) = 4 kg.m/s.$$

$$KE_{Block} = \frac{P^2}{2M} = \frac{4^2}{2 \times 5} = 1.6 J = \mu Mgd$$

$$\mu = \frac{1.6}{Mgd} = \frac{1.6}{5 \times 10 \times 0.2} = 0.16$$

Q4 **A bullet fired horizontally with speed of 400 m/sec. It strikes a wooden block of 2 kg hanging vertically with the help of long string. After striking with bullet, the block rises a height of 20 cm. Calculate the speed with which bullet emerges out from block. Mass of bullet is 20 g. (take g = 10 m/s²)**

Sol. $V = \sqrt{2gh} = \sqrt{2 \times 10 \times 0.2} = 2 m/s.$

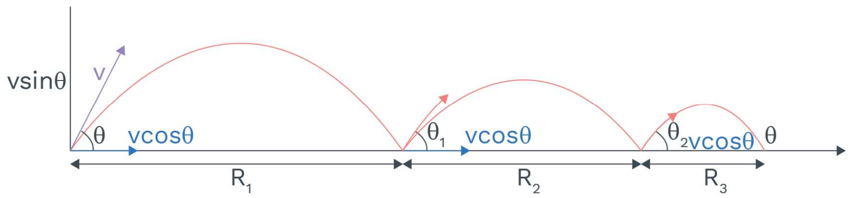
mu = MV + mv

$$v = u - \frac{M}{m}V$$

$$= 400 - \frac{2 \times 2}{20 \times 10^{-3}} = 200 m/s.$$

Q5 **A projectile of mass m is fired with a speed v at an angle θ from a smooth horizontal field. Coefficient of restitution of the collision between projectile and the field is e. How far from starting point, does the projectile make its third collision with the field ?**

Sol.



$R_1 = v\cos\theta T_1$

$R_2 = v\cos\theta T_2$

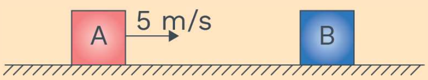
$R_3 = v\cos\theta T_3$

$R = R_1 + R_2 + R_3 = u\cos\theta[T_1 + T_2 + T_3]$

$= v\cos\theta\left[\frac{2u\sin\theta}{g} + \frac{2eu\sin\theta}{g} + \frac{2e^2u\sin\theta}{g}\right]$

$R = \frac{(1+e+e^2)v^2\sin 2\theta}{g}$.

Q6 **Two blocks of equal masses are placed on a horizontal surface. The surface of 'A' is smooth but that of 'B' has a friction coefficient of 0.2 with the floor. Block 'A' is given a speed of 5 m/s, towards 'B' which is kept at rest. Calculate the distance travelled by 'B' if (a) the collision is perfectly elastic and (b) the collision is perfectly inelastic. Take g = 10 m/s².**



Sol.

$v^2 = u^2 + 2as \Rightarrow 0 = (v)^2 - 2as \Rightarrow s = \frac{(v)^2}{2as}$

(a) e = 1 so after collision $v_A = 0$ and $v_B = 5\text{ m/sec}$

So, $s = \frac{(5)^2}{2\times 0.2\times 10}$, $\therefore s = 6.25\text{m}$

(b) when e = 0 applying momentum conservation e = 0

$m\times 5 + 0 = (m+m)\times v \Rightarrow v = 2.5\text{m/sec}$

So $v^2 = u^2 + 2as \Rightarrow s = \dfrac{(2.5)^2 \times 2}{2 \times 0.2 \times 10}$

$a = \dfrac{\mu g}{2} \quad \therefore\ s = 3.12\,m.$

Q7 **A body of mass 5 kg moves along the x-axis with a velocity 2 m/s. A second body of mass 10 kg moves along y-axis with a velocity of √3 m/s. They collide at origin and stick together. Find**
(i) The final speed of the combined mass after collision
(ii) The amount of heat liberated in the collision.

Sol. $P_x = 5 \times 2 = 1$

$P_y = 10\sqrt{3}$

$P_y = 10\sqrt{3} = 10\hat{i} + 10\sqrt{3}\,\hat{j}$

$\vec{v} = \dfrac{10}{15}\left(\hat{i} + \sqrt{3}\hat{j}\right)$

$\vec{v} = \dfrac{2}{3}\left(\hat{i} + \sqrt{3}\hat{j}\right)$

$|\vec{v}| = \dfrac{4}{3}\,m/s.$

$H = \Delta E = E_i - E_f = \left[\dfrac{1}{2}m_1v_1^2 + \dfrac{1}{2}m_2u_2^2\right] - \dfrac{1}{2}(m_1 + m_2)v^2$

$= \dfrac{1}{2} \times 5 \times 2^2 + \dfrac{1}{2} \times 10 \times \left(\sqrt{3}\right)^2 - \dfrac{1}{2} \times (10+5)(4/3)^2 = 25 - \dfrac{40}{3} = \dfrac{35}{3}.$

Q8 **A 3 kg block 'A' moving with 4 m/sec on a smooth table collides inelastically and head on with an 8 kg block 'B' moving with speed 1.5 m/sec towards 'A'. Given e = 1/2**
(a) What is final velocities of both blocks
(b) Find out impulse of reformation and deformation
(c) Find out maximum potential energy of deformation
(d) Find out the loss in kinetic energy of system.

Sol. Using momentum conservation

$$3\times4-8\times1.5=3v_1+8v_2$$

$$12-12=3v_1+8v_2$$

$\therefore \quad 3v_1+8v_2=0$ ...(1)

Coefficient of restitution $\frac{v_2-v_1}{4+1.5}=\frac{1}{2}$ ...(2)

$v_2-v_1=\frac{5.5}{2}$ ...(2)

Using (i) $3\times v_1+8\left(\frac{5.5}{2}+v_1\right)=0$

$\therefore \quad v_2=-\frac{3}{8}v_1=-\frac{3}{8}\times(-2)=\frac{3}{4}\,m/sec$

(b) Applying momentum conservation equation

$m_1v_1+m_2v_2=(m_1+m_2)v \quad \therefore V = 0$ so

$|P_D|=|m_1(\bar{v}-v_1)|=|m_1v_1|=3\times4=12\,Ns$

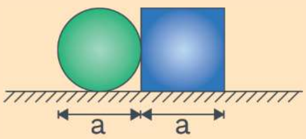
$|J_R|=|eJ_D|=6$

(c) $P.E.=\frac{1}{2}mv_1^2+\frac{1}{2}m_2v_2^2-\frac{1}{2}(m_1+m_2)v^2.$

$=\frac{1}{2}\times3\times4^2+\frac{1}{2}\times8\times(1.5)^2-0=33\,J$

(d) $\Delta K=K_i-K_f=33-\left[\frac{1}{2}\times3(2)^2+\frac{1}{2}\times8\times\left(\frac{3}{4}\right)^2\right]=\frac{99}{4}J.$

Q9 **A square plate of edge ‘a’ and a circular disc of same diameter are put touching each other at midpoint of an edge of the plate as shown. If mass per unit area for two plates are same then find the distance of COM of the system from the centre of disc.**



Sol. Let areal density is ρ

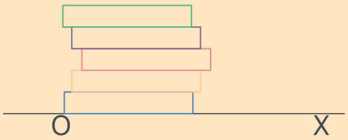
$$M_C = \pi\left(\frac{a}{2}\right)^2 \times \rho$$

$$M_S = a^2 \times \rho.$$

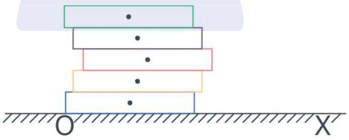
$$X_{cm} = \frac{\rho a^2 (a)}{\rho\frac{\pi a^2}{4} + \rho a^2} = \frac{4a}{(4+\pi)}$$

Right of the disc centre.

Q10 **Five homogeneous bricks, each of length 'L', are arranged as shown in figure. Each brick is displaced w.r.t the one in contact by L/5. Find the x-coordinate of the COM relative to the origin O shown.**



Sol.



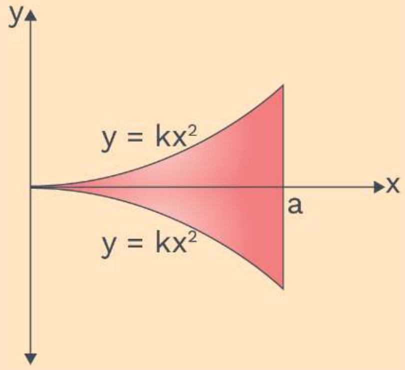
COM of brick 1 and 5 $\rightarrow \frac{L}{2}$

COM of brick 2 and 4 $\rightarrow \frac{L}{2} + \frac{L}{5}$

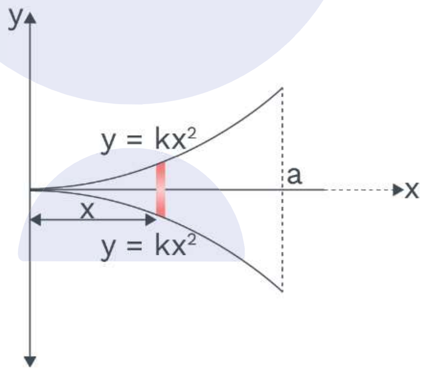
COM of brick 3 $\rightarrow \frac{L}{2} + \frac{2L}{5}$

$$X_{cm} = \frac{2m\frac{L}{2} + 2m\left(\frac{L}{2}+\frac{L}{5}\right) + m\left(\frac{L}{2}+\frac{2L}{5}\right)}{5m} = \frac{33L}{50}.$$

Q11 **A thin uniform sheet of metal of uniform thickness is cut into the shape bounded by the line x = a and y = ± kx², as shown. Calculate the coordinates of the COM.**



Sol.



Length of the shaded region = 2y = $2kx^2$

$dm = 2y\,dx \times \rho$

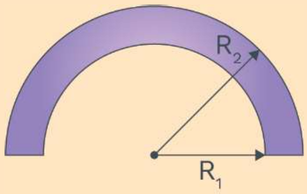
$dm = 2kx^2 \times \rho\,dx$

$$M = \int_0^a dm = \int_0^a 2kx^2 \rho dx = \frac{2k\rho a^3}{3}$$

$$X_{cm} = \frac{\int_0^a x dm}{\int_0^a dm} = \frac{\int_0^a 2kx^3 \rho dx}{\int_0^a xm} = \frac{2k\rho\left(\frac{a^4}{4}\right)}{2k\rho\left(\frac{a^3}{3}\right)} \Rightarrow X_{cm} = \frac{3a}{4}$$

Using symmetry the y-coordinate of the shown plate will be zero.

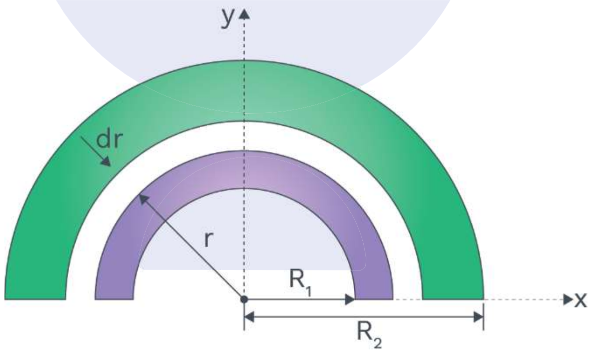
Q12 **Find the centre of mass of an annular half disc shown in figure.**



Sol. Let ρ be mass per unit area of object. To find its COM we consider an element as a half ring of dm mass as shown in figure of radius' 'r and width dr and there we have

Now $dm = \rho\pi r\, dr$

COM of this half ring will be at $\left(0, \frac{2r}{\pi}\right)$



$$y_{cm} = \frac{1}{M}\int_{R_1}^{R_2} (\rho.\pi r\, dr).\frac{2r}{\pi}$$

$$y_{cm} = \frac{2\rho}{\rho\frac{\pi}{2}\left(R_2^2 - R_1^2\right)}\int_{R_1}^{R_2} r^2 dr = \frac{4\left(R_2^2 - R_1^3\right)}{3\pi\left(R_2^2 - R_1^2\right)}$$

Q13 **A man of mass 'M' hanging with a light rope which is connected with a balloon of mass 'm'. the system is at rest in air. When man rises a distance h with respect to balloon Find.**
(a) The distance raised by man
(b) The distance descended by balloon

Sol.

$$Y_{com} = \frac{Mh}{M+m}$$

Since no external force is acting
∴ COM should be at rest.

$$\Delta y_{CM} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}$$

Let balloon descend by a distance x.

$$0 = \frac{m(x) + M(x-h)}{m+M}$$

$$Mh = (m + M)\, x$$

$$x = \frac{Mh}{m+M} \text{ (Distance descend by balloon)}$$

$$h - x = \frac{mh}{m+M} \text{ (Distance raised by man O).}$$

Q14 **In a process a neutron which is initially at rest, decays into a proton, an electron and an antineutrino. The ejected electron has a momentum of $p_1 = 2.4 \times 10^{-26}$ kg-m/s and the antineutrino has $p_2 = 7.0 \times 10^{-27}$ kg-m/s. Calculate the recoil speed of the proton if the electron and the antineutrino are ejected**
(a) along the same direction.
(b) in mutually perpendicular directions. (Mass of the proton $m_p = 1.67 \times 10^{-27}$ kg).

Sol.

(a) $p_1 = 2.4 \times 10^{-26}\,\text{kg} - \text{m/sec}.$

$p_2 = 7.0 \times 10^{-27}\,\text{kg} - \text{m/sec}.$

$p_1 + p_2 + p_3 = 0$

$\therefore\ p_3 = -\left(24 \times 10^{-27} + 7.0 \times 10^{-27}\right)$

$|p_3| = 31 \times 10^{-27}$

$\therefore\quad v_3 = \dfrac{31 \times 10^{-27}}{1.67 \times 10^{-27}} = 18.6\,\text{m/sec}.$

(b) $p_e = 2.4 \times 10^{-26}\,\hat{i}$

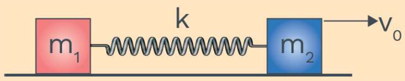
$\vec{p}_{an} = 7.0 \times 10^{-27}\,\hat{j}$

$\vec{p}_p = -(\vec{p}_e + \vec{p}_{an}) = -\left(24 \times 10^{-27}\,\hat{i} + 7.0 \times 10^{-27}\,\hat{j}\right)$

$|\vec{p}_p| = \sqrt{(24)^2 + (7.0)^2} \times 10^{-27}$

$v_p = \dfrac{|\vec{p}_p|}{m_p} = 15.0\,\text{m/sec}.$

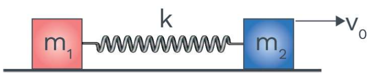
Q15 **Two block of the masses m_1 and m_2 are connected with help of a spring of spring constant 'k' initially the spring in its natural length as shown. A sharp impulse is given to mass m_2 so that it acquires a velocity v_0 towards right. If the system is kept an smooth floor then find**
(a) the velocity of the COM
(b) the maximum elongation that the spring will suffer ?



Sol.

$$x_{cm} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

$$v_{cm} = \frac{m_1v_1 + m_2v_2}{m_1 + m_2}$$



$$v_{cm} = \frac{0 + m_2v_0}{m_1 + m_2}$$

F = 0; p = constant

$$m_2v_0 = (m_1 + m_2)v \;\; \& \;\; \frac{1}{2}m_2v_0^2 = \frac{1}{2}m_1v^2 + \frac{1}{2}m_2v^2 + \frac{1}{2}kx_0^2$$

v_{cm} Remain unchanged

$$\therefore \quad \frac{1}{2}m_2v_0^2 = \frac{1}{2}(m_1 + m_2)\left(\frac{m_2v_0}{m_1 + m_2}\right)^2 + \frac{1}{2}kx_0^2$$

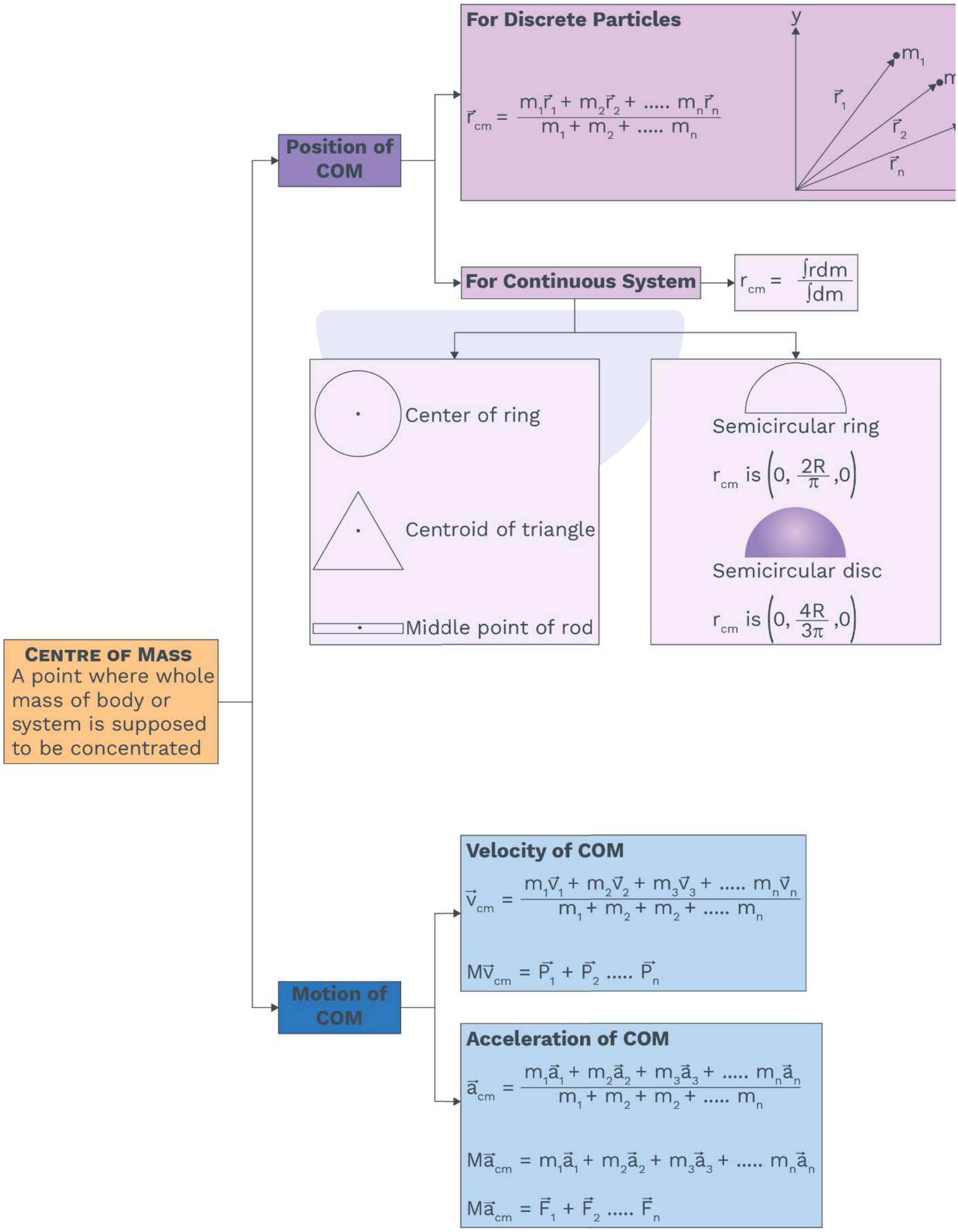
$$m_2v_0^2\left[1 - \frac{m_2 + m_2}{(m_1 + m_2)^2} \times m_2\right] = kx_0^2$$

$$\therefore \quad m_2v_0^2\left[\frac{m_1 + m_2 - m_2}{m_1 + m_2}\right] = kx_0^2$$

$$v_0^2\frac{m_1m_2}{m_1 + m_2} \times \frac{1}{k} = x_0^2$$

$$\Rightarrow \quad x_0 = v_0\sqrt{\frac{m_1m_2}{k(m_1 + m_2)}}.$$

Mind Map



CENTRE OF MASS
A point where whole mass of body or system is supposed to be concentrated
Position of COM
For Discrete Particles
$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + ... m_n\vec{r}_n}{m_1 + m_2 + ... m_n}$
y
m_1
$\vec{r}_1$
$\vec{r}_2$
$\vec{r}_n$
For Continuous System
$r_{cm} = \frac{\int r dm}{\int dm}$
Center of ring
Centroid of triangle
Middle point of rod
Semicircular ring
r_{cm} is $\left(0, \frac{2R}{\pi}, 0\right)$
Semicircular disc
r_{cm} is $\left(0, \frac{4R}{3\pi}, 0\right)$
Motion of COM
Velocity of COM
$\vec{v}_{cm} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + ... m_n\vec{v}_n}{m_1 + m_2 + m_2 + ... m_n}$
$M\vec{v}_{cm} = \vec{P}_1 + \vec{P}_2 ..... \vec{P}_n$
Acceleration of COM
$\vec{a}_{cm} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + ... m_n\vec{a}_n}{m_1 + m_2 + m_2 + ... m_n}$
$M\vec{a}_{cm} = m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + ... m_n\vec{a}_n$
$M\vec{a}_{cm} = \vec{F}_1 + \vec{F}_2 ..... \vec{F}_n$